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A Normal Paracontact Metric Manifold Satisfying Some Conditions on the M-Projective Curvature Tensor

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Abstract

In the present paper we have studied the curvature tensors of a normal paracontact metric manifold satisfying the conditions $R(\xi,Y)W^*=0$, $W^*(\xi,Y)R=0$, $W^*(\xi,Y)\widetilde{Z}=0$, $W^*(\xi,Y)S=0$ and $W^*(\xi,Y)\widetilde{C}=0$, where W^*,R, S, $\widetilde{Z}$ and $\widetilde{C}$ are the M-projective curvature, Riemannian curvature, Ricci, concircular curvature and quasi-conformal curvature tensor, respectively.

Keywords: *Paracontact metric manifold, M-projective curvature tensor, concircular, quasi-conformal*
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1. Introduction

Pokhariyal and Mishra [11] defined a tensor field W^* on a n-dimensional Riemannian manifold as

$$\begin{aligned} W^*(X,Y)Z &= R(X,Y)Z-\frac{1}{4n}\big[S(Y,Z)X-S(X,Z)Y \\ &+ g(Y,Z)QX-g(X,Z)QY\big]. \end{aligned} \tag{1.1}$$

Such a tensor field W^* is known as M-projective curvature tensor, where Q is Ricci operator and S is Ricci tensor. Ojha [9, 10] defined and studied the properties of M-projective curvature tensor in Sasakian and Kähler manifolds. He also showed that it bridges the gap between the conformal curvature tensor, conharmonic curvature tensor, and concircular curvature tensor on one side and H-projective curvature tensor on the other [12]. Recently M-projective curvature tensor has been studied by Chaubey [4], Chaubey et al. [3, 5], Devi and Singh [6], Kumar [8], Singh [13], Vankatesha and Sumangala [14].

The concircular curvature tensor $\widetilde{Z}$ and the quasi-conformal curvature tensor $\widetilde{C}$ of n-dimensional Riemann manifold are given by

$$\widetilde{Z}(X,Y)Z=R(X,Y)Z-\frac{r}{n(n-1)}\big[g(Y,Z)X-g(X,Z)Y\big] \tag{1.2}$$

and

$$\begin{aligned} \widetilde{C}(X,Y)Z &= aR(X,Y)Z+b\big[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY\big] \\ &- \frac{r}{n}\big[\frac{a}{n-1}+2b\big]\big[g(Y,Z)X-g(X,Z)Y\big], \end{aligned} \tag{1.3}$$

where a and b are two scalars, and r is the scalar curvature of the manifold. The notion of quasi-conformal curvature tensor was introduced by Yano and Sawaki [17]. If $a=1$ and $b=-\frac{1}{n-1}$, then quasi-conformal curvature tensor reduces to conformal curvature tensor.

In [1, 2], we have studied the curvature tensors satisfying some conditions on a $C(\alpha)$-manifold and induced cases have been discussed.

The study of paracontact geometry was initiated by Kenayuki and Williams [7]. Zamkovoy studied paracontact metric manifolds and their subclasses [18]. Recently, Welyczko studied slant curves and Legendre curves in 3-dimensional normal almost paracontact metric manifolds

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[15, 16].

Motiveted by these ideas, we have studied the M-projective curvature tensor of normal paracontact metric manifolds. Section 2 deals with some preliminaries on a normal paracontact metric manifold. In section 3, we study a normal paracontact metric manifold satisfying $R(\xi,Y)W^{*}=0$, $W^{*}(\xi,Y)R=0$, $W^{*}(\xi,Y)\widetilde{Z}=0$, $W^{*}(\xi,Y)S=0$ and $W^{*}(\xi,Y)\widetilde{C}=0$.

2. Preliminaries

A $n-$dimensional differentiable manifold (M,g) is said to be an almost paracontact metric manifold if there exist on M a $(1,1)$ tensor field ϕ, a contravariant vector ξ and a 1-form η-such that

$$\phi^2 X = X-\eta(X)\xi, \qquad \phi\xi=0, \qquad \eta(\phi X)=0, \qquad \eta(\xi)=1 \tag{2.1}$$

and

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \qquad \eta(X)=g(X,\xi), \tag{2.2}$$

for any $X,Y\in\chi(M)$.
If in addition to the above relations, we have

$$(\nabla_X\phi)Y=-g(X,Y)\xi-\eta(Y)X+2\eta(X)\eta(Y)\xi, \tag{2.3}$$

then M is called a normal paracontact metric manifold, where ∇ is Levi-Civita connection.
We have also on a normal paracontact metric manifold M

$$\phi X=\nabla_X\xi \tag{2.4}$$

for any $X\in\chi(M)$.

Moreover, if such a manifold has constant sectional curvature equal to c, then the Riemannian curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X-g(X,Z)Y\} \\ &+ \frac{c-1}{4}\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi+g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y-2g(\phi X,Y)\phi Z\} \end{aligned} \tag{2.5}$$

for any vector fields $X,Y,Z\in\chi(M)$.

In a normal paracontact metric space form by direct calculations, we can easily to see that

$$S(X,Y)=(\frac{c(n-5)+3n+1}{4})g(X,Y)+(\frac{(c-1)(5-n)}{4})\eta(X)\eta(Y) \tag{2.6}$$

from which

$$QX=(\frac{c(n-5)+3n+1}{4})X+(\frac{(c-1)(5-n)}{4})\eta(X)\xi \tag{2.7}$$

for any $X,Y\in\chi(M)$, where Q is the Ricci operatory and S is the Ricci tensor of M.

Corollary 2.1. *A normal paracontact metric space form is always an η-Einstein manifold.*

From (2.6) and (2.7), we can easily see

$$S(X,\xi)=(n-1)\eta(X), \tag{2.8}$$

$$Q\xi=(n-1)\xi, \tag{2.9}$$

and

$$r=\frac{n-1}{4}\left[c(n-5)+3n+5\right]. \tag{2.10}$$

Let M be n-dimensional normal paracontact metric space form and we denote the Riemannian curvature tensor of R, then we have from (2.5), for $X=\xi$

$$R(\xi,Y)Z=g(Y,Z)\xi-\eta(Z)Y, \tag{2.11}$$

for $Z=\xi$

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y. \tag{2.12}$$

In (2.12) choosing $Y=\xi$, we get

$$R(X,\xi)\xi = X-\eta(X)\xi. \tag{2.13}$$

Taking the inner product both of the sides (2.5) with $\xi\in\chi(M)$, we obtain

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X)-g(X,Z)\eta(Y). \tag{2.14}$$

In the same way we obtain from (1.2) and (1.3),

$$\widetilde{Z}(\xi,Y)Z = \big[1-\frac{r}{n(n-1)}\big]\big[g(Y,Z)\xi-\eta(Z)Y\big], \tag{2.15}$$

$$\widetilde{Z}(\xi,Y)\xi = \big[1-\frac{r}{n(n-1)}\big]\big[\eta(Y)\xi-Y\big], \tag{2.16}$$

$$\widetilde{C}(\xi,Y)Z = \big[\frac{4a+b(n-5)(1-c)}{4}-\frac{r}{n}\big[\frac{a}{n-1}+2b\big]\big]\big[g(Y,Z)\xi-\eta(Z)Y\big] \tag{2.17}$$

and

$$\widetilde{C}(\xi,Y)\xi = \big[\frac{4a+b(n-5)(1-c)}{4}-\frac{r}{n}\big[\frac{a}{n-1}+2b\big]\big]\big[\eta(Y)\xi-Y\big]. \tag{2.18}$$

Also from (1.1), we have

$$W^*(\xi,Y)Z = \big[\frac{c(5-n)+3(3n+1)}{16n}\big]\big[g(Y,Z)\xi-\eta(Z)Y\big] \tag{2.19}$$

and

$$W^*(X,Y)\xi = \big[\frac{c(5-n)+3(3n+1)}{16n}\big]\big[\eta(Y)X-\eta(X)Y\big]. \tag{2.20}$$

3. M-Projective Curvature Tensor of a Normal Paracontact Metric Manifold

Theorem 3.1. *A $n-$dimensional normal paracontact metric space form is always $R(\xi,Y)W^*=0$.*

Proof: Let M be $n-$dimensional a normal paracontact metric space form. Then, we can easily to see that

$$\begin{aligned}(R(X,Y)W^*)(U,V,Z) &= R(X,Y)W^*(U,V)Z-W^*(R(X,Y)U,V)Z\\ &- W^*(U,R(X,Y)V)Z-W^*(U,V)R(X,Y)Z,\end{aligned} \tag{3.1}$$

for any $X,Y,U,V,Z\in\chi(M)$. Taking $X=\xi$ in (3.1) and using (2.11), we obtain

$$\begin{aligned}(R(\xi,Y)W^*)(U,V,Z) &= g(Y,W^*(U,V)Z)\xi-\eta(W^*(U,V)Z)Y\\ &- g(Y,U)W^*(\xi,V)Z+\eta(U)W^*(Y,V)Z\\ &- g(Y,V)W^*(U,\xi)Z+\eta(V)W^*(U,Y)Z\\ &- g(Y,Z)W^*(U,V)\xi+\eta(Z)W^*(U,V)Y.\end{aligned} \tag{3.2}$$

Using (2.19) and (2.20) in (3.2), and putting $U=\xi$, we obtain

$$(R(\xi,Y)W^*)(\xi,V,Z) = W^*(Y,V)Z-\big[\frac{c(5-n)+3(3n+1)}{16n}\big]\big[g(V,Z)Y-g(Y,Z)V\big]. \tag{3.3}$$

In (3.3), using (1.1) and substituting $Z=\xi$, we conclude

$$R(\xi,Y)W^*=0.$$

The proof is completed. The converse is obvious.

Theorem 3.2. *Let M be $n-$dimensional a normal paracontact metric space form. Then $W^*(\xi,Y)R=0$ if and only if either the scalar curvature of M is $r=(n-1)(3n+2)$ or M reduces real space form with constant sectional curvature $c=1$.*

Proof: Suppose that $W^*(\xi,Y)R=0$. Then, we have

$$\begin{aligned}(W^*(\xi,Y)R)(U,V,Z) &= W^*(\xi,Y)R(U,V)Z-R(W^*(\xi,Y)U,V)Z\\ &- R(U,W^*(\xi,Y)V)Z-R(U,V)W^*(\xi,Y)Z\\ &= 0,\end{aligned} \tag{3.4}$$

for any $Y,U,V,Z\in\chi(M)$. In (3.4), using (2.19) we obtain

$$\begin{aligned}0 &= \big[\frac{c(5-n)+3(3n+1)}{16n}\big]\big[g(Y,R(U,V)Z)\xi-\eta(R(U,V)Z)Y\big]\\ &- g(Y,U)R(\xi,V)Z+\eta(U)R(Y,V)Z\\ &- g(Y,V)R(U,\xi)Z+\eta(V)R(U,Y)Z\\ &- g(Y,Z)R(U,V)\xi+\eta(Z)R(U,V)Y.\end{aligned} \tag{3.5}$$

Substituting $U=\xi$ in (3.5) and using (2.11) and (2.12), we conclude

$$\Big[\frac{c(5-n)+3(3n+1)}{16n}\Big]\big[R(Y,V)Z-g(V,Z)Y+g(Y,Z)V\big]=0. \tag{3.6}$$

From (3.6), we have

$$c=\frac{3(3n+1)}{n-5}. \tag{3.7}$$

In (2.10), using (3.7) we find

$$r=(n-1)(3n+2) \tag{3.8}$$

On the other hand, from (3.6) we get

$$R(Y,V)Z=g(V,Z)Y-g(Y,Z)V. \tag{3.9}$$

This tell us that M reduces real space form with constant sectional curvature $c=1$.
The converse is obvious. The proof is completed.

Theorem 3.3. *Let M be $n-$dimensional a normal paracontact metric space form. Then $W^*(\xi,Y)\widetilde{Z}=0$ if and only if either the scalar curvature of M is $r=(n-1)(3n+2)$ or M reduces real space form with constant sectional curvature $c=1$.*

Proof: Assume that $W^*(\xi,Y)\widetilde{Z}=0$. Then, we have

$$\begin{aligned}(W^*(\xi,Y)\widetilde{Z})(U,V,Z) &= W^*(\xi,Y)\widetilde{Z}(U,V)Z-\widetilde{Z}(W^*(\xi,Y)U,V)Z\\ &- \widetilde{Z}(U,W^*(\xi,Y)V)Z-\widetilde{Z}(U,V)W^*(\xi,Y)Z\\ &= 0,\end{aligned} \tag{3.10}$$

for any $Y,U,V,Z\in\chi(M)$. In (3.10), using (2.19) we obtain

$$\begin{aligned}0 &= \Big[\frac{c(5-n)+3(3n+1)}{16n}\Big]\big[g(Y,\widetilde{Z}(U,V)Z)\xi-\eta(\widetilde{Z}(U,V)Z)Y\big]\\ &- g(Y,U)\widetilde{Z}(\xi,V)Z+\eta(U)\widetilde{Z}(Y,V)Z\\ &- g(Y,V)\widetilde{Z}(U,\xi)Z+\eta(V)\widetilde{Z}(U,Y)Z\\ &- g(Y,Z)\widetilde{Z}(U,V)\xi+\eta(Z)\widetilde{Z}(U,V)Y.\end{aligned} \tag{3.11}$$

Taking $U=\xi$ in (2.15) and using (2.16), we obtain

$$\Big[\frac{c(5-n)+3(3n+1)}{16n}\Big]\big[\widetilde{Z}(Y,V)Z-(1-\frac{r}{n(n-1)})\big[g(V,Z)Y-g(Y,Z)V\big]\big]=0. \tag{3.12}$$

In (2.15), using (1.2) we conclude

$$\Big[\frac{c(5-n)+3(3n+1)}{16n}\Big]\big[R(Y,V)Z-g(V,Z)Y+g(Y,Z)V\big]. \tag{3.13}$$

This proves our assertion. The converse obvious.

Theorem 3.4. *Let M be $n-$dimensional a normal paracontact metric space form. Then $W^*(\xi,Y)S=0$ if and only if either the scalar curvature of M is $r=(n-1)(3n+2)$ or M reduces an Einstein manifold.*

Proof: Suppose that $W^*(\xi,Y)S=0$. Then, we can easily see that

$$S(W^*(\xi,Y)Z,U)+S(Z,W^*(\xi,Y)U)=0. \tag{3.14}$$

In (3.14), using (2.19) we obtain

$$\begin{aligned}0 &= \Big[\frac{c(5-n)+3(3n+1)}{16n}\Big]\big[(n-1)g(Y,Z)\eta(U)-S(Y,U)\eta(Z)\\ &+ (n-1)g(Y,U)\eta(Z)-S(Y,Z)\eta(U)\big].\end{aligned} \tag{3.15}$$

Substituting $Z=\xi$ in (3.15), we find

$$\Big[\frac{c(5-n)+3(3n+1)}{16n}\Big]\big[S(Y,U)-(n-1)g(Y,U)\big]=0. \tag{3.16}$$

From (3.16), we get

$$c=\frac{3(3n+1)}{n-5}. \tag{3.17}$$

This tell us that the scalar curvature of M is $r=(n-1)(3n+2)$. On the other hand, from (3.16) we have

$$S(Y,U)=(n-1)g(Y,U), \tag{3.18}$$

and this implies that M reduces an Einstein manifold.
This proves our assertion. The converse is obvious.

Theorem 3.5. *Let M be $n-$dimensional a normal paracontact metric space form. Then $W^*(\xi,Y)\widetilde{C}=0$ if and only if either the scalar curvature of M is $r=(n-1)(3n+2)$ or M reduces real space form with constant sectional curvature $c=[\frac{4a+b\big[(n-5)(1-3c)-2(3n+1)\big]}{4a}]$.*

Proof: Suppose that $W^*(\xi,Y)\widetilde{C}=0$. Then, we have

$$\begin{aligned}(W^*(\xi,Y)\widetilde{C})(U,V,Z) &= W^*(\xi,Y)\widetilde{C}(U,V)Z-\widetilde{C}(W^*(\xi,Y)U,V)Z\\ &- \widetilde{C}(U,W^*(\xi,Y)V)Z-\widetilde{C}(U,V)W^*(\xi,Y)Z\\ &= 0, \end{aligned} \tag{3.19}$$

for any $Y,U,V,Z\in\chi(M)$. Using (2.19)in (3.19), we obtain

$$\begin{aligned}0 &= [\frac{c(5-n)+3(3n+1)}{16n}]\big[g(Y,\widetilde{C}(U,V)Z)\xi-\eta(\widetilde{C}(U,V)Z)Y\big]\\ &- g(Y,U)\widetilde{C}(\xi,V)Z+\eta(U)\widetilde{C}(Y,V)Z\\ &- g(Y,V)\widetilde{C}(U,\xi)Z+\eta(V)\widetilde{C}(U,Y)Z\\ &- g(Y,Z)\widetilde{C}(U,V)\xi+\eta(Z)\widetilde{C}(U,V)Y. \end{aligned} \tag{3.20}$$

Taking $U=\xi$ in (3.20) and using (2.17) and (2.18), we obtain

$$\begin{aligned}0 &= [\frac{c(5-n)+3(3n+1)}{16n}]\\ &\otimes \big[\widetilde{C}(Y,V)Z-[\frac{4a+b(n-5)(1-c)}{4}]\\ &- \frac{r}{n}[\frac{a}{n-1}+2b]\big]g(V,Z)Y-g(Y,Z)V\big]. \end{aligned} \tag{3.21}$$

In (3.21), substituting $Y\longrightarrow\phi Y$ and $V\longrightarrow\phi V$ we conclude

$$R(\phi Y,\phi V)Z=[\frac{4a+b\big[(n-5)(1-3c)-2(3n+1)\big]}{4a}]\big[g(\phi Z,W)\phi Y-g(\phi Y,Z)\phi W\big]. \tag{3.22}$$

This proves our assertion. The converse is obvious.

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