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Common fixed point of four maps in $b-$metric spaces

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Abstract

In this paper, some common fixed point results for four mappings satisfying generalized contractive condition in a $b-$metric space are proved. Advantage of our work in comparison with studies done in the context of $b-$metric is that, the $b-$metric function used in the theorems and results are not necessarily continuous. So, our results extend and improve several comparable results obtained previously. We also present two examples that show the applicability and validity of our results.

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1. Introduction and preliminaries

Czerwik in [13] introduced the concept of a $b-$metric space. Since then, several research papers have dealt with fixed point theory for single-valued and multivalued operators in $b-$metric spaces (see, e.g., [2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 16, 17, 18, 20, 21, 26, 25]). Pacurar [26] obtained results on fixed point of sequences of almost contractions in $b-$metric spaces. Recently, Hussain and Shah [17] studied KKM mappings in cone $b-$metric spaces. Khamsi [20, 21] showed that each cone metric space over a normal cone induces a b-metric structure (see also [18]).

The aim of this paper is to present some common fixed point results for four mappings satisfying generalized contractive condition in a $b-$metric space, where the $b-$metric is

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not necessarily continuous. Many authors in their work have used the $b-$metric spaces in which $b-$metric function is continuous, but the techniques used here can be employed in the setup of discontinuous $b-$metric spaces. From this point of view the results obtained in this paper generalize and extend several comparable existing results in the framework of $b-$metric spaces. On the other hand, many authors (see, e.g., [1, 22, 24, 27]) have used the Ćirić [10, 11] and Hardy-Rogers [15] type contractions in their works. In this paper we focused on Ćirić and Hardy-Rogers type contractions and present some common fixed point results in b-metric spaces.

Consistent with [13] and [25, p. 264], following definitions and results will be needed in the sequel.

1.1. Definition. ([13]) Let X be a nonempty set and $k \geq 1$ a given real number. A function $d : X \times X \to R^+$ is a $b-$metric iff for each $x, y, z \in X$, following conditions are satisfied:

(b1) $d(x, y) = 0$ iff $x = y$,

(b2) $d(x, y) = d(y, x)$,

(b3) $d(x, z) \leq k[d(x, y) + d(y, z)]$.

A pair (X, d) is called a $b-$metric space.

It should be noted that the class of $b-$metric spaces is effectively larger than that of metric spaces. Indeed, a b-metric is a metric if and only if $k = 1$.

Following is an example which shows that a $b-$metric need not be a metric (see, also [25, p. 264]).

1.2. Example. Let (X, d) be a metric space and $\rho(x, y) = (d(x, y))^p$, where $p > 1$ is a real number. We show that ρ is a $b-$metric with $k = 2^{p-1}$: Obviously, conditions (i) and (ii) of definition 1.1 are satisfied. If $1 < p < \infty$, then convexity of the function $f(x) = x^p$ $(x > 0)$ implies that $\left(\frac{a+b}{2}\right)^p \leq \frac{1}{2}\left(a^p + b^p\right)$, that is, $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ holds.

Thus for each $x, y, z \in X$, we have

$$\begin{aligned} \rho(x, y) &= (d(x, y))^p \leq (d(x, z) + d(z, y))^p \\ &\leq 2^{p-1}\left((d(x, z))^p + (d(z, y))^p\right) = 2^{p-1}(\rho(x, z) + \rho(z, y)). \end{aligned}$$

So condition (iii) of definition 1.1 holds and ρ is a $b-$metric. Note that (X, ρ) is not necessarily a metric space.

For example, if $X = \mathbb{R}$ be the set of real numbers and $d(x, y) = |x - y|$ a usual metric, then $\rho(x, y) = (x - y)^2$ is a $b-$metric on $\mathbb{R}$ with $k = 2$, but not a metric on $\mathbb{R}$, as the triangle inequality for a metric does not hold.

Before stating our results, we present some definitions and propositions in a $b-$metric space.

1.3. Definition. ([8]) Let (X, d) be a $b-$metric space. Then a sequence $\{x_n\}$ in X is called:

(a) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to +\infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.

(b) Cauchy if and only if $d(x_n, x_m) \to 0$, as $n, m \to +\infty$.

1.4. Proposition. (See, remark 2.1 in [8]) In a $b-$metric space (X, d) the following assertions hold:

(i) a convergent sequence has a unique limit,

(ii) each convergent sequence is Cauchy,

(iii) in general, a $b-$metric is not continuous.

1.5. Definition. ([8]) Let (X, d) be a $b-$metric space. If Y is a nonempty subset of X, then the closure $\overline{Y}$ of Y is the set of limits of all convergent sequences of points in Y, i.e.,

$$\overline{Y} = \left\{ x \in X : \text{there exists a sequence } \{x_n\} \text{ in } Y \text{ such that } \lim_{n\to\infty} x_n = x \right\}.$$

1.6. Definition. ([8]) Let (X, d) be a $b-$metric space. Then a subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\}$ in Y which converges to an element x, we have $x \in Y$ (i.e., $\overline{Y} = Y$).

1.7. Definition. ([8]) The $b-$metric space (X, d) is complete if every Cauchy sequence in X converges.

In general a $b-$metric function d for $k > 1$ is not jointly continuous in all of its two variables. Following is an example of a $b-$metric which is not continuous.

1.8. Example. ([16]) Let $X = \mathbb{N} \cup \{\infty\}$ and $D : X \times X \to \mathbb{R}$ defined by

$$D(m, n) = \begin{cases} 0, & \text{if } m = n, \\ \left|\frac{1}{m} - \frac{1}{n}\right|, & \text{if } m, n \text{ are even or } mn = \infty, \\ 5, & \text{if } m \text{ and } n \text{ are odd and } m \neq n, \\ 2, & \text{otherwise.} \end{cases}$$

Then it is easy to see that for all $m, n, p \in X$, we have

$$D(m, p) \leq 3(D(m, n) + D(n, p)).$$

Thus, (X, D) is a $b-$metric space with $k = 3$. If $x_n = 2n$, for each $n \in \mathbb{N}$, then

$$D(2n, \infty) = \frac{1}{2n} \to 0, \text{ as } n \to \infty,$$

that is, $x_n \to \infty$, but $D(x_{2n}, 1) = 2 \nrightarrow D(\infty, 1)$, as $n \to \infty$.

As $b-$metric is not continuous in general, so we need the following simple lemma about the b-convergent sequences.

1.9. Lemma. ([2]) Let (X, d) be a $b-$metric space with $k \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y, respectively. Then, we have

$$\frac{1}{k^2} d(x, y) \leq \liminf_{n\longrightarrow\infty} d(x_n, y_n) \leq \limsup_{n\longrightarrow\infty} d(x_n, y_n) \leq k^2 d(x, y).$$

In particular, if $x = y$, *then we have* $\lim_{n\longrightarrow\infty} d(x_n, y_n) = 0$. *Moreover for each* $z \in X$ *we have*

$$\frac{1}{k} d(x, z) \leq \liminf_{n\longrightarrow\infty} d(x_n, z) \leq \limsup_{n\longrightarrow\infty} d(x_n, z) \leq k d(x, z).$$

Also, we present the following simple lemma needed in the proof of our main result.

1.10. Lemma. Let (X, d) be a $b-$metric space. If there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} x_n = t$ for some $t \in X$ then $\lim_{n\to\infty} y_n = t$.

Proof. By a triangle inequality in a $b-$metric space, we have

$$d(y_n, t) \leq k(d(y_n, x_n) + d(x_n, t)).$$

Now, by taking the upper limit when $n \to \infty$ in the above inequality we get

$$\limsup_{n\longrightarrow\infty} d(y_n, t) \leq k(\limsup_{n\longrightarrow\infty} d(x_n, y_n) + \limsup_{n\longrightarrow\infty} d(x_n, t)) = 0.$$

Hence $\lim_{n\to\infty} y_n = t$.

1.11. Definition. ([19]) Let (X, d) be a $b-$metric space. A pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

2. Common fixed point results

2.1. Theorem. Suppose that f, g, S and T are self mappings on a complete $b-$metric space (X, d) such that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$. If

$$d(fx, gy) \leq \frac{q}{k^4} \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2}(d(Sx, gy) + d(fx, Ty))\}, \tag{2.1}$$

holds for each $x, y \in X$ with $0 < q < 1$, then f, g, S and T have a unique common fixed point in X provided that S and T are continuous and and pairs $\{f, S\}$ and $\{g, T\}$ are compatible.

Proof. Let $x_0 \in X$. As $f(X) \subseteq T(X)$, there exists $x_1 \in X$ such that $fx_0 = Tx_1$. Since $gx_1 \in S(X)$, we can choose $x_2 \in X$ such that $gx_1 = Sx_2$. In general, x_{2n+1} and x_{2n+2} are chosen in X such that $fx_{2n} = Tx_{2n+1}$ and $gx_{2n+1} = Sx_{2n+2}$. Define a sequence $\{y_n\}$ in X such that $y_{2n} = fx_{2n} = Tx_{2n+1}$, and $y_{2n+1} = gx_{2n+1} = Sx_{2n+2}$, for all $n \geq 0$. Now, we show that $\{y_n\}$ is a Cauchy sequence. Consider

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n+1}) \\
&\leq \frac{q}{k^4} \max\{d(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, Sx_{2n}), d(gx_{2n+1}, Tx_{2n+1}), \\
&\quad \frac{1}{2}(d(Sx_{2n}, gx_{2n+1}) + d(fx_{2n}, Tx_{2n+1}))\} \\
&= \frac{q}{k^4} \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), \frac{1}{2}(d(y_{2n-1}, y_{2n+1}) + \\
&\quad + d(y_{2n}, y_{2n}))\} \\
&= \frac{q}{k^4} \max\{(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{d(y_{2n-1}, y_{2n+1})}{2}\} \\
&\leq \frac{q}{k^4} \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), \frac{k}{2}(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}))\}.
\end{aligned}$$

If $d(y_{2n}, y_{2n+1}) > d(y_{2n-1}, y_{2n})$ for some n, then from the above inequality we have

$$d(y_{2n}, y_{2n+1}) < \frac{q}{k^3} d(y_{2n}, y_{2n+1}),$$

a contradiction. Hence $d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$ for all $n \in \mathbb{N}$. Also, by the above inequality we obtain

$$d(y_{2n}, y_{2n+1}) \leq \frac{q}{k^3} d(y_{2n-1}, y_{2n}). \tag{2.2}$$

Similarly,

$$d(y_{2n-1}, y_{2n}) \leq \frac{q}{k^3} d(y_{2n-2}, y_{2n-1}). \tag{2.3}$$

From 2.2 and 2.3 we have

$$d(y_n, y_{n-1}) \leq \lambda d(y_{n-1}, y_{n-2}),$$

where $\lambda = \dfrac{q}{k^3} < 1$ and $n \geq 2$. Hence, for all $n \geq 2$, we obtain

$$d(y_n, y_{n-1}) \leq \cdots \leq \lambda^{n-1} d(y_1, y_0). \tag{2.4}$$

So for all $n > m$, we have

$$d(y_n, y_m) \leq kd(y_m, y_{m+1}) + k^2 d(y_{m+1}, y_{m+2}) + \cdots + k^{n-m-1} d(y_{n-1}, y_n).$$

Now from (4), we have

$$\begin{aligned}d(y_n, y_m) &\leq (k\lambda^m + k^2\lambda^{m+1} + \cdots + k^{n-m-1}\lambda^{n-1})d(y_1, y_0)\\ &\leq k\lambda^m[1 + k\lambda + (k\lambda)^2 + \cdots]d(y_1, y_0)\\ &\leq \frac{k\lambda^m}{1-k\lambda}d(y_1, y_0)\end{aligned}$$

On taking limit as $m, n \to \infty$, we have $d(y_n, y_m) \to 0$ as $k\lambda < 1$. Therefore $\{y_n\}$ is a Cauchy sequence. Since X is a complete $b-$metric space, there is some y in X such that

$$\lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y.$$

We show that y is a common fixed point of f, g, S and T.

Since S is continuous, therefore $\lim_{n\to\infty} S^2x_{2n+2} = Sy$ and $\lim_{n\to\infty} Sfx_{2n} = Sy$.

Since a pair $\{f, S\}$ is compatible, $\lim_{n\to\infty} d(fSx_{2n}, Sfx_{2n}) = 0$. So by lemma 1.10, we have $\lim_{n\to\infty} fSx_{2n} = Sy$.

Putting $x = Sx_{2n}$ and $y = x_{2n+1}$ in 2.1, we obtain

$$\begin{aligned}d(fSx_{2n}, gx_{2n+1}) \quad &\leq \quad \frac{q}{k^4}\max\{d(S^2x_{2n}, Tx_{2n+1}), d(fSx_{2n}, S^2x_{2n}), (gx_{2n+1}, Tx_{2n+1}),\\ &\qquad \frac{1}{2}(d(S^2x_{2n}, gx_{2n+1}) + d(fSx_{2n}, Tx_{2n+1}))\}.\end{aligned} \tag{2.5}$$

Taking the upper limit as $n \to \infty$ in 2.5 and using lemma 1.9, we get

$$\begin{aligned}\frac{d(Sy, y)}{k^2} \quad &\leq \quad \limsup_{n\longrightarrow\infty} d(fSx_{2n}, gx_{2n+1})\\ &\leq \quad \frac{q}{k^4}\max\{\limsup_{n\longrightarrow\infty} d(S^2x_{2n}, Tx_{2n+1}), \limsup_{n\longrightarrow\infty} d(fSx_{2n}, S^2x_{2n}),\\ &\qquad \limsup_{n\longrightarrow\infty} d(gx_{2n+1}, Tx_{2n+1}),\\ &\qquad \frac{1}{2}(\limsup_{n\longrightarrow\infty} d(S^2x_{2n}, gx_{2n+1}) + \limsup_{n\longrightarrow\infty} d(fSx_{2n}, Tx_{2n+1}))\}.\\ &\leq \quad \frac{q}{k^4}\max\{k^2d(Sy, y), 0, 0, \frac{k^2}{2}(d(Sy, y) + d(Sy, y))\}\\ &= \quad \frac{q}{k^4}k^2d(Sy, y) = \frac{q}{k^2}d(Sy, y).\end{aligned}$$

Consequently, $d(Sy, y) \leq qd(Sy, y)$. As $0 < q < 1$, so $Sy = y$.

Using continuity of T, we obtain $\lim_{n\to\infty} T^2x_{2n+1} = Ty$ and $\lim_{n\to\infty} Tgx_{2n+1} = Ty$.

Since g and T are compatible, $\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$. So, by lemma 1.10, we have $\lim_{n\to\infty} gTx_{2n} = Ty$.

Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in 2.1, we obtain

$$\begin{aligned}d(fx_{2n}, gTx_{2n+1}) \quad &\leq \quad \frac{q}{k^4}\max\{d(Sx_{2n}, T^2x_{2n+1}), d(fx_{2n}, Sx_{2n}), d(gTx_{2n+1}, T^2x_{2n+1}),\\ &\qquad \frac{1}{2}(d(Sx_{2n}, gTx_{2n+1}) + d(fx_{2n}, T^2x_{2n+1}))\}.\end{aligned} \tag{2.6}$$

Taking upper limit as $n \to \infty$ in 2.6 and using lemma 1.9, we obtain

$$\begin{aligned}\frac{d(y, Ty)}{k^2} \quad &\leq \quad \limsup_{n\longrightarrow\infty} d(fx_{2n}, gTx_{2n+1}) \leq \frac{q}{k^4}\max\{k^2d(y, Ty), 0, 0, \frac{k^2}{2}(d(y, Ty) + d(y, Ty))\}\\ &= \quad \frac{qd(y, Ty)}{k^2},\end{aligned}$$

which implies that $Ty = y$. Also, we can apply condition 2.1 to obtain

$$\begin{aligned} d(fy, gx_{2n+1}) &\leq \frac{q}{k^4} \max\{d(Sy, Tx_{2n+1}), d(fy, Sy), d(gx_{2n+1}, Tx_{2n+1}), \\ &\quad \frac{1}{2}(d(Sy, gx_{2n+1}) + d(fy, Tx_{2n+1}))\}. \end{aligned} \tag{2.7}$$

Taking upper limit $n \to \infty$ in 2.7, and using $Sy = Ty = y$, we have

$$\begin{aligned} \frac{d(fy, y)}{k^2} &\leq \frac{q}{k^4} \max\{k^2 d(Sy, y), k^2 d(fy, Sy), k^2 d(y, y), \frac{k^2}{2}(d(Sy, y) + d(fy, y))\} \\ &= \frac{q}{k^2} d(fy, y). \end{aligned}$$

which implies that $d(fy, y) = 0$ and $fy = y$ as $0 < q < 1$.

Finally, from condition 2.1, and the fact $Sy = Ty = fy = y$, we have

$$\begin{aligned} d(y, gy) &= d(fy, gy) \\ &\leq \frac{q}{k^4} \max\{d(Sy, Ty), d(fy, Sy), d(gy, Ty), \frac{1}{2}(d(Sy, gy) + d(fy, Ty))\} \\ &= \frac{q}{k^4} d(y, gy) \leq q d(y, gy), \end{aligned}$$

which implies that $d(y, gy) = 0$ and $gy = y$. Hence $Sy = Ty = fy = gy = y$.

If there exists another common fixed point x in X for f, g, S and T, then

$$\begin{aligned} d(x, y) &= d(fx, gy) \\ &\leq \frac{q}{k^4} \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2}(d(Sx, gy) + d(fx, Ty))\} \\ &= \frac{q}{k^4} \max\{d(x, y), d(x, x), d(y, y), \frac{1}{2}(d(x, y) + d(x, y))\} \\ &= \frac{q}{k^4} d(x, y) \leq q d(x, y), \end{aligned}$$

which further implies that $d(x, y) = 0$ and hence, $x = y$. Thus, y is a unique common fixed point of f, g, S and T. □

Now, we give two examples to support our result.

2.2. Example. Let $X = [0, \infty)$ be endowed with $b-$metric $d(x, y) = |x - y|^2 = (x - y)^2$, where $k = 2$. Define f, g, S and T on X by

$$f(x) = \ln(1 + \tfrac{x}{4}), \quad g(x) = \ln(1 + \tfrac{x}{5}),$$

$$S(x) = e^{5x} - 1, \quad T(x) = e^{4x} - 1.$$

Obviously, $f(X) = T(X) = g(X) = S(X) = [0, \infty)$. We show that the pair $\{f, S\}$ is compatible: Let $\{x_n\}$ be a sequence in X such that for some $t \in X$, $\lim_{n\to\infty} d(fx_n, t) = 0$ and $\lim_{n\to\infty} d(Sx_n, t) = 0$. That is, $\lim_{n\to\infty} |fx_n - t| = \lim_{n\to\infty} |Sx_n - t| = 0$. Since f and S are continuous, we have

$$\begin{aligned} \lim_{n\to\infty} d(fSx_n, Sfx_n) &= (\lim_{n\to\infty} |fSx_n - Sfx_n|)^2 = (|ft - St|)^2 \\ &= (\left|\ln(1 + \frac{t}{4}) - e^{5t} + 1\right|)^2. \end{aligned}$$

But $(\left|\ln(1 + \frac{t}{4}) - e^{5t} + 1\right|)^2 = 0 \iff t = 0$, so the pair $\{f, S\}$ is compatible. Similarly $\{g, T\}$ is compatible.

For each $x, y \in X$, the mean value theorem gives

$$\begin{aligned} d(fx, gy) &= (fx - gy)^2 = [\ln(1 + \frac{x}{4}) - \ln(1 + \frac{y}{5})]^2 \\ &\leq (\frac{x}{4} - \frac{y}{5})^2 \leq \frac{1}{20^2}(5x - 4y)^2 \\ &\leq \frac{1}{20^2}(e^{5x} - e^{4y})^2 = \frac{1}{20^2} d(Sx, Ty) \\ &\leq \frac{\frac{1}{25}}{2^4} \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2}(d(Sx, gy) + d(fx, Ty))\}, \end{aligned}$$

where $\frac{1}{25} \leq q < 1$ and $k = 2$. Thus, f, g, S and T satisfy all conditions of Theorem 2.1. Moreover 0 is the unique common fixed point of f, g, S and T.

2.3. Example. Let $X = [0, 1]$ be endowed with $b-$metric $d(x, y) = |x - y|^2 = (x - y)^2$, where $k = 2$. Define f, g, S and T on X by

$$f(x) = (\tfrac{x}{2})^8, \quad g(x) = (\tfrac{x}{2})^4,$$

$$S(x) = (\tfrac{x}{2})^4, \quad T(x) = (\tfrac{x}{2})^2.$$

Obviously, $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$. Furthermore, the pairs $\{f, S\}$ and $\{g, T\}$ are compatible.

For each $x, y \in X$, we have

$$\begin{aligned} d(fx, gy) &= (fx - gy)^2 = ((\frac{x}{2})^8 - (\frac{y}{2})^4)^2 = ((\frac{x}{2})^4 + (\frac{y}{2})^2)^2.((\frac{x}{2})^4 - (\frac{y}{2})^2)^2 \\ &\leq (\frac{1}{16} + \frac{1}{4})^2 d(Sx, Ty) = \frac{\frac{5}{16}}{2^4} d(Sx, Ty) \\ &\leq \frac{q}{k^4} \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{1}{2}(d(Sx, gy) + d(fx, Ty))\}, \end{aligned}$$

where $\frac{5}{16} \leq q < 1$ and $k = 2$. Thus, f, g, S and T satisfy all conditions of Theorem 2.1. Moreover 0 is the unique common fixed point of f, g, S and T.

Now, we get the special cases of Theorem 2.1 as follows:

2.4. Corollary. Let (X, d) be a complete $b-$metric space and $f, g : X \to X$ two mappings such that

$$d(fx, gy) \leq \frac{q}{k^4} \max\{d(x, y), d(fx, x), d(gy, y), \frac{1}{2}(d(x, gy) + d(fx, y))\},$$

holds for all $x, y \in X$ with $0 < q < 1$. Then, there exists a unique point $y \in X$ such that $fy = gy = y$.

Proof. If we take $S = T = I_X$ (identity mapping on X), then Theorem 2.1 gives that f and g have a unique common fixed point.

2.5. Corollary. Let (X, d) be a complete $b-$metric space and $S, T : X \to X$ continuous mappings such that

$$d(x, y) \leq q \max\{d(Sx, Ty), d(x, Sx), d(y, Ty), \frac{d(Sx, y) + d(x, Ty)}{2}\},$$

holds for all $x, y \in X$ with $0 < q < 1$. Then, S and T have a unique common fixed point.

Proof. If we take f and g as identity maps on X, then Theorem 2.1 gives that S and T have a unique common fixed point.

2.6. Corollary. Let (X, d) be a complete $b-$metric space and $f : X \to X$ a mapping such that

$$d(fx, fy) \leq \frac{q}{k^4} \max\{d(x, y), d(fx, x), d(fy, y), \frac{1}{2}(d(x, fy) + d(fx, y))\},$$

holds for all $x, y \in X$, with $0 < q < 1$. Then f has a unique fixed point in X.

Proof. Take S and T as identity maps on X and $f = g$ and then apply Theorem 2.1.

2.7. Theorem. Let $\{f,\ S\}$ and$\{g, T\}$ be compatible self mappings on a complete $b-$metric space (X, d) satisfying

(2.8)

$$d(fx, gy) \leq \frac{1}{k^4}(a_1 d(Sx, Ty)+a_2 d(fx, Ty)+a_3 d(Sx, gy)+a_4 d(fx, Sx)+a_5 d(gy, Ty)),$$

for all $x, y \in X$, where $a_i \geq 0$ $(i = 1, 2, 3, 4, 5)$ are real constants with $a_1 + \alpha a_2 + \beta a_3 + a_4 + a_5 < 1$, where $\alpha + \beta = 2$, for $\alpha, \beta \in \mathbb{N} \cup \{0\}$. If $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ and S and T are continuous, then f, g, S and T have a unique common fixed point.

Proof. Let x_0 in X. Since $f(X) \subseteq T(X)$, Choose $x_1, x_2 \in X$ such that $Tx_1 = fx_0$, and $Sx_2 = gx_1$. This can be done as $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$. In general, $x_{2n+1} \in X$ is chosen such that $Tx_{2n+1} = fx_{2n}$ and $x_{2n+2} \in X$ such that $Sx_{2n+2} = gx_{2n+1}$; $n = 0, 1, 2, \cdots$. Denote $y_{2n} = Tx_{2n+1} = fx_{2n}$, and $y_{2n+1} = Sx_{2n+2} = gx_{2n+1}$, for all $n \geq 0$. Now, we show that $\{y_n\}$ is a Cauchy sequence. For this, consider

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(fx_{2n}, gx_{2n+1}) \\
&\leq \frac{1}{k^4}(a_1 d(Sx_{2n}, Tx_{2n+1}) + a_2 d(fx_{2n}, Tx_{2n+1}) + a_3 d(Sx_{2n}, gx_{2n+1}) \\
&+ a_4 d(fx_{2n}, Sx_{2n}) + a_5 d(gx_{2n+1}, Tx_{2n+1})) \\
&= \frac{1}{k^4}(a_1 d(y_{2n-1}, y_{2n}) + a_2 d(y_{2n}, y_{2n}) + a_3 d(y_{2n-1}, y_{2n+1}) \\
&+ a_4 d(y_{2n}, y_{2n-1}) + a_5 d(y_{2n+1}, y_{2n})) \\
&\leq \frac{1}{k^4}(a_1 d(y_{2n-1}, y_{2n}) + ka_3 d(y_{2n-1}, y_{2n}) + ka_3 d(y_{2n}, y_{2n+1}) \\
&+ a_4 d(y_{2n}, y_{2n-1}) + a_5 d(y_{2n+1}, y_{2n})).
\end{aligned}$$

Hence,

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &\leq \frac{1}{k^4}(a_1 d(y_{2n-1}, y_{2n}) + ka_3 d(y_{2n-1}, y_{2n}) + ka_3 d(y_{2n}, y_{2n+1}) + \\
& \quad +a_4 d(y_{2n}, y_{2n-1}) + a_5 d(y_{2n}, y_{2n+1})) \\
&\leq \frac{1}{k^3}(a_1 d(y_{2n-1}, y_{2n}) + a_3 d(y_{2n-1}, y_{2n}) \\
&+ a_3 d(y_{2n}, y_{2n+1}) + a_4 d(y_{2n}, y_{2n-1}) + a_5 d(y_{2n}, y_{2n+1})).
\end{aligned} \tag{2.9}$$

Now, we prove that $d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$, for each $n \in \mathbb{N}$. If $d(y_{2n-1}, y_{2n}) < d(y_{2n}, y_{2n+1})$, for some $n \in \mathbb{N}$ then from the above inequality we have

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &< \frac{1}{k^3}(a_1 d(y_{2n}, y_{2n+1}) + 2a_3 d(y_{2n}, y_{2n+1}) + a_4 d(y_{2n}, y_{2n+1}) + a_5 d(y_{2n}, y_{2n+1})) \\
&= \frac{1}{k^3}(a_1 + 2a_3 + a_4 + a_5) d(y_{2n}, y_{2n+1}) \\
&< d(y_{2n}, y_{2n+1}),
\end{aligned}$$

a contradiction. So, we have $d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$, for each $n \in \mathbb{N}$. Thus

$$d(y_{2n}, y_{2n+1}) \leq \frac{1}{k^3}(a_1 + 2a_3 + a_4 + a_5) d(y_{2n-1}, y_{2n}). \tag{2.10}$$

Also, we have

$$\begin{aligned}
d(y_{2n},y_{2n-1}) &= d(fx_{2n},gx_{2n-1})\\
&\le \frac{1}{k^4}(a_1d(Sx_{2n},Tx_{2n-1})+a_2d(fx_{2n},Tx_{2n-1})+a_3d(Sx_{2n},gx_{2n-1})\\
&+a_4d(fx_{2n},Sx_{2n})+a_5d(gx_{2n-1},Tx_{2n-1}))\\
&= \frac{1}{k^4}(a_1d(y_{2n-1},y_{2n-2})+a_2d(y_{2n},y_{2n-2})+a_3d(y_{2n-1},y_{2n-1})\\
&+a_4d(y_{2n},y_{2n-1})+a_5d(y_{2n-1},y_{2n-2}))\\
&\le \frac{1}{k^4}(a_1d(y_{2n-1},y_{2n-2})+ka_2d(y_{2n},y_{2n-1})+ka_2d(y_{2n-1},y_{2n-2})\\
&+a_4d(y_{2n},y_{2n-1})+a_5d(y_{2n-1},y_{2n-2})).
\end{aligned}$$

Hence,

$$\begin{aligned}
(2.11)\quad d(y_{2n},y_{2n-1}) &\le \frac{1}{k^3}(a_1d(y_{2n-1},y_{2n-2})+a_2d(y_{2n},y_{2n-1})\\
&+ a_2d(y_{2n-1},y_{2n-2})+a_4d(y_{2n},y_{2n-1})+a_5d(y_{2n-1},y_{2n-2})).
\end{aligned}$$

Similarly, if $d(y_{2n-1},y_{2n-2})\le d(y_{2n},y_{2n-1})$, for some $n\in\mathbb{N}$ then from 2.11 we obtain

$$\begin{aligned}
d(y_{2n},y_{2n-1}) &\le \frac{1}{k^3}(a_1d(y_{2n},y_{2n-1})+2a_2d(y_{2n},y_{2n-1})+a_4d(y_{2n},y_{2n-1})+a_5d(y_{2n},y_{2n-1}))\\
&\le \frac{1}{k^3}(a_1+2a_2+a_4+a_5)d(y_{2n},y_{2n-1})\\
&< d(y_{2n},y_{2n-1}),
\end{aligned}$$

a contradiction. So, we have $d(y_{2n},y_{2n-1})\le d(y_{2n-1},y_{2n-2})$ for each $n\in\mathbb{N}$. Now from 2.11 we get

$$(2.12)\qquad d(y_{2n},y_{2n-1})\le \frac{1}{k^3}(a_1+2a_2+a_4+a_5)d(y_{2n-1},y_{2n-2}).$$

Now, from 2.10 and 2.12 we have

$$d(y_n,y_{n-1})\le \lambda d(y_{n-1},y_{n-2}),\qquad n\ge 2,$$

where $\lambda=\max\{\frac{a_1+2a_2+a_4+a_5}{k^3},\frac{a_1+2a_3+a_4+a_5}{k^3}\}$. As $k\ge 1$, so $\lambda\in(0,1)$. Now for $n\ge 2$, we have

$$(2.13)\qquad d(y_n,y_{n-1})\le \ldots \le \lambda^{n-1}d(y_1,y_0).$$

For $n>m$, we have

$$d(y_n,y_m)\le kd(y_m,y_{m+1})+k^2d(y_{m+1},y_{m+2})+\ldots+k^{n-m-1}d(y_{n-1},y_n).$$

Hence from 2.13 and $k\lambda<1$, we have

$$\begin{aligned}
d(y_n,y_m) &\le (k\lambda^m+k^2\lambda^{m+1}+\ldots+k^{n-m-1}\lambda^{n-1})d(y_1,y_0)\\
&\le k\lambda^m[1+k\lambda+(k\lambda)^2+\cdots]d(y_1,y_0)\\
&\le \frac{k\lambda^m}{1-k\lambda}d(y_1,y_0)\\
&= \frac{k\lambda^m}{1-k\lambda}d(y_1,y_0)\to 0,\quad \text{as}\quad m\to\infty.
\end{aligned}$$

So $\{y_n\}$ is a Cauchy sequence. Let $y\in X$ be such that $\lim_{n\to\infty}fx_{2n}=\lim_{n\to\infty}Tx_{2n+1}=\lim_{n\to\infty}gx_{2n+1}=\lim_{n\to\infty}Sx_{2n+2}=y$. Since S is continuous, so $\lim_{n\to\infty}S^2x_{2n+2}=Sy$ and

$\lim_{n\to\infty} Sfx_{2n} = Sy$. Using compatibility of a pair $\{f, S\}$, we have $\lim_{n\to\infty} d(fSx_n, Sfx_n) = 0$. So, by Lemma 1.10, we obtain $\lim_{n\to\infty} fSx_{2n} = Sy$. From 2.8, we have

$$d(fSx_{2n}, gx_{2n+1}) \leq \frac{1}{k^4}(a_1 d(S^2x_{2n}, Tx_{2n+1}) + a_2 d(fSx_{2n}, Tx_{2n+1}) + a_3 d(S^2x_{2n}, gx_{2n+1}) \\ + a_4 d(fSx_{2n}, S^2x_{2n}) + a_5 d(gx_{2n+1}, Tx_{2n+1})).$$

Taking the upper limit as $n \to \infty$, and using Lemma 1.9 we get

$$\begin{aligned}
\frac{d(Sy, y)}{k^2} &\leq \frac{1}{k^4}(k^2 a_1 d(Sy, y) + k^2 a_2 d(Sy, y) + k^2 a_3 d(Sy, y) + k^2 a_4 d(Sy, Sy) + k^2 a_5 d(y, y)) \\
&\leq \frac{1}{k^4}(k^2 a_1 d(Sy, y) + k^2 a_2 d(Sy, y) + k^2 a_3 d(Sy, y)) \\
&= \frac{1}{k^2}(a_1 + a_2 + a_3) d(Sy, y) \\
&\leq \frac{(a_1 + a_2 + a_3 + a_4 + a_5)}{k^2} d(Sy, y).
\end{aligned}$$

Therefore, $d(Sy, y) \leq (a_1 + a_2 + a_3 + a_4 + a_5)d(Sy, y)$. As $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, so $Sy = y$.

Similarly, using continuouty of T , we obtain that $\lim_{n\to\infty} T^2x_{2n+1} = Ty$ and $\lim_{n\to\infty} Tgx_{2n+1} = Ty$. Since g and T are compatible, $\lim_{n\to\infty} d(gTx_n, Tgx_n) = 0$, so by Lemma 1.10, we have $\lim_{n\to\infty} gTx_{2n} = Ty$.

From 2.8, it follows that

$$d(fx_{2n}, gT_{2n+1}) \leq \frac{1}{k^4}(a_1 d(Sx_{2n}, T^2x_{2n+1}) + a_2 d(fx_{2n}, T^2x_{2n+1}) + a_3 d(Sx_{2n}, gTx_{2n+1}) \\ + a_4 d(fx_{2n}, Sx_{2n}) + a_5 d(gTx_{2n+1}, T^2x_{2n+1})).$$

Taking upper limit as $n \to \infty$ and using Lemma 1.9 we get

$$\begin{aligned}
\frac{d(y, Ty)}{k^2} &\leq \frac{1}{k^4}(k^2 a_1 d(y, Ty) + k^2 a_2 d(y, Ty) + k^2 a_3 d(y, Ty) + k^2 a_4 d(y, y) + k^2 a_5 d(Ty, Ty) \\
&= \frac{1}{k^2}(a_1 + a_2 + a_3) d(y, Ty) \\
&\leq \frac{(a_1 + a_2 + a_3 + a_4 + a_5)}{k^2} d(y, Ty),
\end{aligned}$$

that is, $d(y, Ty) \leq (a_1 + a_2 + a_3 + a_4 + a_5)d(y, Ty)$. Therefore, $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ implies that $Ty = y$. Again from 2.8, it follows that

$$d(fy, gx_{2n+1}) \leq \frac{1}{k^4}(a_1 d(Sy, Tx_{2n+1}) + a_2 d(fy, Tx_{2n+1}) + a_3 d(Sy, gx_{2n+1}) \\ + a_4 d(fy, Sy) + a_5 d(gx_{2n+1}, Tx_{2n+1})).$$

Taking upper limit as $n \to \infty$ and using $Sy = y$ and $Ty = y$, we get

$$\begin{aligned}
\frac{d(fy, y)}{k^2} &\leq \frac{1}{k^4}(k^2 a_1 d(Sy, y) + k^2 a_2 d(fy, y) + k^2 a_3 d(Sy, y) + k^2 a_4 d(fy, y) + k^2 a_5 d(y, y) \\
&\leq \frac{a_1 + a_2 + a_3 + a_4 + a_5}{k^2} d(fy, y).
\end{aligned}$$

Therefore, $d(fy, y) \leq (a_1+a_2+a_3+a_4+a_5)d(fy, y)$. As $a_1+a_2+a_3+a_4+a_5 < 1$, so $fy = y$. Again, from 2.8 we have $d(fy, gy) = 0$, hence $fy = gy$. Thus, $fy = gy = Sy = Ty = y$.

If there exists another common fixed point x in X for f, g, S and T, then

$$\begin{aligned} d(x,y) &= d(fx,gy) \\ &\leq \frac{1}{k^4}(a_1 d(Sx,Sy) + a_2 d(fx,Ty) + a_3 d(Sx,gy) + a_4 d(fx,Sx) + a_5 d(gy,Ty)) \\ &= \frac{1}{k^4}(a_1+a_2+a_3)d(x,y) \\ &\leq (a_1+a_2+a_3+a_4+a_5)d(x,y), \end{aligned}$$

Therefore, $d(x,y) \leq (a_1+a_2+a_3+a_4+a_5)d(x,y)$. As $a_1+a_2+a_3+a_4+a_5 < 1$, so $d(x,y) = 0$, i.e., $x = y$. Therefore y is a unique common fixed point of f, g, S and T. □

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