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A General Formula for Determinants and Inverses of r-circulant Matrices with Third Order Recurrences

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Abstract
This note provides formula for determinant and inverse of r-circulant matrices with general sequences of third order. In other words, the study combines many papers in the literature.

Keywords: *r-circulant; determinant; matrix inverse; third order recurrence.*

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1. Introduction

A r-circulant matrix of order n, $C_n := circ_r(c_0, c_1, \ldots, c_{n-1})$, associated with the numbers $c_0, c_1, \ldots, c_{n-1}$, is defined as

$$C_n = \begin{pmatrix} c_0 & c_1 & \ldots & c_{n-2} & c_{n-1} \\ rc_{n-1} & c_0 & \ldots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rc_2 & rc_3 & \ldots & c_0 & c_1 \\ rc_1 & rc_2 & \ldots & rc_{n-1} & c_0 \end{pmatrix}. \tag{1.1}$$

where each row is a cyclic shift of the row above it [1]. If $r = 1$, then the matrix C_n is ordinary circulant matrix. If $r = -1$, then the matrix C_n is skew-circulant matrix.

Circulant matrices and their applications are a fundamental key in many areas of pure and applied science (see [6, 10], and references there in). Recently, many researcher get very interesting properties of them. For example, in [1], Shen and Cen obtained upper and lower bounds for the spectral norms of r-circulant matrices involving Fibonacci and Lucas numbers. Further, they gave some bounds for the spectral norms of Kronecker and Hadamard products of these matrices. In [2], Shen et al. obtained useful formulas for determinants and inverses of circulant matrices with Fibonacci and Lucas numbers, using properties of circulant matrices and this sequences. In [3], Bozkurt and Tam gave formulas for determinants and inverses of circulant matrices involving Jacobsthal and Jacobsthal-Lucas numbers taking into account the method in [2]. Bozkurt and Tam [4] defined r-circulant matrices with general second order number sequences. Then, the authors obtained formulas for determinant and inverse of this matrix. Moreover, they gave some bounds for norms of r-circulant matrices involving Fibonacci and Lucas numbers. Yazlık and Taskara [5] considered circulant matrices with k-Horadam numbers. Then, the authors obtained formulas for determinant and inverse of this matrix. Liu and Jiang [8] defined Tribonacci circulant matrix, Tribonacci left circulant matrix, Tribonacci g-circulant matrix. Then, the authors acquired determinants and inverses of these matrices. In [9], Bozkurt et al. considered the determinant of circulant and skew-circulant matrices whose entries are Tribonacci numbers. Bozkurt and Yılmaz [11] obtained formulas for determinant and inverse of circulant matrices with Pell and Pell-Lucas numbers.

In this paper, we consider third order linear recurrence for $n > 2$:

$$W_n = pW_{n-1} + qW_{n-2} + tW_{n-3} \tag{1.2}$$

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with initial conditions $W_0 = 0, W_1 = a$ and $W_2 = b$. The first few values are

$$0, a, b, pb + qa, p^2 b + pqa + qb + ta, \dots .$$

Then, we obtain formulas for determinants and inverses of r-circulant matrices E_n, i.e.,

$$E_n := circ_r\,(W_1, W_2, \dots, W_n),$$

where W_n is given by (1.2).

As it can be seen from the definition of the sequence, it is a general form of some well-known sequences. In other words,

♢ If $p = q = a = b = r = 1$ and $t = 0$, then we obtain determinant and inverse of circulant matrices with Fibonacci numbers, as in [2].

♢ If $p = a = b = r = 1, t = 0$ and $q = 2$, then we obtain determinant and inverse of circulant matrices with Jacobsthal numbers, as in [3].

♢ If $q = a = r = 1, p = b = 2$ and $t = 0$, then we obtain determinant and inverse of circulant matrices with Pell numbers, as in [11].

♢ If $p = q = a = b = t = r = 1$, then we obtain determinant and inverse of circulant matrices with Tribonacci numbers, as in [8].

♢ If $p = q = a = b = t = 1$ and $r = -1$, then we obtain determinant and inverse of skew-circulant matrices with Tribonacci numbers, as in [9].

To sum up, the derived formulas combine many of the papers in the literature.

2. Determinant of E_n

This section is dedicated for determinant formula of r-circulant matrices with general third order sequences. Firstly, let us give the following lemmas.

Lemma 2.1. *[9] If*

$$D_n = \begin{pmatrix} d_1 & d_2 & d_3 & \cdots & d_{n-1} & d_n \\ a & b & & & & \\ c & a & b & & & \\ & c & a & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \end{pmatrix}, \tag{2.1}$$

then

$$\det D_n = \sum_{k=1}^{n} d_k b^{n-k} \left(-\sqrt{bc}\right)^{k-1} U_{k-1}\left(\frac{a}{2\sqrt{bc}}\right), \tag{2.2}$$

where $U_k(x)$ is the kth Chebyshev polynomial of second kind.

Lemma 2.2. *If*

$$B_n = \begin{pmatrix} X_1 & d_1 & d_2 & d_3 & \cdots & d_{n-1} & d_n \\ Y_1 & f_1 & f_2 & f_3 & \cdots & f_{n-1} & f_n \\ 0 & a & b & 0 & & & 0 \\ & c & a & b & & & \\ & & c & a & \ddots & & \\ & & & \ddots & \ddots & \ddots & 0 \\ 0 & & & 0 & c & a & b \end{pmatrix},$$

then

$$\det(B_n) = X_1 \sum_{k=1}^{n-1} f_k b^{n-1-k} \left(-\sqrt{bc}\right)^{k-1} U_{k-1}\left(\frac{a}{2\sqrt{bc}}\right) - Y_1 \sum_{k=1}^{n-1} d_k b^{n-1-k} \left(-\sqrt{bc}\right)^{k-1} U_{k-1}\left(\frac{a}{2\sqrt{bc}}\right),$$

where $U_k(x)$ is the kth Chebyshev polynomial of second kind.

Proof. Using the same method in the first Lemma 1, we have

$$\det(B_n) = X_1 \det(F_{n-1}) - Y_1 \det(D_{n-1}) \ .$$

So, this proof is completed. □

Theorem 2.1. *For $n \geq 4$, the determinant of E_n is*

$$W_1 \left[(g_n + j f_n) \left((W_1 - r(pW_n + qW_{n-1}))x_n^{n-3} + rt \sum_{k=2}^{n-2} W_{n-1-k} x_n^{n-2-k} \left(-\sqrt{x_n z_n}\right)^{k-1} U_{k-1}\left(\frac{y_n}{2\sqrt{x_n z_n}}\right)\right) \right.$$
$$\left. - h_n \sum_{k=1}^{n-2} [rW_{n+1-k} - (pr-j)W_{n-k}] x_n^{n-2-k} \left(-\sqrt{x_n z_n}\right)^{k-1} U_{k-1}\left(\frac{y_n}{2\sqrt{x_n z_n}}\right)\right],$$

where $x_n = W_1 - rW_{n+1}$, $y_n = W_2 - rW_{n+2} - p(W_1 - rW_{n+1})$, $z_n = -rtW_n$, $j = -\dfrac{r(W_2 - pW_1)}{W_1}$ *and*

$$\begin{aligned} f_n &= \sum_{i=2}^{n} W_i e^{n-i}, \\ g_n &= r\sum_{i=2}^{n-1} (W_{i+1} - pW_i) e^{n-i} + W_1 - prW_n, \\ h_n &= rt\sum_{i=1}^{n-3} W_i e^{n-1-i} + (W_1 - r(pW_n + qW_{n-1})e + W_2 - pW_1 - qrW_n \ . \end{aligned}$$

Proof. Firstly, let us define n-square matrix

$$F_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & e^{n-2} & 0 & \dots & 0 & 1 \\ 0 & e^{n-3} & 0 & \dots & 1 & 0 \\ 0 & e^{n-4} & 0 & & & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & e & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{2.3}$$

here e is the positive root of the characteristic equation $x_n e^2 + y_n e + z_n = 0$, i.e.,

$$e = \frac{-y_n + \sqrt{y_n^2 - 4x_n z_n}}{2x_n},$$

where

$$x_n = W_1 - rW_{n+1},\ y_n = W_2 - rW_{n+2} - p(W_1 - rW_{n+1}), \text{ and } z_n = -rtW_n \ .$$

Then, consider n-square matrix G_n as below:

$$G_n = \left(\begin{array}{c|cccccc c|ccc} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -pr & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ -qr & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -p \\ -tr & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -p & -q \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -p & -q & -t \\ 0 & 0 & 0 & 0 & 0 & 0 & \iddots & -p & -q & -t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \iddots & -q & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -p & \iddots & -t & 0 & 0 & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -p & -q & -t & \cdots & 0 & \vdots & \vdots & \vdots \\ 0 & 1 & -p & -q & -t & 0 & \cdots & 0 & 0 & 0 & 0 \end{array}\right)$$

It can be seen that for all $n > 3$,

$$\det(G_n) = \det(F_n) = \begin{cases} 1, & n \equiv 1,2 \ (mod4) \\ -1, & n \equiv 1,2 \ (mod4), \end{cases}$$

where F_n is defined in (2.3) and $\det(G_nF_n) = 1$. By matrix multiplication, we get;

$$K_n = G_nE_nF_n, \tag{2.4}$$

i.e.,

$$K_n = \left(\begin{array}{cc|cccccc} W_1 & f_n & W_{n-1} & W_{n-2} & \cdots & & & W_2 \\ r(W_2-pW_1) & g_n & r(W_n-pW_{n-1}) & r(W_{n-1}-pW_{n-2}) & \cdots & & & r(W_3-pW_2) \\ 0 & h_n & W_1-r(pW_n+qW_{n-1}) & rtW_{n-3} & \cdots & & & rtW_1 \\ \hline 0 & 0 & y_n & x_n & & & & \\ 0 & 0 & z_n & y_n & x_n & & & \\ & & & \ddots & \ddots & \ddots & & \\ 0 & 0 & & & z_n & y_n & & x_n \end{array}\right),$$

where

$$\begin{aligned} f_n &= \sum_{i=2}^{n} W_i e^{n-i}, \\ g_n &= W_1 - prW_n + r\sum_{i=2}^{n-1}(W_{i+1}-pW_i)e^{n-i}, \\ h_n &= W_2 - pW_1 - qrW_n + (W_1 - r(pW_n + qW_{n-1})e + rt\sum_{i=1}^{n-3} W_i e^{n-1-i}. \end{aligned}$$

Multiplying the first row with $j = -\dfrac{r(W_2-pW_1)}{W_1}$ and adding it to the second row in K_n, we obtain

$$|K_n| = \left|\begin{array}{cccccccc} W_1 & f_n & W_{n-1} & W_{n-2} & \cdots & & & W_2 \\ 0 & g_n+jf_n & rW_n+(j-rp)W_{n-1} & rW_{n-1}+(j-rp)W_{n-2} & \cdots & & & rW_3+(j-rp)W_2 \\ 0 & h_n & W_1-r(pW_n+qW_{n-1}) & rtW_{n-3} & \cdots & & & rtW_1 \\ 0 & 0 & y_n & x_n & \cdots & & & 0 \\ \vdots & \vdots & z_n & y_n & \ddots & & & \vdots \\ & & & \ddots & \ddots & \ddots & & 0 \\ 0 & 0 & 0 & 0 & z_n & y_n & & x_n \end{array}\right|.$$

By Laplace expansion on the first column

$$\det K_n = W_1 \det Z_n = \det E_n$$

here

$$Z_n = \left(\begin{array}{c|cccccc} g_n+jf_n & rW_n+(j-rp)W_{n-1} & rW_{n-1}+(j-rp)W_{n-2} & & \cdots & & rW_3+(j-rp)W_2 \\ h_n & W_1-r(pW_n+qW_{n-1}) & rtW_{n-3} & & \cdots & & rtW_1 \\ \hline 0 & y_n & x_n & 0 & & & 0 \\ 0 & z_n & y_n & x_n & & & \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & & 0 & z_n & y_n & x_n \end{array}\right).$$

Applying Lemma 2, we complete the proof. □

3. Inverse of E_n

In this section, we compute the inverse of the matrix E_n. Note that, just only for the inverse, we consider $W_2 = pa$. So,

$$G_nE_nF_n = K_n = \left(\begin{array}{cc|cccccc} W_1 & f_n & W_{n-1} & W_{n-2} & \cdots & & & W_2 \\ 0 & g_n & r(W_n - pW_{n-1}) & r(W_{n-1} - pW_{n-2}) & \cdots & & & r(W_3 - pW_2) \\ 0 & h_n & W_1 - r(pW_n + qW_{n-1}) & rtW_{n-3} & \cdots & & & rtW_1 \\ \hline 0 & 0 & y_n & x_n & & & & \\ 0 & 0 & z_n & y_n & x_n & & & \\ & & & \ddots & \ddots & \ddots & & \\ 0 & 0 & & & z_n & y_n & & x_n \end{array}\right).$$

Lemma 3.1. *[8] Let* $\psi = \begin{pmatrix} \alpha & V \\ U & A \end{pmatrix}$ *be an $(n-2)$-square matrix, then*

$$\psi^{-1} = \begin{pmatrix} \frac{1}{l} & -\frac{1}{l}VA^{-1} \\ -\frac{1}{l}A^{-1}U & A^{-1} + \frac{1}{l}A^{-1}UVA^{-1} \end{pmatrix},$$

where $l = \alpha - VA^{-1}U$, V is a row vector and U is a column vector.

Lemma 3.2. *Let us define the matrix $T = [t_{i,j}]_{i,j=1}^{n-3}$ of the form:*

$$t_{ij} = \begin{cases} W_1x_n & , i = j, \\ W_1y_n & , i = j+1, \\ W_1z_n & , i = j+2, \\ 0 & , otherwise. \end{cases}$$

Then, inverse of T is

$$T^{-1} = \left[t'_{i,j}\right]_{i,j=1}^{n-3} = \begin{cases} \frac{1}{W_1x_n} & , i = j \\ -\frac{y_n}{W_1x_n^2} & , i = j+1 \\ -\frac{y_nt'_{i-2,j} + z_nt'_{i-1,j}}{x_n} & , i = j+k (k \geq 2) \\ 0 & , i < j. \end{cases} \tag{3.1}$$

Proof. From matrix multiplication, we can easily see that $TT^{-1} = T^{-1}T = I_{n-3}$, where I_{n-3} is identity matrix. □

Theorem 3.1. *Let $E_n = circ_r(W_1, W_2, \ldots, W_n)$ be r-circulant matrix. Then,*

$$\begin{aligned} E_n^{-1} = circ_r \Big(& c'_2 - \left(p + \tfrac{h_n}{g_n}\right) c'_3 - qc'_4 - tc'_5, \\ & -pc'_2 + \left(\tfrac{ph_n}{g_n} - q\right) c'_3 - tc'_4, \tfrac{c'_n}{r}, \tfrac{c'_{n-1} - pc'_n}{r} \\ & \tfrac{1}{r}\left(c'_{n-2} - pc'_{n-1} - qc'_n\right), \ldots, \tfrac{1}{r}\left(c'_{n-k+3} - pc'_{n-k+4} - qc'_{n-k+5} - tc'_{n-k+6}\right)\Big), \end{aligned}$$

where

$$\begin{aligned} c'_1 &= 0, \\ c'_2 &= W_1^2 g_n, \\ c'_3 &= -\frac{rW_1}{g_n}\sum_{k=0}^{n-3} s_k(W_{n-k} - pW_{n-k-1}), \ (\text{for } s_0 = \tfrac{1}{l}), \\ c'_4 &= -\frac{rp_1W_1(W_n - pW_{n-1})}{g_n} - \frac{rW_1}{g_n}\sum_{k=1}^{n-3} u_{k,1}(W_{n-k} - pW_{n-k-1}), \\ &\vdots \\ c'_t &= -\frac{rp_{t-3}W_1(W_n - pW_{n-1})}{g_n} - \frac{rW_1}{g_n}\sum_{k=1}^{n-3} u_{k,t-3}(W_{n-k} - pW_{n-k-1}), \ (t \geq 4) \end{aligned}$$

and

$$\begin{aligned} g_n &= W_1 - prW_n + r\textstyle\sum_{i=2}^{n-1}(W_{i+1} - pW_i)e^{n-i}, \\ h_n &= W_2 - pW_1 - qrW_n + (W_1 - r(pW_n + qW_{n-1})e + rt\textstyle\sum_{i=1}^{n-3} W_ie^{n-1-i}. \end{aligned}$$

Proof. Firstly, Let us define

$$H_n=\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0\\ 0 & 1 & 0 & 0 & \cdots & 0\\ 0 & -\frac{h_n}{g_n} & 1 & 0 & \cdots & 0\\ 0 & 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

and

$$L_n=\begin{pmatrix} W_1 & -f_n & -W_{n-1}+\frac{rf_n(W_n-pW_{n-1})}{g_n} & -W_{n-2}+\frac{rf_n(W_{n-1}-pW_{n-2})}{g_n} & \cdots & -W_{n-2}+\frac{rf_n(W_3-pW_2)}{g_n}\\ 0 & W_1 & -\frac{rW_1(W_n-pW_{n-1})}{g_n} & -\frac{rW_1(W_{n-1}-pW_{n-2})}{g_n} & \cdots & -\frac{rW_1(W_3-pW_2)}{g_n}\\ 0 & 0 & W_1 & 0 & \cdots & 0\\ 0 & 0 & 0 & W_1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & W_1 \end{pmatrix}.$$

Then, from matrix multiplication, we have

$$H_nG_nE_nF_nL_n=\begin{pmatrix} W_1^2 & 0 & & & & \\ 0 & W_1g_n & & & & \\ & & W_1\rho_3 & W_1\rho_4 & \cdots & W_1\rho_n\\ & & W_1y_n & W_1x_n & & \\ & & W_1z_n & W_1y_n & \ddots & \\ & & & \ddots & \ddots & W_1x_n\\ & & & & W_1z_n & W_1y_n \end{pmatrix}=\mathcal{Y}_1\oplus N,$$

where $\mathcal{Y}_1=diag(W_1^2,W_1g_n)$, $\mathcal{Y}_1\oplus N$ is the direct sum of $\mathcal{Y}_1$ and N,

$$\rho_3=W_1-r\left(W_n\left(p+\frac{h_n}{g_n}\right)-W_{n-1}\left(q+p\frac{h_n}{g_n}\right)\right)$$

and

$$\rho_i=-\frac{rh_n}{g_n}(W_{n-i+3}-pW_{n-i+2})+rtW_{n-i+1}\qquad \text{for } i=4,5,\ldots,n.$$

If we define $P=H_nG_n$ and $Q=F_nL_n$, we get

$$E_n^{-1}=Q\left(\mathcal{Y}_1^{-1}\oplus N^{-1}\right)P.$$

According to Lemma 4, we define $(n-2)$-square matrix

$$N=\begin{pmatrix} W_1\rho_3 & V\\ U & T\end{pmatrix}.$$

Then, we have

$$N^{-1}=\begin{pmatrix} \frac{1}{l} & \frac{-VT^{-1}}{l}\\ \frac{-T^{-1}U}{l} & T^{-1}+\frac{1}{l}T^{-1}UVT^{-1}\end{pmatrix},$$

where

$$\begin{aligned} U &= (W_1y_n,W_1z_n,0,\ldots,0)^T,\\ V &= (W_1\rho_4,W_1\rho_5,\ldots,W_1\rho_n),\\ T &= \begin{cases} W_1x_n & ,i=j\\ W_1y_n & ,i=j+1\\ W_1z_n & ,i=j+2\\ 0 & ,otherwise,\end{cases}\\ l &= W_1\left(\rho_3-W_1y_n\sum_{i=1}^{n-3}t_{i1}\rho_{i+3}-W_1z_n\sum_{i=1}^{n-4}\rho_{i+4}\right). \end{aligned}$$

Let be $R=\frac{-VT^{-1}}{l}$ row vector, $S=\frac{-T^{-1}U}{l}$ column vector and $J=T^{-1}+\frac{1}{l}T^{-1}UVT^{-1}$, where T^{-1} is as in (3.1) Then, we have

$$R=[p_1,p_2,\dots,p_{n-3}],$$

where $p_i=\frac{-W_1}{l}\sum_{k=i}^{n-3}\rho_{k+3}t'_{k,i}$,

$$S=[s_1,s_2,\dots,s_{n-3}]^T,$$

where $s_1=\frac{-W_1}{l}t'_{1,1}y_n$ and for $i\geq 2$, $s_i=\frac{-W_1}{l}\left(y_nt'_{i,1}+z_nt'_{i,2}\right)$,

$$J=u_{i,j}=\begin{cases} t'_{1,j}-\frac{W_1^2}{l}y_nt'_{1,1}\sum_{k=j}^{n-3}\rho_{k+3}t'_{k,j} & \text{, for } i=1\\ t'_{i,j}-\frac{W_1^2}{l}(y_nt'_{i,1}+z_nt'_{i,2})\sum_{k=j}^{n-3}\rho_{k+3}t'_{k,j} & \text{, for } i=2,3,\dots,n-3.\end{cases}$$

So, we obtain

$$N^{-1}=\begin{pmatrix} \frac{1}{l} & p_1 & p_2 & \cdots & p_{n-3}\\ s_1 & u_{1,1} & u_{1,2} & \cdots & u_{1,n-3}\\ s_2 & u_{2,1} & u_{2,2} & \cdots & u_{2,n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ s_{n-3} & u_{n-3,1} & u_{n-3,2} & \cdots & u_{n-3,n-3}\end{pmatrix}_{(n-2)\times(n-2)},$$

where s_i's, p_i's and $u_{i,j}$'s are as in above.

The last row elements of the $Q=F_nL_n$ are $0,\ W_1,-\frac{rW_1(W_n-pW_{n-1})}{g_n},\ -\frac{rW_1(W_{n-1}-pW_{n-2})}{g_n},\dots,\ -\frac{rW_1(W_3-pW_2)}{g_n}$. Then, the last row elements of $Q\left(\mathcal{Y}_1^{-1}\oplus N^{-1}\right)$ are as the following:

$$\begin{aligned} c'_1 &= 0,\\ c'_2 &= W_1^2g_n,\\ c'_3 &= -\frac{rW_1}{g_n}\sum_{k=0}^{n-3}s_k(W_{n-k}-pW_{n-k-1}),\ (\text{for } s_0=\frac{1}{l}),\\ c'_4 &= -\frac{rp_1W_1(W_n-pW_{n-1})}{g_n}-\frac{rW_1}{g_n}\sum_{k=1}^{n-3}u_{k,1}(W_{n-k}-pW_{n-k-1}),\\ &\vdots\\ c'_t &= -\frac{rp_{t-3}W_1(W_n-pW_{n-1})}{g_n}-\frac{rW_1}{g_n}\sum_{k=1}^{n-3}u_{k,t-3}(W_{n-k}-pW_{n-k-1}),\ (t\geq 4).\end{aligned}$$

Since inverse of r-circulant matrix is r-circulant matrix [4], E_n^{-1} matrix is an r-circulant matrix. If $E_n^{-1}=circ_r\left(c_1,c_2,\dots,c_n\right)$, last row elements of the E_n^{-1} matrix are as in below:

$$\begin{aligned} & rc_2=-prc'_2+\left(\frac{prh_n}{g_n}-qr\right)c'_3-trc'_4\\ & rc_3=c'_n\\ & rc_4=c'_{n-1}-pc'_n\\ & rc_5=c'_{n-2}-pc'_{n-1}-qc'_n\\ & \vdots\\ & rc_k=c'_{n-k+3}-pc'_{n-k+4}-qc'_{n-k+5}-tc'_{n-k+6}\quad (\text{for } 5<k\leq n),\\ & c_1=c'_2-\left(p+\frac{h_n}{g_n}\right)c'_3-qc'_4-tc'_5.\end{aligned}$$

Therefore, we complete this proof. □

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