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Some Constructions of Color Hom-Novikov-Poisson Algebras

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Abstract
The aim of this paper is to introduce color Hom-Novikov-Poisson algebras which generalize color Hom-Novikov algebras. Many constructions of color Hom-Novikov-Poisson algebras are given either from color Novikov-Poisson algebras or from ε-commutative Hom-associative color algebras.

Keywords: *Color Hom-Novikov-Poisson algebras; color Hom-Novikov algebras; ε-commutative Hom-associative color algebras.*

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1. Introduction

Novikov-Poisson algebras both generalize commutative associative algebras and Novikov algebras. These algebras were originally introduced by Xu [13], [14], motivated by studying simple Novikov algebras and irreducible modules. Using Poisson algebras as a motivation, he defined a Novikov-Poisson algebra as a Novikov algebra that is also equipped with a commutative associative product, satisfying some compatibility conditions [13]. Novikov-Poisson algebras are closed under some perturbations of structure maps [14]. Every Novikov-Poisson algebras with associative commutative unity can be constructed from an associative commutative derivation algebra [6]. The relationship between Novikov-Poisson algebras and Hamiltonian super-operators was discussed in [15].

A graded version of ordinary algebras called graded or color algebras are considered and discussed in several works. In this scheme, e.g., relations between simple Jordan color algebras and associative graded algebras are discussed in [4], derivations and extensions of Lie color algebras are studied in [11]. The reader is referred to [2] for some characterizations of color Hom-Poisson algebras and to [5] for the introduction of Novikov color algebras.

The purpose of this paper is to introduce color Novikov-Poisson algebras and the twisted version of these color algebras called color Hom-Novikov-Poisson algebras, which are motived by recent works related to Hom-type algebras. The theory of Hom-algebras originated from Hom-Lie algebras introduced by J. T. Hartwig, D. Larsson and S. D. Silvestrov in [7] in the study of quasi-deformations of Lie algebras of vector fields, including q-deformations of Witt algebras and Virasoro algebras. In this scheme, e.g., associative algebras and Leibniz algebras are twisted into Hom-associative algebras and Hom-Leibniz algebras respectively in [10] and likewise, Hom-type analogues of Novikov algebras and Novikov-Poisson algebras are defined and discussed respectively in [17] and [18]. For further informations on other Hom-type algebras, one may refer to e.g., [8], [9],[16].

The notion of color algebras is extended to the Hom-setting by studying Hom-Lie superalgebras, Hom-Lie admissible superalgebras [1], color Hom-Lie algebras [20] and color Hom-Novikov algebras [3]. For further informations on other graded Hom-type algebras, on may refer to, e.g., [12], [19], [21], [22].

A description of the rest of this paper is as it follows.

In section 2, we recall basic notions concerning color algebras and color Hom-algebras and we extend the notion of color algebras [5] to the one of color Novikov-Poisson algebras (Definition 2.4). In section 3, we define color Hom-Novikov Poisson algebras and we point out that color Novikov-Poisson algebras are particular instances of color Hom-Novikov-Poisson algebras and that, these color Hom-algebras generalize color Hom-Novikov algebras. Next,

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we prove some construction methods (Theorem 3.1, Proposition 3.2, Theorem 3.2). Theorem 3.1 and Proposition 3.2 say that color Hom-Novikov-Poisson algebras can be obtained from ε-commutative Hom-associative color algebras. The structure map of a multiplicative color Hom-Novikov-Poisson algebra can be perturbed by a suitable element and its own twisting map(Theorem 3.2).

In this paper, all graded vector spaces are assumed to be over a fixed ground field $\mathbb{K}$ of characteristic 0.

2. Preliminaries

Let G be an abelian group. A vector space V is said to be a G-graded if, there exists a family $(V_a)_{a\in G}$ of vector subspaces of V such that $V = \oplus_{a\in G} V_a$. An element $x \in V$ is said to be homogeneous of degree $a \in G$ if $x \in V_a$. We denote $\mathcal{H}(V)$ the set of all homogeneous elements in V. Let $V = \oplus_{a\in G} V_a$ and $V' = \oplus_{\in G} V'_a$ be two G-graded vector spaces. A linear mapping $f : V \to V'$ is said to be homogeneous of degree $b \in G$ if $f(V_a) \subseteq V'_{a+b}$, $\forall a \in G$. If, f is homogeneous of degree zero i.e. $f(V_a) \subseteq V'_a$ holds for any $a \in G$, then f is said to be even. An algebra (A, μ) is said to be G-graded if its underlying vector space is G-graded i.e. $A = \oplus_{\in G} A_a$, and if furthermore $\mu(A_a, A_b) \subseteq A_{a+b}$, for all $a, b \in G$. Let A' be another G-graded algebra. A morphism $f : A \to A'$ of G-graded algebras is by definition an algebra morphism from A to A' which is, in addition an even mapping.

Definition 2.1. Let G be an abelian group. A mapping $\varepsilon : G \times G \longrightarrow \mathbb{K}^*$ is called a bicharacter on G if the following identities hold for all $a, b, c \in G$:
(i) $\varepsilon(a, b)\varepsilon(b, a) = 1$,
(ii) $\varepsilon(a + b, c) = \varepsilon(a, c)\varepsilon(b, c)$,
(iii) $\varepsilon(a, b + c) = \varepsilon(a, b)\varepsilon(a, c)$.

It is easy to see that $\varepsilon(0, a) = \varepsilon(a, 0) = 1$ and $\varepsilon(a, a) = \pm 1$ for all $a \in G$. In particular, for a fixed $a \in G$, the induced map $\varepsilon_a : G \longrightarrow \mathbb{K}^*$ defined by $\varepsilon_a(b) = \varepsilon(a, b)$ is a homomorphism of groups.

In this paper, for simplicity, the degree of any homogeneous element x will be denoted again by x. Thus if x and y are two homogeneous elements of degree a and b respectively and ε is a bicharacter, then we shorten the notation by writing $\varepsilon(x, y)$ instead of $\varepsilon(a, b)$.
Unless stated, in the sequel all the graded spaces are over the same abelian group G and the bicharacter will be the same for all the structures.

Definition 2.2. (i) A color algebra is a triple (A, μ, ε) consisting of a G-graded vector space A, an even bilinear map $\mu : A \times A \longrightarrow A$ i.e $\mu(A_a, A_b) \subseteq A_{a+b}$ for all $a, b \in G$, and a bicharacter $\varepsilon : G \times G \longrightarrow \mathbb{K}^*$. the color algebra $(A, \mu, \varepsilon, , \alpha)$ is said to be associative if $\mu(\mu(x, y), z) = \mu(x, \mu(y, z))$ and ε-commutative if $\mu(x, y) = \varepsilon(x, y)\mu(y, x)$ for all $x, y, z \in \mathcal{H}(A)$.
(ii) A double color algebra is a quadruple $(A, \mu_1, \mu_2, \varepsilon)$ where (A, μ_i, ε) is a color algebra for $i = 1,\ 2$.

The following notion is due to [5], where it is called a Novikov color algebra. However, we call it a color Novikov algebra in this paper to unify our terminologies.

Definition 2.3. [5] A color Novikov algebra is a color algebra $(V, \cdot, \varepsilon)$ satisfying:

$$(x \cdot y) \cdot z = \varepsilon(y, z)(x \cdot z) \cdot y \tag{2.1}$$
$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = \varepsilon(x, y)((y \cdot x) \cdot z - y \cdot (x \cdot z)) \tag{2.2}$$

for all $x, y, z \in \mathcal{H}(V)$.

We give a generalization of color Novikov algebra as it follows.

Definition 2.4. A color Novikov-Poisson algebra is a double color algebra $(V, \mu_1, \mu_2, \varepsilon)$ consisting of an ε-commutative associative color algebra (A, μ_1, ε), a color Novikov algebra (A, μ_2, ε) such that the following compatibility conditions

$$\mu_1(\mu_2(x, y), z) - \mu_2(x, \mu_1(y, z)) = \varepsilon(x, y)(\mu_1(\mu_2(y, x), z)) - \mu_2(y, \mu_1(x, z)) \tag{2.3}$$
$$\mu_2(\mu_1(x, y), z) = \varepsilon(y, z)\mu_1(\mu_2(x, z), y) \tag{2.4}$$

for all $x, y, z \in \mathcal{H}(V)$.

Example 2.1. (i) Every Novikov-Poisson algebra [13],[14] is a color Novikov-Poisson algebra where $\varepsilon(x, y) = 1$ for all homogeneous elements x, y.
(ii) Let $(A, \cdot, \varepsilon)$ be a color Novikov algebra in which $\cdot$ is ε-commutative. Then $(A, \cdot, *, \varepsilon)$ is a Novikov-Poisson algebra where "$* = \cdot$".
(iii) Every Novikov-Poisson superalgebra [20] is a color Novikov-Poisson algebra in which $G = \mathbb{Z}_2$ and $\varepsilon(x, y) = (-1)^{xy}$ for all homogeneous elements x, y.

For the rest of this section, we give basic facts about color Hom-algebras [2], [3], [20] as well as examples of color Hom-Novikov algebras.

Definition 2.5. (i) By a color Hom-algebra, we mean a quadruple $= \mathcal{A} = (A, \mu, \varepsilon, \alpha)$ consisting of a G-graded vector space A, an even bilinear map $\mu : A \times A \longrightarrow A$ i.e $\mu(A_a, A_b) \subseteq A_{a+b}$ for all $a, b \in G$, a bicharacter $\varepsilon : G \times G \longrightarrow \mathbb{K}^*$ and an even linear map $\alpha : A \longrightarrow A$. A color Hom-algebra $(A, \mu, \varepsilon, , \alpha)$ is said to be multiplicative if $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ and ε-commutative if $\mu(x, y) = \varepsilon(x, y)\mu(y, x)$ for all $x, y \in \mathcal{H}(A)$.
(ii) The Hom-associator of a color algebra $\mathcal{A}$ is the trilinear map $as_{\mathcal{A}} : A^{\otimes 3} \longrightarrow A$ defined as $as_{\mathcal{A}}(x, y, z) = \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z))$ for all $x, y, z \in \mathcal{H}(A)$. $\mathcal{A}$ is said to be Hom-associative color algebra is $as_{\mathcal{A}}(x, y, z) = 0$ for all $x, y, z \in \mathcal{H}(A)$.
(iii) A double color Hom-algebra is a quintuple $\mathcal{V} = (V, \mu_1, \mu_2, \varepsilon, \alpha)$ in which $(V, \mu_i, \varepsilon, \alpha)$ is a color Hom-algebra for $i = 1, 2$. If furthermore $\alpha \circ \mu_i = \mu_i \circ \alpha^{\otimes 2}$ for $i = 1, 2$ then $\mathcal{V}$ is said to be multiplicative.
(iv) A mixed Hom-associator of $\mathcal{V}$ is the trilinear map $a\tilde{s}_{\mathcal{V}} : V^{\otimes 3} \longrightarrow V$ defined as $a\tilde{s}_{\mathcal{V}}(x, y, z) = \mu_1(\mu_2(x, y), \alpha(z)) - \mu_2(\alpha(x), \mu_1(y, z))$ for all $x, y, z \in \mathcal{H}(V)$.

Remark 2.1. If $\varepsilon(x, y) = 1$ for all homogeneous elements $x \; and \; y$ then the notion of color Hom-algebra and double color Hom-algebra is reduced to the one of Hom-algebra [7] and double Hom-algebra [18] respectively.

Definition 2.6. (ii) A weak morphism $f : (A, \mu, \varepsilon, \alpha) \longrightarrow (A', \mu', \varepsilon, \alpha')$ of two color Hom-algebras is an even linear map $f : A \longrightarrow A'$ of the underlying G-graded vector spaces, satisfying $f \circ \mu = \mu' \circ f^{\otimes 2}$. If furthermore $f \circ \alpha = \alpha' \circ f$, then f is said to be a morphism.
(ii) A weak morphism $f : (A, \mu_1, \mu_2, \varepsilon, \alpha) \longrightarrow (A', \mu_1', \mu_2', \varepsilon, \alpha')$ of two double color Hom-algebras is an even linear map $f : A \longrightarrow A'$ of the underlying G-graded vector spaces, satisfying $f \circ \mu_i = \mu_i' \circ f^{\otimes 2}$ for $i = 1, 2$. If furthermore $f \circ \alpha = \alpha' \circ f$, then f is said to be a morphism.
(iii) A derivation of degree $d \in G$ of a color Hom-algebra $(A, \cdot, \varepsilon, \alpha)$ is a linear map $D : A \longrightarrow A$ such that $D(x \cdot y) = D(x) \cdot y + \varepsilon(d, x)x \cdot D(y)$. In particular, an even derivation is a derivation of degree zero [3].

Similarly in the Definition 2.3, the following notion is due to [3], where it s called a Hom-Novikov color algebra. In this paper, we called it a color Hom-Novikov algebra.

Definition 2.7. A color Hom-Novikov algebra is a color Hom-algebra $\mathcal{A} = (A, \cdot, \varepsilon, \alpha)$ satisfying:

$$(x \cdot y) \cdot \alpha(z) = \varepsilon(y, z)(x \cdot z) \cdot \alpha(y) \tag{2.5}$$
$$as_{\mathcal{A}}(x, y, z) = \varepsilon(x, y)as_{\mathcal{A}}(y, x, z) \tag{2.6}$$

for all $x, y, z \in \mathcal{H}(A)$ where $as_{\mathcal{A}}$ is the Hom-associator (Definition 2.5 (ii)).

Remark 2.2. Every Hom-Novikov superalgebra [22] is a color Hom-Novikov algebra in which $G = \mathbb{Z}_2$ and $\varepsilon(x, y) = (-1)^{xy}$ for all homogeneous elements $x, \; y$.

Below is an example of color Hom-Novikov algebra for $G = \mathbb{Z}_2$ given in [21].

Example 2.2. There is a three-dimensional multiplicative color Hom-Novikov algebra
$\tilde{A} = (A = A_0 \oplus A_1, \tilde{\mu}, \alpha)$ where the non-zero products are: $\tilde{\mu}(e_1, e_2) = \frac{1}{2}\lambda e_1 = -\tilde{\mu}(e_2, e_1)$,
$\tilde{\mu}(e_2, e_2) = \frac{1}{2}e_2, \; \tilde{\mu}(e_3, e_2) = \frac{1}{2}\lambda e_3, \; \tilde{\mu}(e_3, e_3) = \frac{1}{2}\lambda^2 e_1$ and the morphism α is defined as:
$\alpha(e_1) = \lambda^2 e_1, \; \alpha(e_2) = e_2, \; \alpha(e_3) = \lambda e_3$.

Proposition 2.1. *[3] Let $\mathcal{A} = (A, \mu, \varepsilon, \alpha)$ be a multiplicative color Hom-Novikov algebra. Then $\mathcal{A}_{\alpha^n} = (A, \alpha^n \circ \mu, \varepsilon, \alpha^{n+1})$ is also a multiplicative color Hom-Novikov algebra for each $n \in \mathbb{N}$.*

This allows to give the following example.

Example 2.3. From the example of a multiplicative color Hom-Novikov algebra $\tilde{A}$ in Example 2.2, we get the family of multiplicative color Hom-Novikov algebras $(\tilde{A}_{\alpha^n})n \in \mathbb{N}$ where the each $n \in \mathbb{N}$, $\tilde{A}_{\alpha^n} = (A, \tilde{\mu}_{\alpha^n}, \alpha^{n+1})$ with the following non-zero products $\tilde{\mu}_{\alpha^n}(_1, e_2) = \frac{1}{2}\lambda^{2n+2}e_1 = -\tilde{\mu}_{\alpha^n}(_2, e_1), \; \tilde{\mu}_{\alpha^n}(e_2, e_2) = \frac{1}{2}e_2, \; \tilde{\mu}_{\alpha^n}(e_3, e_2) = \frac{1}{2}\lambda^{n+1}e_3, \; \tilde{\mu}_{\alpha^n}(e_3, e_3) = \frac{1}{2}\lambda^{2n+2}e_1$ and the morphism defined by $\alpha^{n+1}(e_1) = \lambda^{2n+2}e_1, \; \alpha^{n+1}(e_2) = e_2, \; \alpha^{n+1}(e_3) = \lambda^{n+1}e_3$.

3. Definition and constructions

The purposes of this section are to present color Hom-Novikov-Poisson algebras and to discuss some basic properties and examples of these objects. It turns out that color Hom-Novikov-Poisson algebras generalize color Novikov-Poisson algebras as the same way as Hom-Novikov-Poisson algebras generalize Novikov-Poisson algebras. We point out once again that Hom-Novikov-Poisson superalgebras [19] are particular instances of color Hom-Novikov-Poisson algebras.

Definition 3.1. A color Hom-Novikov-Poisson algebra is a double color Hom-algebra $\mathcal{A} = (A, \cdot, *, \varepsilon, \alpha)$ consisting of an ε-commutative Hom-associative color algebra $\mathcal{A} = (A, \cdot, \varepsilon, \alpha)$, a color Hom-Novikov algebra $\mathcal{A} = (A, *, \varepsilon, \alpha)$ such that the following compatibility conditions:

$$(x \cdot y) * \alpha(z) = \varepsilon(y, z)(x * z) \cdot \alpha(y) \tag{3.1}$$
$$a\tilde{s}_{\mathcal{A}}(x, y, z) = \varepsilon(x, y)a\tilde{s}_{\mathcal{V}}(y, x, z) \tag{3.2}$$

hold for all $x, y, z \in \mathcal{H}(A)$ where $a\tilde{s}_{\mathcal{A}}$ is the mixed Hom-associator (Definition 2.5 (iv)).

Remark 3.1. (i) Every color Novikov-Poisson algebra is a color Hom-Novikov-Poisson algebra with identity as twisting map.
(ii) Let $\mathcal{A} = (A, \mu_1, \mu_2, \varepsilon, \alpha)$ be a multiplicative Hom-Novikov-Poisson algebra and β be a morphism of $\mathcal{A}$. Then $\mathcal{A}_\beta = (A, \beta \circ \mu_1, \beta \circ \mu_2, \varepsilon, \beta \circ \alpha)$ is a multiplicative Hom-Novikov-Poisson algebra.

Example 3.1. There is a tree-dimensional color Hom-Novikov-Poisson algebra $\mathcal{A} = (A_1 \oplus A_2, \cdot, \mu, \varepsilon, \alpha)$ with $A_0 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $A_2 = \mathbb{C}e_3$ such that the non-zero products are $\mu(e_1, e_1) = e_2,\ \mu(e_2, e_2) = e_1,\ \mu(e_1, e_3) = e_3,\ \mu(e_2, e_3) = e_3$ and $\alpha(e_1) = e_2,\ \alpha(e_2) = e_1,\ \alpha(e_3) = 0$. In fact $\mathcal{A} = (A_1 \oplus A_2, \cdot, \mu, \varepsilon, \alpha)$ is a non trivial Hom-Novikov-Poisson superalgebra [21]. According to Remark 3.1 (ii), we get the following color Hom-Novikov-Poisson algebras: $\mathcal{A}_\alpha = (A_1 \oplus A_2, \cdot_\alpha, \mu_\alpha, \varepsilon, \alpha^2)\ with\ \alpha^2(e_1) = e_1,\ \alpha^2(e_2) = e_2,\ \alpha^2(e_3) = 0$ where the non-zero products are $\mu_\alpha(e_1, e_1) = e_1,\ \mu_\alpha(e_2, e_2) = e_2$ and $\mathcal{A}_{\alpha^2} = (A_1 \oplus A_2, \cdot_{\alpha^2}, \mu_{\alpha^2}, \varepsilon, \alpha)\ (since\ \alpha^3 = \alpha)$ where the non-zero products are $\mu_{\alpha^2}(e_1, e_1) = e_2,\ \mu_{\alpha^2}(e_2, e_2) = e_1$. Actually, we have $\mathcal{A}_{\alpha^{2n-1}} = \mathcal{A}_\alpha$ and $\mathcal{A}_{\alpha^{2n}} = \mathcal{A}_{\alpha^2}$ for all $n \in \mathbb{N}^*$.

The following useful observation says that there is another way to state the compatibility condition (3.1) in a color Hom-Novikov-Poisson algebra.

Lemma 3.1. *Let $\mathcal{A} = (A, \cdot, *, \varepsilon, \alpha)$ be a double color Hom-algebra in which $\cdot$ is ε-commutative. Then*

$$(x \cdot y) * \alpha(z) = \varepsilon(y, z)(x * z) \cdot \alpha(y) \tag{3.3}$$

holds for all $x, y, z \in \mathcal{H}(A)$ if and only if

$$(x \cdot y) * \alpha(z) = \alpha(x) \cdot (y * z) \tag{3.4}$$

for all $x, y, z \in \mathcal{H}(A)$. In particular, if $\mathcal{A}$ is a color Hom-Novikov-Poisson algebra, then

$$(x \cdot y) * \alpha(z) = \varepsilon(y, z)(x * z) \cdot \alpha(y) = \alpha(x) \cdot (y * z) \textit{ for all } x, y, z \in \mathcal{H}(A). \tag{3.5}$$

Proof. Pick $x, y, z \in \mathcal{H}(A)$. Then the ε-commutativity of " $\cdot$ " implies $(x \cdot y) * \alpha(z) = \varepsilon(x, y)(y \cdot x) * \alpha(z)$ and $(y * z) \cdot \alpha(x) = \varepsilon(y + z, x)\alpha(x) \cdot (y * z)$. Therefore

$$(x \cdot y) * \alpha(z) = \alpha(x) \cdot (y * z)$$

holds for all $x, y, z \in \mathcal{H}(A)$ if and only if

$$(y \cdot x) * \alpha(z) = \varepsilon(x, z)(y * z) \cdot \alpha(x)$$

for all $x, y, z \in \mathcal{H}(A)$.
Finally remarking that $(y \cdot x) * \alpha(z) = \varepsilon(x, z)(y * z) \cdot \alpha(x)$ is equivalent to (3.3), we conclude that (3.3) holds for all $x, y, z \in \mathcal{H}(A)$ if and only if (3.4) holds for all $x, y, z \in \mathcal{H}(A)$.
If $\alpha = Id_A$, we get the following result which will be used below.

Corollary 3.1. *Let $(A, \cdot, *, \varepsilon)$ be a color Novikov-Poisson algebra such that $(A, \cdot, \varepsilon)$ is an ε-commutative associative color algebra and $(A, *, \varepsilon)$ is a color Novikov algebra. then*

$$(x \cdot y) * z = x \cdot (y * z) \tag{3.6}$$

holds for all $x, y \in \mathcal{H}(A)$.

The following result shows that every ε-commutative Hom-associative color algebra with an even derivation commutating with the twisting map, has a color Hom-Novikov-Poisson algebra structure.

Theorem 3.1. *Let $(A, \cdot, \varepsilon, \alpha)$ be an ε-commutative Hom-associative color algebra and $\partial : \mathcal{A} \to \mathcal{A}$ be an even derivation commutating with α. Then $\mathcal{A} = (A, \cdot, *, \varepsilon, \alpha)$ is a color Hom-Novikov-Poisson algebra*

where $x * y = x \cdot \partial y$, for all $x, y \in \mathcal{H}(A)$.

Proof. It follows from [3] (Theorem 3.12) that $(A, *, \varepsilon, \alpha)$ is a color Hom-Poisson algebra. We need to check the two compatibility conditions (3.1) and (3.2). Pick $x, y, z \in \mathcal{H}(A)$, then we have:

$$\begin{aligned}
(x \cdot y) * \alpha(z) &= (x \cdot y) \cdot \partial(\alpha(z)) \\
&= \varepsilon(x, y)(y \cdot x) \cdot \partial(\alpha(z)) (\text{by the } \epsilon\text{-commutativity of } \cdot) \\
&= \varepsilon(x, y)(y \cdot x) \cdot \alpha(\partial(z)) \\
&= \varepsilon(x, y)\alpha(y) \cdot (x \cdot \partial(z)) (\text{by the Hom-associativity}) \\
&= \varepsilon(x, y)\varepsilon(y, x + z)(x \cdot \partial z) \cdot \alpha(y) (\text{by the } \varepsilon\text{-commutativity of } \cdot) \\
&= \varepsilon(y, z)(x * z) \cdot \alpha(y)
\end{aligned}$$

which proves the equality (3.1). Furthermore we get

$$\begin{aligned}
(x * y) \cdot \alpha(z) - \alpha(x) * (y \cdot z) &= (x \cdot \partial y) \cdot \alpha(z) - \alpha(x) \cdot \partial(y \cdot z) \\
&= (x \cdot \partial y) \cdot \alpha(z) - \alpha(x) \cdot (\partial y \cdot z + y \cdot \partial z) \\
&= \alpha(x) \cdot (\partial y \cdot z) - \alpha(x) \cdot (\partial y \cdot z + y \cdot \partial z) \\
&= -\alpha(x) \cdot (y \cdot \partial z) \\
&= -(x \cdot y) \cdot \alpha(\partial z)
\end{aligned}$$

Similarly, we get $(y * x) \cdot \alpha(z) - \alpha(y) * (x \cdot z) = -(y \cdot x) \cdot \alpha(\partial z)$. Then (3.2) is obtained from the ε-commutativity of " $\cdot$ ". Below, an example of a color Hom-Novikov-Poisson algebra is given for $G = \mathbb{Z}_2$.

Example 3.2. Let $(A, \cdot, \alpha)$ be a commutative Hom-associative algebra with a derivation d such that $\alpha \circ d = d \circ \alpha$. Set $\tilde{A} = A \times A = \tilde{A}_0 \oplus \tilde{A}_1 \ with\ \tilde{A}_0 = A \times \{0\},\ \tilde{A}_1 = \{0\} \times A$. For any x_0, x_1, y_0, y_1 in A, define the operation $\diamond$ on $\tilde{A}$ by

$$(x_0, x_1) \diamond (y_0, y_1) = (x_0 \cdot x_1, 0)$$

and two even linear maps $\tilde{\alpha} : \tilde{A} \longrightarrow \tilde{A}$ and $\partial : \tilde{A} \longrightarrow \tilde{A}$ by $\tilde{\alpha}(x_0, x_1) = (\alpha(x_0), \alpha(x_1))$ and $\partial(x_0, x_1) = (d(x_0)), x_1)$. then $(\tilde{A}, \diamond, \tilde{\alpha})$ is a commutative Hom-associative superalgebra (an ε-commutative Hom-associative color algebra with $G = \mathbb{Z}_2$) and ∂ is an even derivation of $(\tilde{A}, \diamond, \tilde{\alpha})$ such that $\partial \circ \tilde{\alpha} = \tilde{\alpha} \circ \partial$ [21]. By Theorem 3.1 $(\tilde{A}, \diamond, *, \alpha)$ is a color Hom-Novikov-Poisson algebra where

$$(x_0, x_1) * (y_0, y_1) = (x_0, x_1) \diamond \partial(y_0, y_1) := (x_0 \cdot d(y_0), 0)$$

We have the following way for constructing color Novikov-Poisson algebras from ε-commutative associative color algebras.

Corollary 3.2. *Let $\mathcal{A} = (A, \cdot, \varepsilon)$ be an ε-commutative associative color algebra and $\partial : A \longrightarrow A$ be an even derivation of $\mathcal{A}$. Then the quadruple $(A, \cdot, *, \varepsilon)$ is a color Novikov-Poisson algebra where*

$$x * y = x \cdot \partial y \ for\ all\ x,\ y\ in\ \mathcal{H}(A).$$

The following result will be used to give an example of color Hom-Novikov-Poisson algebra. It is a color graded generalization of Proposition 2.2 in [14].

Proposition 3.1. *Let $(A, \cdot, *, \varepsilon)$ be a color Novikov-Poisson algebra, such that $(A, \cdot, \varepsilon)$ is an ε-commutative associative color algebra and $(A, *, \varepsilon)$ is a color Novikov algebra. Suppose that $(A, \cdot, \varepsilon)$ contains an identity element 1_A, i.e. $1_A \cdot x = x \cdot 1_A = x$ for all $x \in \mathcal{H}(A)$. Define a linear map $\partial : A \longrightarrow A$ by*

$$\partial(x) = 1_A * x - (1_A * 1_A) \cdot x, \text{ for all } x \in \mathcal{H}(A) \tag{3.7}$$

Then ∂ is an even derivation of $(A, \cdot, \varepsilon)$.

Proof. It is clear that ∂ is an even linear map.
Next by (3.6) we have:

$$x * y = (x \cdot 1_A) * y = x \cdot (1_A * y) \text{ for all } x, y \in \mathcal{H}(A) \tag{3.8}$$

Note that $1_A \in A_0$(homogeneous element of degree zero) and then for $x, y \in \mathcal{H}(A)$,

$$\begin{aligned}
\partial(x \cdot y) &= 1_A * (x \cdot y) - (1_A * 1_A) \cdot (x \cdot y) \\
&= (1_A * x) \cdot y + x * (1_A \cdot y) - (x * 1_A) \cdot y - (1_A * 1_A) \cdot (x \cdot y) \text{ (by (2.3))} \\
&= (1_A * x) \cdot y + x \cdot (1_A * y) - (x * 1_A) \cdot y - (1_A * 1_A) \cdot (x \cdot y) \text{ (by (3.8))} \\
&= (1_A * x) \cdot y + x \cdot (1_A * y) - \varepsilon(x, y) y \cdot (x * 1_A) - (1_A * 1_A) \cdot (x \cdot y) \\
&\quad \text{(since } x * 1_A \in A_x \text{ and } \cdot \text{ is } \varepsilon\text{-commutative)} \\
&= (1_A * x) \cdot y + x \cdot (1_A * y) - \varepsilon(x, y)(y \cdot x) * 1_A - (1_A * 1_A) \cdot (x \cdot y) \text{(by (3.6))} \\
&= (1_A * x) \cdot y + x \cdot (1_A * y) - (x \cdot y) * 1_A - (1_A * 1_A) \cdot (x \cdot y) \\
&\quad \text{(by the } \varepsilon\text{-commutativity of } \cdot) \\
&= (1_A * x) \cdot y + x \cdot (1_A * y) - (x \cdot y) \cdot (1_A * 1_A) - (1_A * 1_A) \cdot (x \cdot y) \text{ (by (3.8))} \\
&= (1_A * x) \cdot y + x \cdot (1_A * y) - 2(1_A * 1_A) \cdot (x \cdot y) \\
&\quad \text{(since } 1_A * 1_A \in A_0 \text{ and } \cdot \text{ is } \varepsilon\text{-commutative)} \\
\partial(x) \cdot y + x \cdot \partial(y) &= (1_A * x - (1_A * 1_A) \cdot x) \cdot y + x \cdot (1_A * y - (1_A * 1_A) \cdot y) \\
&= (1_A * x) \cdot y + x \cdot (1_A * y) - ((1_A * 1_A) \cdot x) \cdot y - x \cdot ((1_A * 1_A) \cdot y) \\
&= (1_A * x) \cdot y + x \cdot (1_A * y) - ((1_A * 1_A) \cdot x) \cdot y - x \cdot (y \cdot (1_A * 1_A)) \\
&= (1_A * x) \cdot y + x \cdot (1_A * y) - (1_A * 1_A) \cdot (x \cdot y) - (x \cdot y) \cdot (1_A * 1_A) \\
&\quad \text{(by the Hom-associativity)} \\
&= (1_A * x) \cdot y + x \cdot (1_A * y) - 2(1_A * 1_A) \cdot (x \cdot y)
\end{aligned}$$

Thus we have $\partial(x \cdot y) = \partial(x) \cdot y + x \cdot \partial(y)$ and therefore ∂ is an even derivation of $(A, \cdot, \varepsilon)$.
Now, we give the following example of color Hom-Novikov-Poisson algebra by using Theorem 3.1 and Proposition 3.1.

Example 3.3. Assume that $(A, \cdot, *, \varepsilon)$ is a color Novikov-Poisson algebra, where $(A, \cdot, \varepsilon)$ is an ε-commutative associative color algebra with unity 1_A and $(A, *, \varepsilon)$ is a color Novikov algebra. Since $\cdot$ is ε-commutative and $1_A * 1_A \in A_0$, we have by (3.6), $(1_A * 1_A) \cdot x = x \cdot (1_A * 1_A) = (x \cdot 1_A) * 1_A = x * 1_A \; for \; all \; x \in \mathcal{H}(A)$ and then the map ∂ defined in (3.7) takes the form

$$\partial(x) = 1_A * x - x * 1_A \; for \; all \; x \in \mathcal{H}(A).$$

Let α be a morphism of the double color algebra $(A, \cdot, *, \varepsilon)$. Define a new multiplication on A by

$$x \diamond y = \alpha(x \cdot y) \; for \; all \; x \in \mathcal{H}(A).$$

Then $(A, \diamond, \alpha, \varepsilon)$ is an ε-commutative Hom-associative color algebra. Obviously $\alpha(1_A) = 1_A$. As a consequence, α commutes with ∂. Furthermore, ∂ is an even derivation of $(A, \diamond, \varepsilon)$ since it is an even derivation of $(A, \cdot, \varepsilon)$ by Proposition 3.1. Now, define a new operation on A by

$$x * y = x \diamond \partial(y) \; for \; all \; x \in \mathcal{H}(A).$$

Then thanks to Theorem 3.1, $(A, \diamond, *, \varepsilon, \alpha)$ is a color Hom-Novikov-Poisson algebra

We need the following observation which will be used below.

Lemma 3.2. *Let $(A, \mu, \varepsilon, \alpha)$ be an ε-commutative Hom-associative color algebra. Then*

$$\mu(\mu(x, y), \alpha(z)) = \varepsilon(y, z)\mu(\mu(x, z), \alpha(y)) \tag{3.9}$$

holds for all $x, y, z \in \mathcal{H}(A)$.

Proof. Let $x, y, z \in \mathcal{H}(A)$. Then

$$\begin{aligned}
\mu(\mu(x, y), \alpha(z)) &= \varepsilon(x, y)\mu(\mu(y, x), \alpha(z)) \text{ (by the } \varepsilon\text{-commutativity)}\\
&= \varepsilon(x, y)\mu(\alpha(y), \mu(x, z)) \text{ (by the Hom-associativity)}\\
&= \varepsilon(x, y)\varepsilon(y, x + z)\mu(\mu(x, z), \alpha(y)) \text{ (by the } \varepsilon\text{-commutativity)}\\
&= \varepsilon(x, y)\varepsilon(y, x)\varepsilon(y, z)\mu(\mu(x, z), \alpha(y)) \text{ (by (iii) of Definition 2.1)}\\
&= \varepsilon(y, z)\mu(\mu(x, z), \alpha(y)) \text{ (by (i) of Definition 2.1)}
\end{aligned}$$

Another way for constructing color Hom-Novikov-Poisson algebra from ε-commutative Hom-associative color algebras is given as it follows;

Proposition 3.2. *Let $\mathcal{A} = (A, \mu, \varepsilon, \alpha)$ be an ε-commutative Hom-associative color algebra. then $(A, \mu, \mu, \varepsilon, \alpha)$ is a color Hom-Novikov-Poisson algebra.*

Proof. Indeed, in this case the defining identity (2.6) of a color Hom-Novikov algebra coincide with the compatibility condition (3.2). The condition (2.6) holds because $as_{\mathcal{A}} = 0$ by the Hom-associativity. The condition (3.1) holds by Lemma 3.2.

The purpose of the following result is to show that certain perturbations preserve color Novikov-Poisson algebra structures. We need the following preliminary observation about perturbing the structure maps in an ε-commutative Hom-associative color algebra.

Lemma 3.3. *Let $\mathcal{A} = (A, \mu, \varepsilon, \alpha)$ be an ε-commutative Hom-associative color algebra and t be an element in $\mathcal{H}(A)$ of degree 0 i.e. $t \in A_0$. Define the operation $\mu_t : A^{\otimes 2} \longrightarrow A$ by*

$$\mu_t(x, y) = \mu(t, \mu(x, y)) \tag{3.10}$$

for all $x, y \in \mathcal{H}(A)$. Then $\mathcal{A}_t = (A, \mu_t, \varepsilon, \alpha^2)$ is an ε-commutative Hom-associative color algebra. Moreover if $\mathcal{A}$ is multiplicative and $\alpha^2(t) = t$, then $\mathcal{A}_t$ is also multiplicative.

Proof. The ε- commutativity of μ_t follows from the one of μ. The multiplicativity is straightforward to check. To prove that $\mathcal{A}_t$ is Hom-associative, pick $x, y, z \in \mathcal{H}(A)$ and abbreviate $\mu(x, y)$ to xy. Then using the fact that $t(yz) \in A_{y+z}$, $\alpha^2(x) \in A_x$ and using repeatedly the condition $\varepsilon(t, w) = \varepsilon(w, t) = 1$ which holds for all $w \in \mathcal{H}(A)$ (since $t \in A_0$), we have:

$$\begin{aligned}
\mu_t(\mu_t(x, y), \alpha^2(z)) &= t\{(t(xy))\alpha^2(z)\}\\
&= t\{\alpha(t)((xy)\alpha(z))\} \text{ (by Hom-associativity)}\\
&= t\{\alpha(t)(\alpha(x)(yz))\} \text{ (by Hom-associativity)}\\
&= \varepsilon(x, y + z)t\{\alpha(t)((yz)\alpha(x))\} \text{ (by } \varepsilon\text{-commutativity)}\\
&= \varepsilon(x, y + z)t\{((yz)\alpha(x))\alpha(t)\} \text{ (by } \varepsilon\text{-commutativity)}\\
&= \varepsilon(x, y + z)t\{((yz)t)\alpha^2(x)\}\text{(by Lemma 3.2)}\\
&= \varepsilon(x, y + z)t\{(t(yz))\alpha^2(x)\} \text{ (by } \varepsilon\text{-commutativity)}\\
&= \varepsilon(y + z, x)\varepsilon(x, y + z)t\{\alpha^2(x)(t(yz))\} \text{ (by } \varepsilon\text{-commutativity)}\\
&= \mu_t(\alpha^2(x), \mu_t(y, z)) \text{ (by Definition 2.1 (i))}
\end{aligned}$$

This shows that $\mathcal{A}_t$ is Hom-associative.

The structure maps of a multiplicative color Hom-Novikov-Poisson algebra can be perturbed by a suitable element and its own twisting map as it follows.

Theorem 3.2. *Let $\mathcal{A} = (A, \mu, \mu', \varepsilon, \alpha)$ be a multiplicative color Hom-Novikov-Poisson algebra where $(A, \mu, \varepsilon, \alpha)$ is an ε-commutative Hom-associative color algebra, $(A, \mu', \varepsilon, \alpha)$ a color Hom-Novikov algebra and t be an homogeneous element in A_0 such that $\alpha^2(t) = t$. Then $\mathcal{A}_t = (A, \mu_t, \mu'_\alpha, \varepsilon, \alpha^2)$ is also a multiplicative color Hom-Novikov-Poisson algebra, where*

$$\begin{aligned} \mu_t(x,y) &= \mu(t, \mu(x,y)) \\ \mu'_\alpha(x,y) &= \mu'(\alpha(x), \alpha(y)) \end{aligned}$$

for all $x, y \in \mathcal{H}(A)$.

Proof. By Lemma 3.3, we know that $(A, \mu_t, \varepsilon, \alpha^2)$ is a multiplicative ε-commutative Hom-associative color algebra. Also, $(A, \mu'_\alpha, \varepsilon, \alpha^2)$ is a multiplicative color Hom-Novikov algebra by Proposition 2.1. It remains to prove the compatibility conditions (3.1) and (3.2) for $\mathcal{A}_t$. To prove the compatibility condition (3.1) for $\mathcal{A}_t$, pick $x, y, z \in \mathcal{H}(A)$. With $\mu(x,y)$ and $\mu'(x,y)$ written as xy and $x * y$ respectively, we have:

$$\begin{aligned} \mu'_\alpha(\mu_t(x,y), \alpha^2(z)) &= (\alpha(t)\alpha(xy)) * \alpha^3(z) \text{ (by the multiplicativity)} \\ &= \alpha^2(t)(\alpha(xy) * \alpha^2(z)) \text{ (by (3.4) in } \mathcal{A}) \\ &= t((\alpha(x)\alpha(y)) * \alpha^2(z)) \text{ (by the multiplicativity)} \\ &= t\{\alpha^2(x)(\alpha(y) * \alpha(z))\} \text{ (by (3.4) in } \mathcal{A}) \\ &= \mu_t(\alpha^2(x), \mu'_\alpha(y,z)). \end{aligned}$$

This proves that $\mathcal{A}_t$ satisfies (3.4), which is equivalent to (3.1) by Lemma 3.1.

To prove the compatibility condition (3.2) for $\mathcal{A}_t$, we consider the mixed Hom-associator $a\tilde{s}_{\mathcal{A}_t}$ (Definition 2.5) and we use the condition $\varepsilon(t, w) = \varepsilon(w, t) = 1$ which holds for all $w \in \mathcal{H}(A)$ (since $t \in A_0$)

$$a\tilde{s}_{\mathcal{A}_t}(x,y,z) = \mu_t(\mu'_\alpha(x,y), \alpha^2(z)) - \mu'_\alpha(\alpha^2(x), \mu_t(y,z)) \tag{3.11}$$

for all $x, y, z \in \mathcal{H}(A)$. The first term in the mixed Hom-associator is:

$$\begin{aligned} \mu_t(\mu'_\alpha(x,y), \alpha^2(z)) &= t\{(\alpha(x) * \alpha(y))\alpha^2(z)\} \\ &= \{(\alpha(x) * \alpha(y))\alpha^2(z)\}t \text{ (by the } \varepsilon\text{-commutativity)} \\ &= \{(\alpha(x) * \alpha(y))\alpha^2(z)\}\alpha^2(t) \text{ (by the the condition } \alpha^2(t) = t \text{)} \\ &= (\alpha^2(x) * \alpha^2(y))(\alpha^2(z)\alpha(t)) \text{ (by the Hom-associativity)} \\ &= (\alpha^2(x) * \alpha^2(y))\alpha(\alpha(z)t) \end{aligned} \tag{3.12}$$

The second term in the mixed Hom-associator is:

$$\begin{aligned} \mu'_\alpha(\alpha^2(x), \mu_t(y,z)) &= \alpha^3(x) * \alpha(t(yz)) \\ &= \alpha^3(x) * \{\alpha(t)(\alpha(y)\alpha(z))\} \\ &= \alpha^3(x) * \{(\alpha(y)\alpha(z))\alpha(t)\} \text{ (by the } \varepsilon\text{-commutativity)} \\ &= \alpha^3(x) * (\alpha^2(y)(\alpha(z)t)) \text{ (by the Hom-associativity)} \end{aligned} \tag{3.13}$$

Using (3.11), (3.12) and (3.13), it follows that the mixed Hom-associators of $\mathcal{A}_t$ and $\mathcal{A}$ are related as follows:

$$\begin{aligned} a\tilde{s}_{\mathcal{A}_t}(x,y,z) &= (\alpha^2(x) * \alpha^2(y))\alpha(\alpha(z)t) - \alpha^3(x) * (\alpha^2(y)(\alpha(z)t)) \\ &= a\tilde{s}_{\mathcal{A}}(\alpha^2(x), \alpha^2(y), \alpha(z)t) \end{aligned}$$

Then the condition (3.2) in $\mathcal{A}_t$ follows from the one in $\mathcal{A}$. Setting $\alpha = Id_A$ in Theorem 3.2, we get the following result which is a color graded generalization of Lemma 2.4 in [14].

Corollary 3.3. *Let $(A, \cdot, *, \varepsilon)$ be a color Novikov-Poisson algebra and $a \in A$ be an arbitrary homogeneous element of degree zero. Then $(A, \diamond, *, \varepsilon)$ is also a color Novikov-Poisson algebra, where*

$$x \diamond y = a \cdot x \cdot y$$

for all $x, y \in \mathcal{H}(A)$.

References

[1] Ammar,F. and Makhlouf, A., Hom-Lie and Hom-Lie-admissible superalgebras, *J. algebra.* 32(2010), no. 7, 1513-1528.

[2] Attan, S., Some characterisations of color Hom-Poisson algebras, *Hacet. J. Math. Stat.* 47(2018), no. 6, 1552-1563

[3] Bakayoko, I., Hom-Novikov color algebras, Arxiv: 1609.07813v1.

[4] Bergen, J. and Grzeszuk, P., Simple Jordan color algebras arising from associative graded algebras, *J. algebra.* 246(2001), no. 2, 915-950.

[5] Gao, X. and Xu, L., Novikov color algebras and Torkten color algebras, *Journal of Jilin University.* 54(2016) , no. 2, 197-201.

[6] Gel'fand, I.M. and Dorfman, I.Ya., Hamiltonian operators and algebraic structures related to them, *Funct. Anal, App.* 13(1979), 248-262.

[7] Hartwig, J. T., Larsson, D. and Silvestrov, S.D., Deformations of Lie algebras using σ-derivations, *J. Algebra.* 295(2006), no. 2, 314-361.

[8] Larsson, D. and Silvestrov, S.D., Quasi-Hom-Lie algebras, central extensions and 2-cycle-like identities, *J. Algebra.* 288(2005), no. 2, 321-344.

[9] Larsson, D. and Silvestrov, S.D., Quasi-Lie algebras, *Noncommutative Geometry and representation Theory in Mathematical physics. cotemp. Math., 391, Amer. Math. Soc., Providence, RI.* 2005, 241-248.

[10] Makhlouf, A. and Silvestrov, S.D., Hom-algebras structures, *J. Gen. Lie theory Appl.* 2(2008), no. 2, 51-64.

[11] Quingcheng, Z. and Yongzeng, Z., Derivations and extensions of lie color algebra, *Acta Mathematica Scienta.* 28(2008), no. 4, 933-948.

[12] Wang, C., Zhang, Q. and Wei, Z., Hom-Leibniz superalgebras and Hom-Leibniz-Poisson superalgebras, *Hacet. J. Math. Stat.* 44(2015), no.5, 1163-1179.

[13] Xu, X., On simple Novikov algebras and their irreducible modules, *J. Algebra.* 185(1996), no. 3, 905-934.

[14] Xu, X., Novikov-Poisson algebras, *J. algebra.* 190(1997), no. 2, 253-279.

[15] Xu, X., Variational calculus of supervariables and related algebraic structures, *J. Algebra.* 223(2000), no. 2, 396-437.

[16] Yau, D., Hom-algebras and homology, *J. Lie theory.* 19(2009), no. 2, 409-421.

[17] Yau, D., Hom-Novikov algebras, *J. Phys. A.* 44(2011), no. 8 085202.

[18] Yau, D., A twisted generalization of Novikov-poisson algebras, Arxiv: 1010.3410v1.

[19] Yu, H., Zhang, W., Jia, Z. and Zhang, Q., Hom-Novikov-Poisson superalgebras, *Journal of Jilin University (Science Edition).* 53(2015), no. 4, 606-610.

[20] Yuan, L., Hom-Lie color algebra structures, *Comm. Alg.* 40(2012), no. 2, 575-592.

[21] Yuan, L. M., Chen, S. and He, C.X., Hom-Gel'fand-Dorfman super-bialgebras and Hom-Lie conformal superalgebras, *Acta mathematica Sinica.* 33(2017), no. 1, 96-116.

[22] Zhang, R., Hou, D. and Bai, C., A Hom-version of the affinizations of balinskii-novikov and Novikov superalgebras, *J. math. Phys.* 52(2011), no. 2, 023505, 19 pp.

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