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Absolute Minimum and Maximum of the Probability Mass Functions and Limit of Generalized Renyi Entropy

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Abstract
Alencar and Francisco (1998) proved the absolute minimum and maximum value of a given probability mass function is related to the limit of the Renyi entropy.
In this paper, we have determined the minimum or maximum of some probability mass function (PMF) using this point. In the following, it will be shown that the absolute minimum value of a given PMF is related to the limit of the Generalized Renyi entropy, as $\alpha \to -\infty$ and β is fixed or ($\beta \to -\infty$ and α is fixed).The absolute maximum value of a PMF is related to the limit of the Generalized Renyi entropy, as $\alpha \to \infty$ and β is fixed or ($\beta \to \infty$ and α is fixed).

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1. Introduction

Information theory is related to the concept of irregularity in statistical thermodynamics and mechanics. Hartley (1928) presented definitions of information in communication engineering, for this reason, information theory has been considered as a subcategory of the communications engineering. The first time, Shannon (1948) considered the communication as a mathematical problem and revealed the method of determination the capacity of a channel. Renyi (1961) generalized entropy to one parameter family of entropies and defined Renyi entropy. This entropy soon after found application in graph theory. The original reason for Renyi to introduce his new entropy to be that he planed to use it in an information theoretic proof of the central limit theorem. But later, other applications of Renyi entropy were found. There are many different areas such as (physics, chemistry, statistics, economics and computer sciences and also data extraction, signal processing, pattern recognition, testing hypothesis and image registration processing). The problem of finding the absolute minimum and maximum values of a function is one of the applications of Renyi entropy which distinguished it. Alencar and Francisco (1998) showed that the absolute minimum and maximum value of a given probability density function can be related to the limit of the Renyi entropy. Section 2 relates to the problem of minimum and maximum of a PMF and Renyi entropy. Also some examples are presented which are calculated the absolute minimum and maximum values of the some probability distributions using this point, including skew discrete Laplace, symmetric discrete Laplace, symmetric discrete distributions, Logarithmic series distribution, discrete uniform distribution, etc.
Generalized exponential entropy introduced by Koski and Persson (1992) and the logarithm of it was named generalized Renyi entropy. In section 3, we show that the absolute minimum and maximum value of a given PMF is related to the limit of the generalized Renyi entropy in section 3. This point is expressed in two theorems and their proofs are presented.

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2. The minimum and maximum of a probability mass function and Renyi entropy

In this section we present definitions of the Renyi entropy and generalized Renyi entropy. One of the applications of the Renyi entropy is to find minimum and maximum values of a PMF. This topic was proposed by Alencar and Francisco (1998), they have shown when α tends to $-\infty$ and ∞ respectively, minimum and maximum of a PMF is related to limit of the Renyi entropy of it . Using that point, absolute minimum or maximum of some PMF are calculated.

Definition 2.1. Let X be a random variable with probability mass function P,

$$P = \{(p_1, p_2, ..., p_n); \ \sum_{i=1}^{n} p_i = 1, \ p_i \geq 0, \ i = 1, 2, ..., n\} \tag{2.1}$$

The Renyi entropy is defined as:

$$H_\alpha(P) = \frac{1}{1-\alpha} \log \sum_{i=1}^{n} p_i^\alpha, \quad \alpha > 0, \ \alpha \neq 1. \tag{2.2}$$

Also Generalized Renyi entropy for random variable X , is introduced as follows:

$$H_{\alpha,\beta}^{GR}(X) = \begin{cases} \frac{1}{\beta-\alpha} ln \frac{\sum_{i=1}^{n} p_i^\alpha}{\sum_{i=1}^{n} p_i^\beta} & \alpha \neq \beta \\ \frac{\sum_{i=1}^{n} p_i^\alpha ln p_i}{\sum_{i=1}^{n} p_i^\alpha} & \alpha = \beta \end{cases} \tag{2.3}$$

The results of Alencar and Francisco are presented in the following.

Theorem 2.1. *Consider that $P = \{p_1, ..., p_n\}$ is a probability density function where n could be infinite. The absolute minimum value of probability in the set, $m = min\{p_1, ..., p_n\}$ is given by*

$$m = 2^{-\lim_{\alpha \to -\infty} H_\alpha(P)}. \tag{2.4}$$

The absolute Maximum value of probability in the set, $M = max\{p_1, ..., p_n\}$ is given by

$$M = 2^{-\lim_{\alpha \to \infty} H_\alpha(P)}. \tag{2.5}$$

Proof of Theorem 2.2 has been presented in Alencar and Francisco (1998) completely.

Some examples

One of the discrete distributions is skew Laplace distribution that is defined as follows and you can see it in Kozubowski and Inusah (2006), Inusah and Kozubowski (2006).

Definition 2.2. A random variable X has a discrete skew Laplace distribution with parameters $p \in (0, 1)$ and $q \in (0, 1)$ denoted by $DSL(p, q)$, if

$$f(k) = P(X = k) = \frac{(1-p)(1-q)}{1-pq} \begin{cases} p^k & k = 0, 1, 2, ... \\ p^{|k|} & k = 0, -1, -2, ... \end{cases} \tag{2.6}$$

Example 2.1. Let X be a variable with skew discrete laplace distribution:

$$\begin{aligned} \lim_{\alpha \to \infty} H_\alpha(f(k)) &= \lim_{\alpha \to \infty} \frac{1}{1-\alpha} log[(\frac{(1-p)(1-q)}{1-pq})^\alpha (\sum_{k=-\infty}^{-1} p^{|k|\alpha} + 1 + \sum_{k=1}^{\infty} p^{k\alpha})] \\ &= \lim_{\alpha \to \infty} \frac{\alpha}{1-\alpha} log \frac{(1-p)(1-q)}{1-pq} + \lim_{\alpha \to \infty} \frac{1}{1-\alpha} log[1 + 2\sum_{k=1}^{\infty} p^{\alpha k}] \end{aligned} \tag{2.7}$$

The second equation of the righthand of (7) when α tends to ∞ is zero and we have:

$$\begin{aligned} \lim_{\alpha \to \infty} H_\alpha(f(k)) &= -log(\frac{(1-p)(1-q)}{1-pq}) \\ \Rightarrow M &= 2^{-\lim_{\alpha \to \infty} H_\alpha(f(k))} = \frac{(1-p)(1-q)}{1-pq} \end{aligned} \tag{2.8}$$

The maximum of skew discrete Laplace distribution is equal to $M = \frac{(1-p)(1-q)}{1-pq}$. In relation (6) if $p = q$ we have symmetric Laplace distribution and denoted by $DL(p)$,

$$f(k) = P(X = k) = \frac{1-p}{1+p}p^{|k|}, \quad p \in (0,1), \quad k \in Z = 0, \pm 1, \pm 2, ...$$

The Renyi entropy for X is

$$H_\alpha(f(k)) = \frac{\alpha log(\frac{1-p}{1+p})}{1-\alpha} + \frac{1}{1-\alpha} log[1 + \frac{2p^\alpha}{1-p^\alpha}] \tag{2.9}$$

When $\alpha \to \infty$, limit of the second equation of the right hand (9) is zero and with L'Hopital's rule, limit of the first equation, is $-log\frac{1-p}{1+p}$ so we obtain:

$$M = 2^{-\lim_{\alpha\to\infty} H_\alpha(P)} = 2^{log\frac{1-p}{1+p}} = \frac{1-p}{1+p}$$

The maximum of the symmetric discrete Laplace distribution occurs in $k = 0$ and equals to $\frac{1-p}{1+p}$.

Example 2.2. For discrete and continuous distributions, symmetry version is important. Other example that we calculate its maximum value is symmetric discrete distributions family

$$f(k) = P(X = k) = \frac{1}{1+2S(p)}p^{|k|^m}, \quad k \in Z, \quad S(p) = \sum_{k=1}^{\infty} p^{|k|^m}. \tag{2.10}$$

If $m = 1$ distribution given in (10) is symmetric discrete Laplace distribution that was discussed in example 2.4. limit of the Renyi entropy for this distribution is:

$$\begin{aligned}
\lim_{\alpha\to\infty} H_\alpha(p) &= \lim_{\alpha\to\infty} \frac{1}{1-\alpha} \log(\frac{1}{1+2S(p)})^\alpha [1 + 2\sum_{k=1}^{\infty} p^{k^{m\alpha}}] \\
&= \lim_{\alpha\to\infty} \frac{-\alpha}{1-\alpha} \log(1+2S(p)) + \frac{1}{1-\alpha} \log(1 + 2\sum_{k=1}^{\infty} p^{k^{m\alpha}}) \\
&= \log(1+2S(p)).
\end{aligned} \tag{2.11}$$

Then we obtain

$$M = 2^{-\lim_{\alpha\to\infty} H_\alpha(p)} = 2^{-\log(1+2S(p))} = \frac{1}{1+2S(p)} = \frac{1}{1+2\sum_{k=1}^{\infty} p^{k^m}}. \tag{2.12}$$

Example 2.3. The PMF of discrete uniform distribution is

$$f_\theta(x) = P(X = x) = \frac{1}{\theta+1}, \quad x = 0, 1, ..., \theta, \ \theta > 0, \ \theta \in Z.$$

The Renyi entropy is:

$$\begin{aligned}
H_\alpha(P) &= \frac{1}{1-\alpha} \log \sum_{k=0}^{\theta} (\frac{1}{\theta+1})^\alpha \\
&= \frac{1}{1-\alpha} \log(\frac{1}{\theta+1})^{\alpha-1} = -\log(\frac{1}{\theta+1}).
\end{aligned} \tag{2.13}$$

Here Renyi entropy does not depend on α and the limit of it is equal to $-log\frac{1}{\theta+1}$, thus minimum and maximum are equal:

$$m = M = 2^{-\lim_{\alpha\to\pm\infty} H_\alpha(P)} = \frac{1}{\theta+1}.$$

It is clear that minimum and maximum for discrete uniform distribution is $\frac{1}{\theta+1}$.

Example 2.4. The Logarithmic series distribution and its Renyi entropy are respectively:

$$f_\theta(x) = P(X = x) = \frac{\beta\theta^x}{x}, \;\; x = 0, 1, ..., \;\; 0 < \theta < 1, \;\; \beta = -(ln(1-\theta))^{-1},$$

and:

$$\begin{aligned} H_\alpha(P) &= \frac{1}{1-\alpha} \log \sum_{x=1}^{\infty} (\frac{\beta\theta^x}{x})^\alpha \\ &= \frac{\alpha \log \beta}{1-\alpha} + \frac{1}{1-\alpha} \log \sum_{x=1}^{\infty} (\frac{\theta^x}{x})^\alpha. \end{aligned} \tag{2.14}$$

respectively. Since $\theta \in (0, 1)$, the limit of second equation of the right hand of (14) is zero and limit of first equation of that is $-log\beta$, thus we have:

$$M = 2^{-\lim_{\alpha\to\infty} H_\alpha(P)} = 2^{\log \beta} = \beta.$$

See that maximum value of this distribution occurs in $x = 0$ and is equal to $\beta\theta$, since $\theta \in [0, 1]$, maximum value is β and this conforms with the result that obtained from limit of the Renyi entropy.

Example 2.5. Let X has geometric distribution with parameter p:

$$P_X(x) = pq^x, \quad x = 0, 1, ..., 0 < p \leq 1,$$

For this distribution, limit of the Renyi entropy is

$$\begin{aligned} H_\alpha(P) &= \frac{1}{1-\alpha} \log \sum_x p^\alpha q^{\alpha x} \quad \alpha \geq 0, \alpha \neq 1 \\ &= \frac{\alpha \log p}{1-\alpha} + \frac{1}{1-\alpha} \log \sum_{x=0}^{\infty} q^{\alpha x} \\ &\Rightarrow \lim_{\alpha\to\infty} H_\alpha(P) = \lim_{\alpha\to\infty} \frac{\alpha \log p}{1-\alpha} + \lim_{\alpha\to\infty} \log \frac{1}{1-q^\alpha} \frac{1}{1-\alpha} = -\log p \\ &\Rightarrow 2^{-\lim_{\alpha\to\infty} H_\alpha(P)} = 2^{\log p} = p. \end{aligned} \tag{2.15}$$

We know that the maximum value of Geometric distribution occurs in $x = 0$ and it is equal p, this conforms with the result of the limit of the Renyi entropy.

Example 2.6. For random variable X with poisson distribution, PMF and its Renyi entropy respectively are as follows :

$$p_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, ... \;\; , \lambda > 0,$$

and

$$\begin{aligned} H_\alpha(P) &= \frac{1}{1-\alpha} \log \sum_{x=0}^{\infty} \frac{\lambda^{\alpha x} e^{-\alpha\lambda}}{(x!)^\alpha}, \alpha \neq 1 \\ &= \frac{\alpha\lambda}{\alpha-1} \log e + \frac{1}{1-\alpha} \log \sum_{x=0}^{\infty} (\frac{\lambda^x}{x!})^\alpha, \;\; \alpha \neq 1. \end{aligned}$$

Taking the limit as $\alpha \to \infty$, and verifying that the second term on the right hand side vanishes, and obtains

$$\lim_{\alpha\to\infty} H_\alpha(P) = \lambda \log e,$$

Finally using this result,

$$M = 2^{-\lim_{\alpha\to\infty} H_\alpha(P)} = 2^{-\lambda \log e} = e^{-\lambda}. \tag{2.16}$$

3. The minimum and maximum value of a probability mass function and generalized Renyi entropy

Application of the generalized Renyi entropy of order α, β is the problem of finding the absolute minimum and maximum values of a PMF. In the following, we introduce two theorems similar to Theorem 2.3 for generalized Renyi entropy. We can determine minimum and maximum of a PMF using the limit of the generalized Renyi entropy instead of the Renyi entropy.

Theorem 3.1. *Consider that $P = \{p_1, ..., p_n\}$ is a probability density function where n could be infinite. The absolute minimum value of probability in the set, $m = min\{p_1, ..., p_n\}$ is given by*

$$m = 2^{-\lim_{\alpha\to-\infty} H^{GR}_{\alpha,\beta}(P)}, \quad when \;\; \beta \;\; is \; fixed \tag{3.1}$$

Proof Firstly, we calculate the limit of equation (3) as $\alpha \to -\infty$

$$\begin{aligned}
\lim_{\alpha\to-\infty} H^{GR}_{\alpha,\beta}(P) &= \lim_{\alpha\to-\infty} \frac{1}{\beta-\alpha} \log \frac{\sum_{i=1}^{n} p_i^{\alpha}}{\sum_{i=1}^{n} p_i^{\beta}} \\
&= \lim_{\alpha\to-\infty} \frac{1}{\beta-\alpha} [\log \sum_{i=1}^{n} p_i^{\alpha} - \log \sum_{i=1}^{n} p_i^{\beta}] \\
&= \lim_{\alpha\to-\infty} \frac{1}{\beta-\alpha} \log \sum_{i=1}^{n} p_i^{\alpha} - \lim_{\alpha\to-\infty} \frac{1}{\beta-\alpha} \log \sum_{i=1}^{n} p_i^{\beta} \qquad (3.2)
\end{aligned}$$

Secondly, when $\alpha \to -\infty$, the second term on the right hand of equation (18) tends to zero, this is sufficient to find

$$\lim_{\alpha\to-\infty} \frac{1}{\beta-\alpha} \log \sum_{i=1}^{n} p_i^{\alpha}.$$

Assume that there exists a minimum probability in the set, given by $m = min\{p_1, ..., p_n\} = p_k$

$$\begin{aligned}
\lim_{\alpha\to-\infty} \frac{1}{\beta-\alpha} \log \sum_{i=1}^{n} p_i^{\alpha} &= \lim_{\alpha\to-\infty} \frac{1}{\beta-\alpha} \log m^{\alpha}[(\frac{p_1}{m})^{\alpha} + ... + 1 + ... + (\frac{p_n}{m})^{\alpha}] \\
&= \lim_{\alpha\to-\infty} \frac{\alpha}{\beta-\alpha} \log m + \lim_{\alpha\to-\infty} \frac{1}{\beta-\alpha} \log[\sum_{i=1, i\neq k}^{n} (\frac{p_i}{m})^{\alpha}] \qquad (3.3)
\end{aligned}$$

The rightmost limit is clearly zero, which yields finally

$$\lim_{\alpha\to-\infty} H^{GR}_{\alpha,\beta}(P) = \lim_{\alpha\to-\infty} \frac{\alpha \log m}{\beta-\alpha} = -\log m.$$

The extension of the proof for two or more minima can be done as follow, Assume we have l minima in the set, let $m = min\{p_1, p_2, ..., p_n\} = p_i = p_{i+1} = ... = p_k$ where $l = k - i + 1$, as before we have:

$$\begin{aligned}
\lim_{\alpha\to-\infty} H^{GR}_{\alpha,\beta}(P) &= \lim_{\alpha\to-\infty} \frac{1}{\beta-\alpha} \log[\sum_{i=1}^{n} p_i^{\alpha}] \\
&= \lim_{\alpha\to-\infty} \frac{1}{\beta-\alpha} \log m^{\alpha}[(\frac{p_1}{m})^{\alpha} + (\frac{p_{i-1}}{m})^{\alpha} + l + (\frac{p_{k+1}}{m})^{\alpha} + ... + (\frac{p_n}{m})^{\alpha}] \\
&= \lim_{\alpha\to-\infty} \frac{\alpha}{\beta-\alpha} \log m + \lim_{\alpha\to-\infty} \frac{1}{\beta-\alpha} \log l + 0 \\
&= -\log m \\
\Rightarrow \quad & m = 2^{-\lim_{\alpha\to-\infty} H^{GR}_{\alpha,\beta}(P)}. \qquad (3.4)
\end{aligned}$$

The results of theorem 3.1 when α is fixed and $\beta \to -\infty$ is well established.

Theorem 3.2. *Consider that $P = \{p_1, ..., p_n\}$ is a probability density function where n could be infinite. The absolute maximum value of probability in the set, $M = max\{p_1, ..., p_n\}$ is given by*

$$M = 2^{-\lim_{\alpha\to\infty} H^{GR}_{\alpha,\beta}(P)}, \quad when \;\; \beta \; is \; fixed \tag{3.5}$$

Proof The limit of equation (3) as $\alpha \to \infty$ is:

$$\begin{aligned}
\lim_{\alpha\to\infty} H^{GR}_{\alpha,\beta}(P) &= \lim_{\alpha\to\infty} \frac{1}{\beta-\alpha}[\log\sum_{i=1}^{n} p_i^{\alpha} - \log\sum_{i=1}^{n} p_i^{\beta}] \\
&= \lim_{\alpha\to\infty} \frac{1}{\beta-\alpha} \log\sum_{i=1}^{n} p_i^{\alpha} \\
&= \lim_{\alpha\to\infty} \frac{1}{\beta-\alpha} \log M^{\alpha}[(\frac{p_1}{M})^{\alpha} + ... + 1 + ... + (\frac{p_n}{M})^{\alpha}] \\
&= \lim_{\alpha\to\infty} \frac{\alpha}{\beta-\alpha} \log M \\
&= -\log M.
\end{aligned} \tag{3.6}$$

The proof for two or more maxima can be done as follows:
Assume, the existence of l maxima in the set given by $p_i = ... = p_j = M$ then $k = j - i + 1$ and:

$$\begin{aligned}
\lim_{\alpha\to\infty} H^{GR}_{\alpha,\beta}(P) &= \lim_{\alpha\to\infty} \frac{1}{\beta-\alpha} \log M^{\alpha}[(\frac{p_1}{M})^{\alpha} + ... + (\frac{p_{i-1}}{M})^{\alpha} + ... + k + ... + (\frac{p_{j+1}}{M})^{\alpha} + ... + (\frac{p_n}{M})^{\alpha}] \\
&= \lim_{\alpha\to\infty} \frac{\alpha\log M}{\beta-\alpha} + \frac{\log k}{\beta-\alpha} \\
&= -\log M
\end{aligned} \tag{3.7}$$

Finally, solving this equation, we obtain

$$M = 2^{-\lim_{\alpha\to\infty} H^{GR}_{\alpha,\beta}(P)}.$$

The results of Theorem 3.2 when α is fixed and $\beta \to \infty$ is well fixed.

4. Conclusion

We recalled the results of Alencar and Francisco (1998), using their results we calculated the absolute minimum and maximum for some given probability mass functions. We also showed that the minimum value of a PMF is limit of the generalized Renyi entropy of order α, β when $\alpha \to -\infty$ and β is fixed and the absolute maximum value is related to limit of the generalized renyi entropy when $\alpha \to \infty$ and β is fixed.
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