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Fractional Differential Inclusions with Non Instantaneous Impulses in Banach Spaces

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Abstract

This paper is devoted to study the existence of solutions for a class of fractional differential inclusions with non instantaneous impulses involving the Caputo fractional derivative in a Banach space. The arguments are based upon Mönch's fixed point theorem and the technique of measures of noncompactness.

Keywords: Inclusions, impulses, Caputo fractional derivative, measure of noncompactness, fixed point, Banach space..
2010 MSC: 26A33, 34A37, 34G20, 47H08.

1. Introduction

The theory of fractional differential equations is an important branch of differential equation theory, which has an extensive physical, chemical, biological, and engineering background, and hence has been emerging as an important area of investigation in the last few decades; see the monographs of Abbas *et al.* [3, 4, 5], Kilbas *et al.* [25], Podlubny [30], and Zhou [35], and the references therein. On the other hand, the theory of impulsive differential equations has undergone rapid development over the years and played a very important role in modern applied mathematical models of real processes rising in phenomena studied in physics, population dynamics, chemical technology, biotechnology and economics; see for instance the monographs by Bainov and Simeonov [15], Benchohra *et al.* [16], Lakshmikantham *et al.* [21], and Samoilenko and Perestyuk [31] and references therein. Moreover, fractional differential equations and inclusions present a natural framework for mathematical modeling of several real-world problems. Some interesting results about impulsive fractional differential equations are given in the papers [32, 33, 34].

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In pharmacotherapy, instantaneous impulses cannot describe the dynamics of certain evolution processes. For example, when one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are a gradual and continuous process. In [1, 2, 6, 23, 29] the authors initially studied some new classes of fractional differential equations with non instantaneous impulses in Banach spaces.

However, the theory for fractional differential equations in Banach spaces has yet been sufficiently developed. Recently, Benchohra *et al.* [17] applied the measure of noncompactness to a class of Caputo fractional differential equations of order $r \in (0,1]$ in a Banach space. Let E be a separable Banach space with norm $\|\cdot\|$.

In this paper, we study the following fractional differential inclusions with non instantaneous impulses

$$^cD^r y(t) \in F(t, y(t)), \quad \text{for a.e. } t \in (s_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < r \leq 1, \tag{1.1}$$

$$y(t) = g_k(t, y(t)),\ t \in (t_k, s_k],\ k = 1, \dots, m, \tag{1.2}$$

$$y(0) = y_0, \tag{1.3}$$

where $^cD^r$ is the Caputo fractional derivative, $F : [0,T] \times E \to \mathcal{P}(E)$ is a multivalued map, $g_k : (t_k, s_k] \times E \to E$, $k = 1, \dots, m$, is a given function, and $y_0 \in E, 0 = s_0 < t_1 < s_1 < \cdots < t_m < s_m < t_{m+1} = T$.

To our knowledge no paper has been considered for impulsive fractional differential inclusions in abstract spaces. This paper fills the gap in the literature. To investigate the existence of solutions of the problem above, we use analogie Mönch's fixed point theorem combined with the technique of measures of noncompactness, which is an important method for seeking solutions of differential equations. See Akhmerov *et al.* [8], AlvÃărez [9], Banaś *et al.* [12, 13, 14], Guo *et al.* [20], MÂĺonch [26], Mönch and Von Harten [28].

2. Preliminaries

In this section, we first state the following definitions, lemmas and some notation. By $C(J, E)$ we denote the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{\|y(t)\| : t \in J\}.$$

Let $L^1(J, E)$ be the Banach space of measurable functions $y : J \to E$ which are Bochner integrable, equipped with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\| dt.$$

Let $L^\infty(J, E)$ be the Banach space of measurable functions $y : J \to E$ which are essentially bounded.

$$PC(J, E) = \{y : J \to E : y \in C([0, t_1] \cup (t_k, s_k] \cup (s_k, t_{k+1}], E),\ k = 1, \dots, m \text{ and there exist } y(t_k^-),\ y(t_k^+), y(s_k^-) \text{ and } y(s_k^+)\ k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k) \text{ and } y(s_k^-) = y(s_k)\}.$$

$PC(J, E)$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t \in J} \|y(t)\|.$$

Set

$$J' = [0, T] \setminus \cup_{k=1}^m (t_k, s_k].$$

Moreover, for a given set V of functions $v : J \to E$ let us denote by

$$V(t) = \{v(t),\ v \in V\},\ t \in J$$

and

$$V(J) = \{v(t),\ v \in V,\ t \in J\}.$$

$AC^1(J, E)$ is the space of continuously differentiable functions whose first derivative is absolutely continuous. We use the notations $\mathcal{P}(E)$ is the collection of all nonempty subsets of E.

$$\mathcal{P}_C\ (E) = \{A \subset E : A \text{ is nonempty, convex}\},$$
$$\mathcal{P}_{KC}\ (E) = \{A \subset E : A \text{ is nonempty, compact, convex}\}.$$

Let X, Y be two sets, $N : X \to Y$ a set-valued map, and $A \to Y$. We define

$$graph(N) = \{(x, y) : x \in X, y \in N(X)\} \quad \text{(the graph of N)}.$$

Let $R > 0$, and let

$$B = \{x \in E : |x| \leq R\},$$

and

$$U = \{x \in C(J, E) : \|x\|_\infty < R\}.$$

Clearly $\bar{U} = C(J, B)$.

For more details on multi-valued maps see [10, 11, 18, 19, 27].

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.1. ([12]). Let X be a Banach space and Ω_X the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \to [0, \infty]$ defined by

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \ and\ diam(B_i) \leq \epsilon\}; hereB \in \Omega_X.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for more details see [12])

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(b) $\alpha(B) = \alpha(\overline{B})$.

(c) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

(e) $\alpha(cB) = |c|\alpha(B); c \in \mathbb{R}$.

(f) $\alpha(convB) = \alpha(B)$.

For completeness we recall the definition of Caputo derivative of fractional order.

Definition 2.2. ([25]). The fractional (arbitrary) order integral of the function $h \in L^1([0, T], E)$ of order $r \in \mathbb{R}_+$ is defined by

$$I^r h(t) = \frac{1}{\Gamma(r)} \int_0^t (t - s)^{r-1} h(s) ds,$$

where Γ is the Euler gamma function defined by $\Gamma(r) = \displaystyle\int_0^\infty t^{r-1} e^{-t} dt,\ r > 0$.

Definition 2.3. ([25]). For a function $h \in AC^n(J, E)$, the Caputo fractional-order derivative of order r of h is defined by

$$({}^c D_0^r h)(t) = \frac{1}{\Gamma(n - r)} \int_0^t (t - s)^{n-r-1} h^{(n)}(s) ds,$$

where $n = [r] + 1$.

We need the following auxiliary lemmas [25].

Lemma 2.4. *Let $r > 0$ and $h \in AC^n(J, E)$. Then the differential equation*

$$^cD_0^r h(t) = 0, \quad \text{for a.e. } t \in J$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}, c_i \in \mathbb{R},\ i = 0, 1, 2, \dots, n-1,\ n = [r] + 1.$

Lemma 2.5. *Let $r > 0$ and $h \in AC^n(J, E)$. Then*

$$I^{r\,c}D_0^r h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}, \quad \text{for a.e. } t \in J$$

for some $c_i \in \mathbb{R},\ i = 0, 1, 2, \dots, n-1,\ n = [r] + 1.$

Definition 2.6. . A multivalued map $F : J \times E \to \mathcal{P}(E)$ is said to be Carathéodory if

i $t \to F(t, u)$ is measurable for each $u \in E$

ii $u \to F(t, u)$ is upper semicontinuous for almost all $t \in J$.

For each $y \in C(J, E)$, define the set of selections of F by

$$S_{F,y} = \{f \in L^1(J, E), f(t) \in F(t, y(t)) \ \text{ a.e. } t \in J\}.$$

For our purpose we will only need the following fixed point theorem, and the important Lemma.

Theorem 2.7. *([7, 26]). Let E be a Banach space and $f \in L^1(J, E)$ countable with $|u(t)| \leq h(t)$ for a.e.$t \in J$, and each $u \in C$; where $h \in L^1(J, \mathbb{R}_+)$ then the function $\phi(t) = \alpha(C(t))$ belongs to $L^1(J, \mathbb{R}_+)$ and satisfies*

$$\alpha\left(\left\{\int_0^T u(s)ds : u \in C\right\}\right) \leq 2\int_0^T \alpha(C(s)ds.$$

Theorem 2.8. *([7, 26]) (the set-valued analog of **Mönch's** fixed point theorem).*
Let K be a closed, convex subset of a Banach space E ; U a relatively open subset of K, and $N : \overline{U} \to \mathcal{P}_C\ (K)$. Assume graph (N) is closed, N maps compact sets into relatively compact sets, and that for some $0 \in U$; the following two conditions are satisfied:

$$\left(\begin{array}{c} M \subset U, M \subset conv(\{0\} \bigcup N(M)) \\ and\ \overline{M} = \overline{C}\ with\ C \subset M\ countable \end{array}\right) \Rightarrow \overline{M} compact. \tag{2.1}$$

$$x \notin \lambda N(x) for\ all\ x \in \overline{U} \backslash U, \lambda \in (0, 1) \tag{2.2}$$

Then there exists $x \in U$ with $x \in N(x)$.

Lemma 2.9. *[22] Let J be a compact real interval. Let F be a multivalued be a Carathéodory multivalued map and let Θ be a linear continuous map from $L^1(J, E) \to C(J, E)$. Then the operator*

$$\Theta \circ S_{F,y} : C(J, E) \to \mathcal{P}_{KC}\ (C(J, E)), y \mapsto (\Theta \circ S_{F,y})(y) = \Theta(S_{F,y})$$

is a closed graph operator in $C(J, E) \times C(J, E)$.

3. Existence of Solutions

First of all, we define what we mean by a solution of the problem (1.1)-(1.3).

Definition 3.1. A function $y \in PC(J,E) \cap AC(J',E)$ is said to be a solution of (1.1)-(1.3) if there exists a function $f \in L^1(J,E)$ with $f(t) \in F(t,y(t))$, for a.e. $t \in J$, such that

$$^cD^r y(t) = f(t), for\ a.e.\ t \in J, 0 < r \leq 1,$$

and the function y satisfies conditions (1.2)-(1.3).

To prove the existence of solutions to (1.1)-(1.3), we need the following auxiliary lemma.

Lemma 3.2. *Let $0 < r \leq 1$ and let $h : J \to E$ be integrable. Then linear problem*

$$^cD^r y(t) = h(t), \quad t \in J_k, \quad k = 0,\dots,m, \tag{3.1}$$

$$y(t) = g_k(t), \quad t \in J_k', \quad k = 1,\dots,m, \tag{3.2}$$

$$y(0) = y_0, \tag{3.3}$$

has a unique solution given by :

$$y(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}h(s)ds & if\ t \in [0,t_1], \\ g_k(t), & if\ t \in J_k' := (t_k, s_k], \\ g_k(s_k) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1}h(s)ds, & if\ t \in J_k := (s_k, t_{k+1}]. \end{cases} \tag{3.4}$$

We are now in a position to state and prove our existence result for the problem (1.1)–(1.3) based on Mönch's fixed point. Let us list the following conditions.

(H1) $F : J \times E \to \mathcal{P}_{kc}(E)$ is a Carathéodory multi-valued map.

(H2) There exists a function $p \in L^\infty(J, \mathbb{R}_+)$ such that

$$\|F(t,y)\|_{\mathcal{P}} = sup\{\|v\|_\infty : v(t) \in F(t,y)\} \leq p(t), \text{ for each } (t,y) \in J \times E.$$

(H3) For each bounded and measurable set $B \subset C(J,E)$ and for each $t \in J$, we have

$$\alpha(F(t,B(t)) \leq p(t)\alpha(B(t)),$$

where $B(t) = \{u(t) : u \in B\}$.

(H4) g_k are uniformly continuous functions and there exists $c_k \in C(J, \mathbb{R}_+)$ such that

$$\|g_k(t,y)\| \leq c_k(t)\|y\|, \ for\ each\ y \in E\ and\ t \in J, \ k = 1,\dots,m.$$

(H5) The function $\phi \equiv 0$ is the unique solution in $PC(J;E)$ of the inequality

$$\Phi(t) \leq \frac{2p^*}{1-c^*}\int_{s_k}^t \Phi(s)ds, k = 0,\dots,m.$$

(H6) For each bounded set $B \subset E$ we have

$$\alpha(g_k(t,B)) \leq c_k(t)\alpha(B), t \in J.$$

Let

$$p^* = esssup_{t\in J} p(t),$$
$$c^* = \max\{\sup_{t\in J} c_k(t) : k = 1, \dots, m\} < 1. \tag{3.5}$$

Remark 3.3. In (H3) and (H6), α is the Kuratowski measure of noncompactness on the space E.

Theorem 3.4. *Assume that assumptions (H1) - (H6) hold. Then the problem (1.1)–(1.3) has at least one solution J.*

Proof. Transform the problem (1.1)–(1.3) into a fixed point problem. Consider the multi-valued map $N : PC(J,E) \to \mathcal{P}(PC(J,E))$ defined by
$N(y)(t) = \{h \in PC(J,E) :$

$$h(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f(s)ds & \text{if } t \in [0,t_1], \\ g_k(t, y(t)), & \text{if } t \in J'_k := (t_k, s_k], \quad f \in S_{F,y}\} \\ g_k(s_k, y(s_k)) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f(s))ds, & \text{if } t \in J_k := (s_k, t_{k+1}]. \end{cases}$$

Clearly, the fixed points of operator N are solutions of problem (1.1)–(1.3). The proof will be given in a couple of steps.

Step 1: *N is convex for each $y \in PC(J,E)$.*
If h_1, h_2 belong to $N(y)$, then there exist $f_1, f_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_i(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f_i(s)ds & \text{if } t \in [0,t_1], \\ g_k(t, y(t)), & \text{if } t \in J'_k := (t_k, s_k], \quad i = 1,2 \\ g_k(s_k, y(s_k)) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f_i(s))ds, & \text{if } t \in J_k := (s_k, t_{k+1}]. \end{cases}$$

Let $0 \le \lambda \le 1$. For each $t \in J$, we have
$(\lambda h_1 + (1-\lambda)h_2)(t) =$

$$\begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} (\lambda f_1 + (1-\lambda) f_2)(s)ds & \text{if } t \in [0,t_1], \\ g_k(t, y(t)), & \text{if } t \in J'_k, \\ g_k(s_k, y(s_k)) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} (\lambda f_1 + (1-\lambda) f_2)(s))ds, & \text{if } t \in J_k. \end{cases}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$(\lambda h_1 + (1-\lambda)h_2) \in N(y).$$

Step 2: *For each compact $M \in \bar{U}$, $N(M)$ is relatively compact.*
To prove this, let $M \in \bar{U}$ be a compact set and let (h_n) be any sequence of elements $N(M)$. We show that (h_n) has a convergent subsequence by using Arzela-Ascoli criterion of noncompactness in $PC(J,E)$. Since $(h_n) \in N(M)$ there exist $(y_n) \in M$ and $(f_n) \in S_{F,y_n}$ such that

$$h_n(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f_n(s)ds & \text{if } t \in [0,t_1], \\ g_k(t, y_n(t)), & \text{if } t \in J'_k := (t_k, s_k], \\ g_k(s_k, y_n(s_k)) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f_n(s))ds, & \text{if } t \in J_k := (s_k, t_{k+1}]. \end{cases}$$

Using Theorem 2.7, the properties of measure of α and (H5), we obtain that

$$\alpha(\{h_n(t)\}) = \alpha\left(\begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f_n(s)ds & \text{if } t \in [0,t_1], \\ g_k(t, y_n(t)), & \text{if } t \in J'_k := (t_k, s_k], \\ g_k(s_k, y_n(s_k)) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f_n(s))ds, & \text{if } t \in J_k := (s_k, t_{k+1}]. \end{cases}\right)$$

If $t \in [0, t_1]$

$$\begin{aligned} \alpha(\{h_n(t)\}) &\leq \frac{2}{\Gamma(r)} \int_0^t \alpha\{(t-s)^{r-1} f_n(s) ds\} \\ &= \frac{2}{\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha\{f_n(s) ds\}. \end{aligned} \tag{3.6}$$

If $t \in J_k$, we have

$$\begin{aligned} \alpha(\{h_n(t)\}) &\leq \alpha\{g_k(s_k, y_n(s_k))\} + \frac{2}{\Gamma(r)} \int_{s_k}^t \alpha\{(t-s)^{r-1} f_n(s)\} ds \\ &\leq c_k(t)\alpha\{y_n(s_k)\} + \frac{2}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} \alpha\{f_n(s)\} ds \\ &\leq c^* \alpha\{y_n(t)\} + \frac{2}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} \alpha\{f_n(s)\} ds. \end{aligned} \tag{3.7}$$

If $t \in J'_k$

$$\begin{aligned} \alpha(\{h_n(t)\}) &= \alpha(g_k(t, y_n(t)) \\ &\leq c_k(t)\alpha\{y_n(t)\} \\ &\leq c^* \alpha\{y_n(t)\}. \end{aligned} \tag{3.8}$$

On the other hand, since M is compact in $\overline{U}$, the sets $\{f_n(s), n \geq 1\}, \{y_n(t), n \geq 1\}$ are compact. Consequently, $\alpha\{f_n(s), n \geq 1\} = 0$ for a.e. $s \in J$ and $\alpha\{y_n(t), n \geq 1\} = 0$ for a.e. $t \in J$. we conclude that $\{h_n(t), n \geq 1\}$ is relatively compact in E , for each $t \in J$. In addition let τ_1 and τ_2 from J with $\tau_1 < \tau_2$. Then, for $\tau_1, \tau_2 \in J_k$, we have

$$\begin{aligned} \|h_n(\tau_2) - |h_n(\tau_1)\| &= \|\frac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} ((\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}) f_n(s)\| ds \\ &\leq \frac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}| p(s) ds, \end{aligned} \tag{3.9}$$

for $\tau_1, \tau_2 \in [0, t_1]$, we have

$$\begin{aligned} \|h_n(\tau_2) - |h_n(\tau_1)\| &= \|\frac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} ((\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}) f_n(s)\| ds, \\ &\leq \frac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}| p(s) ds, \end{aligned} \tag{3.10}$$

and for $\tau_1, \tau_2 \in J'_k$, we have

$$\|h_n(\tau_2) - |h_n(\tau_1)\| = \|g_k(\tau_2, y_n(\tau_2)) - g_k(\tau_1, y_n(\tau_1))\|. \tag{3.11}$$

As $\tau_2 \to \tau_2$, the right hand side of the above inequality tends to zero. This shows that $\{h_n(t), n \geq 1\}$ is equicontinuous. Consequently, $\{h_n, n \geq 1\}$ is relatively compact in $PC(J, E)$.

Step 3: *N has a closed graph.*
Let $(y_n, h_n) \in graph(N), n \geq 1\}$, with $\|y_n - y\|, \|h_n - h\| \to 0$ as $n \to \infty$. We must show that $(y, h) \in graph(N)$. $(y_n, h_n) \in graph(N)$ means that $h_n \in N(y_n)$ which means that there exists $f_n \in S_{F, y_n}$, such that for each $t \in J$,

$$h_n(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f_n(s) ds & \text{if } t \in [0, t_1], \\ g_k(t, y_n(t)), & \text{if } t \in J'_k := (t_k, s_k], \\ g_k(s_k, y_n(s_k)) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f_n(s)) ds, & \text{if } t \in J_k := (s_k, t_{k+1}]. \end{cases}$$

Let

$$a_n(t)=\begin{cases} y_0 & \text{if } t\in[0,t_1],\\ g_k(t,y_n(t)), & \text{if } t\in J'_k:=(t_k,s_k],\\ g_k(s_k,y_n(s_k)), & \text{if } t\in J_k:=(s_k,t_{k+1}].\end{cases}$$

$$a(t)=\begin{cases} y_0 & \text{if } t\in[0,t_1],\\ g_k(t,y(t)), & \text{if } t\in J'_k:=(t_k,s_k],\\ g_k(s_k,y(s_k)), & \text{if } t\in J_k:=(s_k,t_{k+1}].\end{cases}$$

We have $\|a_n-a\|\to 0$ as $n\to\infty$.
Consider the continuous linear operator

$$\Theta: L^\infty(J,E)\longrightarrow C(J,E)$$
$$f\longmapsto \Theta(f)(t)=\begin{cases} \frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}f(s)ds & \text{if } t\in[0,t_1],\\ 0, & \text{if } t\in J'_k:=(t_k,s_k],\\ \frac{1}{\Gamma(r)}\int_{s_k}^t(t-s)^{r-1}f(s))ds, & \text{if } t\in J_k:=(s_k,t_{k+1}].\end{cases}$$

We have

$$\begin{aligned}\|(h_n-a_n)(t)-(h-a)(t)\| &= \|(h_n-h)(t)+(a-a_n)(t)\|.\\ &\le \|(h_n-h)(t)\|+\|(a-a_n)(t)\|,\end{aligned}$$

implies that

$$\|(h_n-a_n)(t)-(h-a)(t)\|\longrightarrow 0 \text{ as } n\to\infty.$$

From Lemma 2.9 it follows that $\Theta\circ S_F$ is a closed graph operator. Moreover, we have

$$(h_n-a_n)(t)\in\Theta(S_{F,y_n})$$

Since $y_n\longrightarrow y$, Lemma 2.9 implies that

$$(h_n-a_n)(t)=\begin{cases} \frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}f(s)ds & \text{if } t\in[0,t_1],\\ 0, & \text{if } t\in J'_k:=(t_k,s_k],\\ \frac{1}{\Gamma(r)}\int_{s_k}^t(t-s)^{r-1}f(s)ds, & \text{if } t\in J_k:=(s_k,t_{k+1}],\end{cases}$$

for some $f\in S_{F,y}$.

Step 4: *M relatively compact in $PC(J,E)$*

Let $M\subset U$, where $M\subset\text{conv}(\{0\}\cup N(M))$ and for some countable set $C\subset M$ let $\overline{M}=\overline{C}$. Taking into account (3.9)-(3.11) , it is easily seen that $N(M)$ is equicontinuous. Therefore, $M\subset\text{conv}(\{0\}\cup N(M))$ implies that M is equicontinuous. It remains to apply the ArzÂťela-Ascoli theorem to show that for each $t\in I$ the set $M(t)$ is relatively compact. By taking into account that C is countable and $C\subset M\subset\text{conv}(0\cup N(M))$, we can find a countable set $H=\{h_n: n\ge 1\}\subset N(M)$ such that $C\subset conv(\{0\}\cup H)$. Then, there are $y_n\in M$ and $f_n\in S_{F,y_n}$ with

$$h_n(t)=\begin{cases} y_0+\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}f_n(s)ds & \text{if } t\in[0,t_1],\\ g_k(t,y_n(t)), & \text{if } t\in J'_k:=(t_k,s_k],\\ g_k(s_k,y_n(s_k))+\frac{1}{\Gamma(r)}\int_{s_k}^t(t-s)^{r-1}f_n(s))ds, & \text{if } t\in J_k:=(s_k,t_{k+1}].\end{cases}$$

Taking into account Theorem 2.7 and the fact that $M \subset \overline{C} \subset \overline{conv}(\{0\} \cup H)$, we obtain

$$\alpha(M(t)) \leq (\alpha(\overline{C}(t)) \leq \alpha(H(t)) = \alpha\{h_n(t) : n \leq 1\}).$$

Using (3.6)-(3.8), we obtain
if $t \in [0, t_1]$

$$\alpha(\{M(t)\}) \leq \frac{2}{\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha\{f_n(s)ds, n \geq 1\},$$

if $t \in J_k$, we have

$$\alpha(\{M(t)\}) \leq c^* \alpha\{y_n(t)\} + \frac{2}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} \alpha\{f_n(s), n \geq 1\} ds$$

,

if $t \in J'_k$

$$\alpha(\{M(t)\}) \leq c^* \alpha\{y_n(t), n \geq 1\}.$$

Also, since $f_n \in S_{F,y_n}$ and $y_n(s) \in M(s)$, then from (H3) we have

$$(t-s)^{r-1} \alpha\{f_n(s)ds, n \geq 1\} = (t-s)^{r-1} p(s) \alpha(M(s)) ds.$$

It follows that
if $t \in [0, t_1]$

$$\begin{aligned} \alpha(\{M(t)\}) &\leq \frac{2p^*}{\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha(M(s)) ds, \\ &\leq \frac{2p^*}{(1-c^*)\Gamma(r)} \int_0^t (t-s)^{r-1} \alpha(M(s)) ds, \end{aligned}$$

if $t \in J_k$, we have

$$\alpha(\{M(t)\}) \leq \frac{2p^*}{(1-c^*)\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} \alpha(M(s)) ds$$

,

if $t \in J'_k$

$$\alpha(\{M(t)\}) \leq c^* \alpha(M(t)) \Rightarrow (1-c^*) \alpha(M(t)) \leq 0.$$

Consequently, if $t \in [0, t_1] \cup J_k$, we have by (H5), that the function Φ given by $\Phi(t) = \alpha(M(t))$ satisfies $\Phi \equiv 0$; that is, $\alpha(M(t)) = 0$ for all $t \in J$. Now, by the ArzÂt'ela-Ascoli theorem, M is relatively compact in $PC(J, E)$.

Step 5: *A priori estimate.*
Let $y \in PC(J, E)$ be such that $y \in \lambda N(y)$ for some $\lambda \in (0, 1)$. Then for each $t \in J$ we have

$$y(t) = \begin{cases} \lambda y_0 + \frac{\lambda}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s) ds & \text{if } t \in [0, t_1], \\ \lambda g_k(t, y(t)), & \text{if } t \in J'_k := (t_k, s_k], \\ \lambda g_k(s_k, y(s_k)) + \frac{\lambda}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f(s) ds, & \text{if } t \in J_k := (s_k, t_{k+1}], \end{cases}$$

for some $f \in S_{F,y}$. On the other hand we have,

$$\begin{aligned}
\|y(t)\| &\leq \|g_k(t,y(t))\| + \|y_0\| + \frac{1}{\Gamma(r)}\int_{s_k}^{t_{k+1}}(t-s)^{r-1}\|f(s)\|ds \\
&\leq c_k(t)\|y(t)\| + \|y_0\| + \frac{1}{\Gamma(r)}\int_{s_k}^{t_{k+1}}(t-s)^{r-1}p(s)ds \\
&\leq c_*\|y(t)\| + \|y_0\| + \frac{p^*T^r}{\Gamma(r+1)} \\
&\leq c^*\|y(t)\| + \|y_0\| + \frac{p^*T^r}{\Gamma(r+1)}.
\end{aligned}$$

Then

$$\|y\| \leq \frac{1}{1-c^*}\left(\|y_0\| + \frac{p^*T^r}{\Gamma(r+1)}\right) := d.$$

Set

$$U = \{y \in PC(J,E) : \|y\| < d+1\}.$$

Condition (2.2) is satisfied by our choice of the open set U. From Theorem 3.4, we conclude that N has at least one fixed point $y \in PC(J,E)$ being a solution of problem (1.1)-(1.3).

4. An Example

Let us consider the following fractional differential inclusions with non instantaneous impulses,

$${}^cD^{\frac{1}{2}}y_n(t) \in \frac{1}{(9+n+e^t)(1+\|y(t)\|)}[y_n(t)-1, y_n(t)], \text{ for each } t \in \left(0,\frac{1}{3}\right] \cup \left(\frac{1}{2},1\right], \tag{4.1}$$

$$y_n(t) = \frac{1}{4+n+e^t}\sin|y_n(t)|, t \in \left(\frac{1}{3},\frac{1}{2}\right], \tag{4.2}$$

$$y_n(0) = 0. \tag{4.3}$$

Set

$$E = l^1 = \{y = (y_1, y_2, \dots, y_n, \dots), \sum_{n=1}^{\infty}|y_n| < \infty\},$$

E is a Banach space with the norm

$$\|y\| = \sum_{n=1}^{\infty}|y_n|.$$

Let

$$F(t,y) = (F_1(t,y), F_2(t,y), \dots, F_n(t,y), \dots),$$

$$F_n(t,y) = \frac{1}{(9+n+e^t)(1+\|y(t)\|)}[y_n(t)-1, y_n(t)],$$

and

$$g_1(t,y) = ({g_1}_1(t,y), {g_1}_2(t,y), \dots, {g_1}_n(t,y), \dots),$$

$${g_1}_n(t,y) = \frac{\sin|y_n(t)|}{4+n+e^t}.$$

Clearly F is closed and convex valued. For each $y \in E$ and $t \in [0,1]$, we have

$$\|F(t,y)\|_{\mathcal{P}} \leq \frac{1}{9+e^t}.$$

Hence, the hypothesis (H2) is satisfied with $p(t) = \frac{1}{9+e^t}$ and $c_1(t) = \frac{1}{4+e^t}$.

Since all conditions of Theorem 3.4 are satisfied with $p^* = \frac{1}{10}$ and $c^* = \frac{1}{5}$, problem (4.1)-(4.3) has at least one solution.

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