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On the Fibonacci and Lucas numbers, their sums and permanents of one type of Hessenberg matrices

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Abstract

At this paper, we derive some relationships between permanents of one type of lower-Hessenberg matrix family and the Fibonacci and Lucas numbers and their sums.

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1. Introduction

The well-known Fibonacci and Lucas sequences are recursively defined by

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1},\ n \geq 1 \\ L_{n+1} &= L_n + L_{n-1},\ n \geq 1 \end{aligned}$$

with initial conditions $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$. The first few values of the sequences are given below:

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34
L_n	2	1	3	4	7	11	18	29	47	76

The permanent of a matrix is similar to the determinant but all of the signs used in the Laplace expansion of minors are positive. The permanent of an n-square matrix is defined by

$$perA = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_i \sigma(i)$$

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where the summation extends over all permutations σ of the symmetric group S_n [1].

Let $A = [a_{ij}]$ be an $m \times n$ matrix with row vectors $r_1, r_2, \ldots, r_m$. We call A is *contractible* on column k, if column k contains exactly two non zero elements. Suppose that A is contractible on column k with $a_{ik} \neq 0, a_{jk} \neq 0$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the *contraction* of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0, a_{kj} \neq 0$ and $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij:k}]^T$ is called the contraction of A on row k relative to columns i and j. We know that if B is a contraction of A[6], then

$$perA = perB. \tag{1.1}$$

It is known that there are a lot of relationships between determinants or permanents of matrices and well-known number sequences. For example, the authors [2] investigate relationships between permanents of one type of Hessenberg matrix and the Pell and Perrin numbers.

In [3], Lee defined a $(0-1)-$ matrix whose permanents are Lucas numbers.

In [4], the author investigate general tridiagonal matrix determinants and permanents. Also he showed that the permanent of the tridiagonal matrix based on $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ is equal to the determinant of the matrix based on $\{-a_i\}$, $\{b_i\}$, $\{c_i\}$.

In [5], the authors give $(0, 1, -1)$ tridiagonal matrices whose determinants and permanents are negatively subscripted Fibonacci and Lucas numbers. Also, they give an $n \times n$ $(-1, 1)$ matrix S, such that per$A$=det$(A \circ S)$, where $A \circ S$ denotes Hadamard product of A and S.

In the present paper, we consider a particular case of lower Hessenberg matrices. We show that the permanents of this type of matrices are related with Fibonacci and Lucas numbers and their sums.

2. Determinantal representation of Fibonacci and Lucas numbers and their sums

Let $H_n = [h_{ij}]_{n \times n}$ be an n-square lower Hessenberg matrix as below:

$$H_n = [h_{ij}]_{n \times n} = \begin{cases} 2, \text{ if } i = j, \text{ for } i, j = 1, 2, \ldots, n-1 \\ 1, \text{ if } j = i-2 \text{ and } i = j = n \\ (-1)^i, \text{ if } j = i+1 \\ 0, \text{ otherwise} \end{cases} \tag{2.1}$$

Then we have the following theorem.

2.1. Theorem. *Let H_n be as in (2.1), then*

$$perH_n = perH_n^{(n-2)} = F_{n+1}$$

where F_n is the nth Fibonacci number.

Proof. By definition of the matrix H_n, it can be contracted on column n. Let $H_n^{(r)}$ be the rth contraction of H_n. If $r = 1$, then

$$H_n^{(1)} = \begin{pmatrix} 2 & -1 & & & & & \\ 0 & 2 & 1 & & & & \\ 1 & 0 & 2 & -1 & & & \\ & 1 & 0 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 2 & (-1)^{n-2} \\ & & & & 1 & (-1)^{n-1} & 2 \end{pmatrix}.$$

Since $H_n^{(1)}$ also can be contracted according to the last column,

$$H_n^{(2)} = \begin{pmatrix} 2 & -1 & & & & & \\ 0 & 2 & 1 & & & & \\ 1 & 0 & 2 & -1 & & & \\ & 1 & 0 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 2 & (-1)^{n-3} \\ & & & & 2 & (-1)^{n-2} & 3 \end{pmatrix}.$$

Continuing this method, we obtain the rth contraction

$$H_n^{(r)} = \begin{pmatrix} 2 & -1 & & & & & \\ 0 & 2 & 1 & & & & \\ 1 & 0 & 2 & -1 & & & \\ & 1 & 0 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 2 & (-1)^{r-1} \\ & & & & F_{r+1} & (-1)^r(F_{r+2}-F_{r+1}) & F_{r+2} \end{pmatrix}, \quad n \text{ is even}$$

$$H_n^{(r)} = \begin{pmatrix} 2 & -1 & & & & & \\ 0 & 2 & 1 & & & & \\ 1 & 0 & 2 & -1 & & & \\ & 1 & 0 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 2 & (-1)^{r} \\ & & & & F_{r+1} & (-1)^{r-1}(F_{r+2}-F_{r+1}) & F_{r+2} \end{pmatrix}, \quad n \text{ is odd}$$

where $2 \le r \le n-4$. Hence

$$H_n^{(n-3)} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ F_{n-2} & (F_{n-2}-F_{n-1}) & F_{n-1} \end{pmatrix}$$

by contraction of $H_n^{(n-3)}$ on column 3,

$$H_n^{(n-2)} = \begin{pmatrix} 2 & -1 \\ F_{n-2} & F_n \end{pmatrix}.$$

By (1.1), we have $per H_n = per H_n^{(n-2)} = F_{n+1}$. □

Let $K_n = [k_{ij}]_{n\times n}$ be an n-square lower Hessenberg matrix in which the superdiagonal entries are alternating -1s and 1s starting with 1, except the first one which is -3, the main diagonal entries are 2s, except the last one which is 1, the subdiagonal entries are 0s, the lower-subdiagonal entries are 1s and otherwise 0. Clearly:

$$(2.2)\qquad K_n = \begin{pmatrix} 2 & -3 & & & & & & \\ 0 & 2 & 1 & & & & & \\ 1 & 0 & 2 & -1 & & & & \\ & 1 & 0 & 2 & 1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & 0 & 2 & (-1)^{n-1} \\ & & & & 1 & 0 & 1 \end{pmatrix}$$

2.2. Theorem. *Let K_n be as in (2.2), then*

$$perK_n = perK_n^{(n-2)} = L_{n-2}$$

where L_n is the nth Lucas number.

Proof. By definition of the matrix K_n, it can be contracted on column n. By consecutive contraction steps, we can write down,

$$K_n^{(r)} = \begin{pmatrix} 2 & -3 & & & & & \\ 0 & 2 & 1 & & & & \\ 1 & 0 & 2 & -1 & & & \\ & 1 & 0 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 2 & (-1)^{r-1} \\ & & & & F_{r+1} & (-1)^{r-2}(F_{r+2}-F_{r+1}) & F_{r+2} \end{pmatrix}, \ n \text{ is even}$$

$$K_n^{(r)} = \begin{pmatrix} 2 & -3 & & & & & \\ 0 & 2 & 1 & & & & \\ 1 & 0 & 2 & -1 & & & \\ & 1 & 0 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 2 & (-1)^{r} \\ & & & & F_{r+1} & (-1)^{r-1}(F_{r+2}-F_{r+1}) & F_{r+2} \end{pmatrix}, \ n \text{ is odd}$$

for $1 \leq r \leq n-4$. Hence

$$K_n^{(n-3)} = \begin{pmatrix} 2 & -3 & 0 \\ 0 & 2 & 1 \\ F_{n-2} & F_{n-2}-F_{n-1} & F_{n-1} \end{pmatrix}$$

by contraction of $K_n^{(n-3)}$ on column 3, gives

$$K_n^{(n-2)} = \begin{pmatrix} 2 & -3 \\ F_{n-2} & F_n \end{pmatrix}.$$

By applying (1.1), we have $perK_n = perK_n^{(n-2)} = 2F_n - 3F_{n-2} = L_{n-2}$, which is desired. □

Let $M_n = [m_{ij}]_{n\times n}$ be an n-square lower Hessenberg matrix as below:

$$(2.3)\qquad M_n = [m_{ij}]_{n\times n} = \begin{cases} 2, \text{ if } i=j, \text{ for } i,j=1,2,\ldots,n \\ 1, \text{ if } j=i-2 \\ (-1)^i, \text{ if } j=i+1 \\ 0, \text{ otherwise} \end{cases}$$

2.3. Theorem. *Let M_n be as in (2.3), then*

$$perM_n = perM_n^{(n-2)} = \sum_{i=0}^{n+1} F_i = F_{n+3} - 1$$

where F_n is the nth Fibonacci number.

Proof. By contraction method on column n, we have

$$M_n^{(r)} = \begin{pmatrix} 2 & -1 & & & & & \\ 0 & 2 & 1 & & & & \\ 1 & 0 & 2 & -1 & & & \\ & 1 & 0 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 2 & (-1)^r \\ & & & & \sum_{i=0}^{r+1} F_i & (-1)^{r-1}\sum_{i=0}^{r} F_i & \sum_{i=0}^{r+2} F_i \end{pmatrix}, \ n \text{ is odd}$$

$$M_n^{(r)} = \begin{pmatrix} 2 & -1 & & & & & \\ 0 & 2 & 1 & & & & \\ 1 & 0 & 2 & -1 & & & \\ & 1 & 0 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 2 & (-1)^{r-1} \\ & & & & \sum_{i=0}^{r+1} F_i & (-1)^{r-2}\sum_{i=0}^{r} F_i & \sum_{i=0}^{r+2} F_i \end{pmatrix}, \ n \text{ is even}$$

for $1 \le r \le n-4$. Hence

$$M_n^{(n-3)} = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ \sum_{i=0}^{n-2} F_i & -\sum_{i=0}^{n-3} F_i & \sum_{i=0}^{n-1} F_i \end{pmatrix}$$

by contraction of $M_n^{(n-3)}$ on column 3, gives

$$M_n^{(n-2)} = \begin{pmatrix} 2 & -1 \\ \sum_{i=0}^{n-2} F_i & \sum_{i=0}^{n} F_i \end{pmatrix}.$$

By applying (1.1), we have

$$perM_n = perM_n^{(n-2)} = \sum_{i=0}^{n+1} F_i = F_{n+3} - 1$$

which is desired. □

Let $N_n = [n_{ij}]_{n\times n}$ be an n-square lower Hessenberg matrix in which the superdiagonal entries are alternating -1s and 1s starting with 1, except the first one which is -2, the main diagonal entries are 2s, except the first one is 3, the subdiagonal entries are 0s, the lower-subdiagonal entries are 1s and otherwise 0. In this content:

$$
(2.4) \qquad N_n = \begin{pmatrix}
3 & -2 & & & & & \\
0 & 2 & 1 & & & & \\
1 & 0 & 2 & -1 & & & \\
 & 1 & 0 & 2 & 1 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \\
 & & & 1 & 0 & 2 & (-1)^{n-1} \\
 & & & & 1 & 0 & 2
\end{pmatrix}
$$

2.4. Theorem. *Let N_n be an n-square matrix $(n \geq 2)$ as in (2.4), then*

$$
perN_n = perN_n^{(n-2)} = \sum_{i=0}^{n} L_i = L_{n+2} - 1
$$

where L_n is the nth Lucas number.

Proof. By contraction method on column n, we have

$$
N_n^{(r)} = \begin{pmatrix}
3 & -2 & & & & & \\
0 & 2 & 1 & & & & \\
1 & 0 & 2 & -1 & & & \\
 & 1 & 0 & 2 & 1 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \\
 & & & 1 & 0 & 2 & (-1)^{r} \\
 & & & & \sum_{i=0}^{r+1} F_i & (-1)^{r-1} \sum_{i=0}^{r} F_i & \sum_{i=0}^{r+2} F_i
\end{pmatrix}, \ n \text{ is odd}
$$

$$
N_n^{(r)} = \begin{pmatrix}
3 & -2 & & & & & \\
0 & 2 & 1 & & & & \\
1 & 0 & 2 & -1 & & & \\
 & 1 & 0 & 2 & 1 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \\
 & & & 1 & 0 & 2 & (-1)^{r-1} \\
 & & & & \sum_{i=0}^{r+1} F_i & (-1)^{r} \sum_{i=0}^{r} F_i & \sum_{i=0}^{r+2} F_i
\end{pmatrix}, \ n \text{ is even}
$$

for $1 \leq r \leq n-4$. Hence

$$
N_n^{(n-3)} = \begin{pmatrix}
3 & -2 & 0 \\
0 & 2 & 1 \\
\sum_{i=0}^{n-2} F_i & -\sum_{i=0}^{n-3} F_i & \sum_{i=0}^{n-1} F_i
\end{pmatrix}
$$

by contraction of $N_n^{(n-3)}$ on column 3, gives

$$
N_n^{(n-2)} = \begin{pmatrix}
3 & -2 \\
\sum_{i=0}^{n-2} F_i & \sum_{i=0}^{n} F_i
\end{pmatrix}.
$$

By applying (1.1), we have

$$
perN_n = perN_n^{(n-2)} = \sum_{i=0}^{n} L_i = L_{n+2} - 1
$$

by the identity $F_{n-1} + F_{n+1} = L_n$. $\square$

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