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Original Article

Certain Relations of Gegenbauer and Modified Gegenbauer Matrix Polynomials by Lie Algebraic Method

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Abstaract — The object of the present paper is to derive the generating formulae for the Gegenbauer and modified Gegenbauer matrix polynomials by introducing a partial differential operator and constructing the Lie algebra representation formalism of special linear algebra by using Weisner's group-theoretic approach. Application of our results is also pointed out.

Keywords — *Gegenbauer matrix polynomials; Generating matrix functions; Matrix differential equations; Differential operator; Group-theoretic method.*

1 Introduction

The study of special matrix polynomials is an important due to their applications in certain areas of statistics, physics, engineering, Lie group theory and number theory. Group theoretic methods have played an important role in the modern theory of special functions. Lie algebraic methods for computing eigenvalues and recurrence relations have been developed and the methods developed in the present paper provide a more flexible and direct treatment than the standard Lie algebraic treatment used recently in [1, 5, 14, 15, 21, 23, 24, 32, 34]. The reason of interest for this family of Gegenbauer matrix polynomials (GMPs) and their associated operational formalism is due to their intrinsic mathematical importance and the fact that these polynomials have important applications in physics. Motivated and inspired by the work of Jódar et. al. and his co-authors on Gegenbauer matrix polynomials, see for example [2, 3, 4, 6, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 22, 25, 26, 33] and due to make use of the Lie group-theoretic method (see [1, 23, 24, 27, 28, 29, 30, 31]). In this paper, we introduce the differential operators for 2-variable Gegenbauer and modified Gegenbauer matrix polynomials (MGMPs) and derive their many new and known generating matrix relations by using Lie algebraic techniques.

1.1 Preliminaries

For the sake of clarity in the presentation we recall some generating matrix relations for the Gegenbauer matrix polynomials and some notations which will be used throughout the next section. Throughout this paper, we assume that A is a positive stable matrix in $\mathbb{C}^{N\times N}$; that is, the matrix A satisfies the following condition

$$Re(\mu) > 0 \text{ for all } \mu \in \sigma(A),\ \sigma(A) := \textit{ spectrum of } A. \tag{1}$$

Definition 1.1. (Jódar et al. [16]) Let A be a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition

$$(-\frac{z}{2}) \notin \sigma(A) \text{ for all } z \in \mathbb{Z}^+ \cup \{0\}. \tag{2}$$

The Gegenbauer matrix polynomials (GMPs) are defined by

$$C_n^A(x) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}(A)_{n-k}, \tag{3}$$

and the generating matrix functions

$$F(x,t,A) = (1-2xt+t^2)^{-A} = \sum_{n=0}^{\infty} C_n^A(x)t^n. \tag{4}$$

If r_1 and r_2 are the roots of the quadratic equation $1 - 2xt + yt^2 = 0$ and r is the minimum of the set $\{r_1, r_2\}$, then the matrix function $F(x,t,A)$ regarded as a function of t, is analytic in the disk $|t| < r$ for every real number in $|x| \leq 1$.

We recall that the Gegenbauer's matrix polynomials (GMPs) satisfy the pure and differential matrix recurrence relations by each element of this set [22]:

$$nC_n^A(x) = 2x(A+(n-1)I)C_{n-1}^A(x) - (2A+(n-2)I)C_{n-2}^A(x); n \geq 2, \tag{5}$$

where I is the identity matrix in $\mathbb{C}^{N\times N}$, and

$$(1-x^2)\frac{d}{dx}C_n^A(x) = (2A+(n-1)I)C_{n-1}^A(x) - nxC_n^A(x). \tag{6}$$

From (5) and (6), we obtain the matrix differential recurrence relations:

$$(1-x^2)\frac{d}{dx}C_n^A(x) = (2A+nI)xC_n^A(x) - (n+1)C_{n+1}^A(x). \tag{7}$$

Gegenbauer matrix polynomials $C_n^A(x)$ is a solution of the following matrix differential equation:

$$(1-x^2)\frac{d^2}{dx^2}C_n^A(x) - x(2A+I)\frac{d}{dx}C_n^A(x) + n(2A+nI)C_n^A(x) = \mathbf{0}, n \geq 0, \tag{8}$$

where $\mathbf{0}$ is the null matrix in $\mathbb{C}^{N\times N}$.

Theorem 1.2. [7] Let A, B and C are matrices in $\mathbb{C}^{N\times N}$ such that $C + nI$ is an invertible matrix for all integers $n \geq 0$. Suppose that C, $C - A$ and $C - B$ are positive stable matrices with $BC = CB$, the relation

$${}_2F_1\Big(A, B; C; z\Big) = (1-z)^{C-A-B}\ {}_2F_1\Big(C-A, C-B; C; z\Big) \tag{9}$$

is valid for $|z| < 1$.

2 Group-theoretic Method for Gegenbauer Matrix Polynomials

From (8) we construct a partial differential equation, replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $y\frac{\partial}{\partial y}$, and $C_n^A(x)$ by $C_n^A(x,y)$:

$$(1-x^2)\frac{\partial^2}{\partial x^2}C_n^A(x,y)-(2A+I)x\frac{\partial}{\partial x}C_n^A(x,y)+y\frac{\partial}{\partial y}(2A+y\frac{\partial}{\partial y}I)C_n^A(x,y)=\mathbf{0}. \quad (10)$$

Therefore $C_n^A(x,y)=C_n^A(x)y^n$ is a solution of the matrix partial differential equation Eq. (10), since $C_n^A(x)$ is a solution of matrix differential equation Eq. (8). We may rewrite (10) in the following form:

$$\begin{aligned}&(1-x^2)\frac{\partial^2}{\partial\, x^2}C_n^A(x,y)+y^2\frac{\partial^2}{\partial\, y^2}C_n^A(x,y)-(2A+I)x\frac{\partial}{\partial x}C_n^A(x,y)\\&+(2A+I)y\frac{\partial}{\partial y}C_n^A(x,y)=\mathbf{0}.\end{aligned}$$

Let $\mathbb{L}$ represent the differential operators of (10), i.e.,

$$\mathbb{L}=(1-x^2)\frac{\partial^2}{\partial x^2}I+y^2\frac{\partial^2}{\partial y^2}I-(2A+I)x\frac{\partial}{\partial x}+(2A+I)y\frac{\partial}{\partial y}.$$

Next, using the matrix recurrence relations (6) and (7), we determine the first-order linear partial differential operators with the aid of $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ the differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ such that

$$\mathbb{B}\Big[C_n^A(x)y^n\Big]=-(2A+(n-1)I)C_{n-1}^A(x)y^{n-1},$$

and

$$\mathbb{C}\Big[C_n^A(x)y^n\Big]=(n+1)C_{n+1}^A(x)y^{n+1},$$

where

$$\mathbb{A}=y\frac{\partial}{\partial y}I,$$

$$\mathbb{B}=\frac{x^2-1}{y}\frac{\partial}{\partial x}I-x\frac{\partial}{\partial y}I,$$

and

$$\mathbb{C}=(x^2-1)y\frac{\partial}{\partial x}I+xy^2\frac{\partial}{\partial y}I+2xyA,$$

where the linear differential operators $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ satisfy the following commutation relations

$$[\mathbb{A},\mathbb{B}]=-\mathbb{B},\quad [\mathbb{A},\mathbb{C}]=\mathbb{C},\quad [\mathbb{B},\mathbb{C}]=-2\mathbb{A}-2A\mathbb{I},$$

where the commutator notation is defined as $[\mathbb{A}, \mathbb{B}] = \mathbb{A}\mathbb{B} - \mathbb{B}\mathbb{A}$. Therefore, we will show that these differential operators generate a three-parameter Lie group.

The second order differential operator $\mathbb{L}$ satisfies the differential operator identity

$$(1 - x^2)\mathbb{L} = \mathbb{B}\mathbb{C} + \mathbb{A}^2 + (2A - I)\mathbb{A}.$$

By means of this identity and the commutator relations we prove that $(1 - x^2)\mathbb{L}$ commutes with each of the differential operators $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$,

$$[(1 - x^2)\mathbb{L}, \mathbb{A}] = [(1 - x^2)\mathbb{L}, \mathbb{B}] = [(1 - x^2)\mathbb{L}, \mathbb{C}] = \mathbf{0}.$$

Then for arbitrary constants b and c the differential operator $e^{c\mathbb{C}}e^{b\mathbb{B}}$ will transform solutions of $\mathbb{L}$ into solutions of $\mathbb{L}$; in other words,

$$e^{c\mathbb{C}}e^{b\mathbb{B}}(1 - x^2)\mathbb{L}C_n^A(x, y) = (1 - x^2)\mathbb{L}\Big(e^{c\mathbb{C}}e^{b\mathbb{B}}C_n^A(x, y)\Big) = \mathbf{0}.$$

if and only if $\mathbb{L}C_n^A(x, y) = \mathbf{0}$.

To accomplish our task of obtaining the generating matrix relations, we search for matrix function $f(x, y, A)$ and extended forms of transformation groups generated by differential operators $\mathbb{B}$ and $\mathbb{C}$ expressed as follows:

$$e^{b\mathbb{B}}f(x, y, A) = f\left(\frac{xy - b}{\sqrt{y^2 - 2bxy + b^2}}, \sqrt{y^2 - 2bxy + b^2}, A\right),$$

and

$$e^{c\mathbb{C}}f(x, y, A) = \Big(c^2y^2 - 2cxy + 1\Big)^{-A} f\left(\frac{x - cy}{\sqrt{c^2y^2 - 2cxy + 1}}, \frac{y}{\sqrt{c^2y^2 - 2cxy + 1}}, A\right),$$

where b, c are arbitrary constants and $f(x, y, A)$ is an arbitrary matrix function. We know that $\mathbb{B}$ and $\mathbb{C}$ commute operators and we find

$$e^{c\mathbb{C}}e^{b\mathbb{B}}[C_n^A(x)y^n] = \Big(c^2y^2 - 2cxy + 1\Big)^{-A} C_n^A(\xi)\eta^n, \tag{11}$$

where

$$\xi = \frac{(1 + 2bc)xy - c(1 + bc)y^2 - b}{\sqrt{c^2y^2 - 2cxy + 1}\sqrt{(1 + bc)^2y^2 - 2b(1 + bc)xy + b^2}},$$

and

$$\eta = \frac{\sqrt{(1 + bc)^2y^2 - 2b(1 + bc)xy + b^2}}{\sqrt{c^2y^2 - 2cxy + 1}}.$$

2.1 Generating Matrix Functions for Gegenbauer Matrix Polynomials

In this subsection, some special cases of the generating matrix functions for Gegenbauer matrix polynomials are derived from the differential operator $(\mathbb{A} - A\mathbb{I})$.

If we choose $b = 1$, $c = 0$ and $C_n^A(x,y) = C_n^A(x)y^n$ in (11), we find

$$e^{\mathbb{B}}[C_n^A(x)y^n] = \Big(y^2 - 2xy + 1\Big)^{\frac{1}{2}n} C_n^A\Big(\frac{xy-1}{\sqrt{y^2-2xy+1}}\Big).$$

By expanding this Ggegenbauer matrix polynomials, we get

$$\Big(y^2 - 2xy + 1\Big)^{\frac{1}{2}n} C_n^A\Big(\frac{xy-1}{\sqrt{y^2-2xy+1}}\Big) = \sum_{k=0}^{n} \frac{((1-n)I-2A)_k}{k!} C_{n-k}^A(x)y^{n-k}.$$

If we divide by y^n and let $t = \frac{1}{y}$, we get

$$\Big(1 - 2xt + t^2\Big)^{\frac{1}{2}n} C_n^A\Big(\frac{x-t}{\sqrt{1-2xt+t^2}}\Big) = \sum_{k=0}^{n} \frac{((1-n)I-2A)_k}{k!} C_{n-k}^A(x)t^{k}. \tag{12}$$

Secondly, if we choose $b = 0$ and $c = 1$, we get

$$e^{\mathbb{C}}[C_n^A(x)y^n] = y^n\Big(y^2 - 2xy + 1\Big)^{-A-\frac{1}{2}nI} C_n^A\Big(\frac{x-y}{\sqrt{y^2-2xy+1}}\Big).$$

If we expand this generating matrix function for Gegenbauer matrix polynomials and divide by y^n, we get the generating matrix relation

$$\Big(y^2 - 2xy + 1\Big)^{-A-\frac{1}{2}nI} C_n^A\Big(\frac{x-y}{\sqrt{y^2-2xy+1}}\Big) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!n!} C_{n+k}^A(x)y^{k}. \tag{13}$$

Thirdly, for $bc \neq 0$ we choose $b = -1$ and $c = 1$, (this choice is suggested by the frequency of occurrence in (11) of the factor $1 + bc$), we have

$$e^{c\mathbb{C}}e^{b\mathbb{B}}[C_n^A(x)y^n] = \Big(c^2y^2 - 2cxy + 1\Big)^{-A} C_n^A(\xi)\eta^n,$$

where $\xi = \frac{1-xy}{\sqrt{1-2xy+y^2}}$ and $\eta = \frac{1}{\sqrt{1-2xy+y^2}}$.

We expand this generating matrix function for Gegenbauer matrix polynomials as follows:

$$\Big(1 - 2xy + y^2\Big)^{-A-\frac{1}{2}nI} C_n^A\Big(\frac{1-xy}{\sqrt{1-2xy+y^2}}\Big) = \sum_{k=0}^{\infty} \frac{(2A+kI)_n}{k!} C_k^A(x)y^{k}. \tag{14}$$

If we let $\rho = \sqrt{1-2xy+y^2}$ we can rewrite (14) in the form

$$\rho^{-2A-nI} C_n^A\Big(\frac{1-xy}{\rho}\Big) = \sum_{k=0}^{\infty} \frac{(2A+kI)_n}{k!} C_k^A(x)y^{k}.$$

In order to express the left member of (14) in hypergeometric matrix functions form we use

$$C_n^A(x) = \frac{x^n}{n!}(2A)_n\, {}_2F_1\Big(-\frac{1}{2}nI, \frac{1}{2}(1-n)I; A + \frac{1}{2}nI; \frac{x^2-1}{x^2}\Big), \Big|\frac{x^2-1}{x^2}\Big| < 1.$$

Then after some simplification, Eq. (14) yields

$$\begin{aligned}&(1-2xy+y^2)^{-A-nI}(1-xy)^n\ {}_2F_1\Big(-\frac{1}{2}nI,\frac{1}{2}(1-n)I;A+\frac{1}{2}nI;\frac{y^2(x^2-1)}{(1-xy)^2}\Big)\\&=\sum_{k=0}^{\infty}(2A+nI)_k[(2A)_k]^{-1}C_k^A(x)y^k,\ \Big|\frac{y^2(x^2-1)}{(1-xy)^2}\Big|<1,\Big|xy\Big|<1.\end{aligned}\tag{15}$$

By applying the Theorem 1.1 and letting $B=2A+nI$, in the left member of (15), we obtain

$$\begin{aligned}&(1-xy)^{-B}\ {}_2F_1\Big(\frac{1}{2}B,\frac{1}{2}(B+I);A+\frac{1}{2}nI;\frac{y^2(x^2-1)}{(1-xy)^2}\Big)\\&=\sum_{k=0}^{\infty}(B)_k[(2A)_k]^{-1}C_k^A(x)y^k,\ \Big|\frac{y^2(x^2-1)}{(1-xy)^2}\Big|<1.\end{aligned}\tag{16}$$

2.2 Generating Matrix Functions Annulled by not Conjugate of $(\mathbb{A}-A\mathbb{I})$

In this subsection, the generating matrix functions for Gegenbauer matrix polynomials are derived from the differential operators not conjugate to $(\mathbb{A}-A\mathbb{I})$. The three generating matrix functions of (12), (13), and (14) have been obtained by transforming $C_n^A(x)y^n$ which is a solution of the system

$$\mathbb{L}C_n^A(x,y)=\mathbf{0}\quad\text{and}\quad(\mathbb{A}-n\mathbb{I})C_n^A(x,y)=\mathbf{0}.$$

If we wish to obtain additional generating matrix functions for the Gegenbauer matrix polynomials, we need to find differential operators which are not conjugate to $(\mathbb{A}-n\mathbb{I})$; i.e., we wish to find first order differential operators R such that for all choices of b and c;

$$e^{c\mathbb{C}}e^{b\mathbb{B}}(\mathbb{A}-n\mathbb{I})e^{-b\mathbb{B}}e^{-c\mathbb{C}}\neq R.$$

We take the set of linear differential operators $R=r_1\mathbb{A}+r_2\mathbb{B}+r_3\mathbb{C}+r_4\mathbb{I}$, for all combinations of zero and nonzero coefficients except for $r_1=r_2=r_3=0$. We find that

$$e^{c\mathbb{C}}e^{b\mathbb{B}}(\mathbb{A}-n\mathbb{I})e^{-b\mathbb{B}}e^{-c\mathbb{C}}=(1+2bc)\mathbb{A}+b\mathbb{B}-c(1+bc)\mathbb{C}+(2bcA-nI)\mathbb{I}.$$

Then for $r_1=1+2bc$, $r_2=b$, $r_3=c(1+bc)$, we have $r^2+4r_2r_3=1$.

Therefore, $\mathbb{A}-n\mathbb{I}$ is not conjugate to differential operators for which $r_1^2+4r_2r_3=0$ in the following cases:

If $r_1=0$, $r_2=1$, and $r_3=0$, we seek a solution of the system

$$\mathbb{L}u(x,y,A)=\mathbf{0}\quad\text{and}\quad(\mathbb{B}+\mathbb{I})u(x,y,A)=\mathbf{0}.$$

A solution of this system is

$$u(x,y,A)=e^{xy}\ {}_0F_1\Big(-;A+\frac{1}{2}I;\frac{y^2(x^2-1)}{4}\Big).$$

If we expand this matrix function, we get

$$e^{xy}\,{}_0F_1\left(-;A+\frac{1}{2}I;\frac{y^2(x^2-1)}{4}\right)=\sum_{k=0}^{\infty}[(2A)_k]^{-1}C_k^A(x)y^k. \tag{17}$$

For $r_1=0$, $r_2=0$, and $r_3\neq 0$, we seek a solution of the system

$$\mathbb{L}u=\mathbf{0}\quad\text{and}\quad(\mathbb{C}+\lambda\mathbb{I})u=\mathbf{0},$$

where λ is an arbitrary constant. We may avoid actually solving this system by noting that

$$e^{b\mathbb{B}}e^{c\mathbb{C}}(\mathbb{B}+\mathbb{I})e^{-c\mathbb{C}}e^{-b\mathbb{B}}=2c(1+bc)\mathbb{A}+(1+bc)^2\mathbb{I}-c^2\mathbb{C}+2c(1+bc)A\mathbb{I}+\mathbb{I}.$$

If we choose $b=1$ and $c=-1$, we get

$$e^{\mathbb{B}}e^{-\mathbb{C}}(\mathbb{B}+I)e^{\mathbb{C}}e^{-\mathbb{B}}=-\mathbb{C}+\mathbb{I}.$$

Therefore, we can obtain a solution of the system $\mathbb{L}u=\mathbf{0}$ and $(\mathbb{C}-\mathbb{I})u=\mathbf{0}$, by transforming the generating matrix functions of (17) as follows:

$$e^{B}e^{-C}e^{xy}\,{}_0F_1\left(-;A+\frac{1}{2}I;\frac{y^2(x^2-1)}{4}\right)=y^{-2A}\exp\left(\frac{y-x}{y}\right)\,{}_0F_1\left(-;A+\frac{1}{2}I;\frac{x^2-1}{4y^2}\right).$$

If we let $t=-\frac{1}{y}$, we get

$$e(-t)^{2A}e^{xt}\,{}_0F_1\left(-;A+\frac{1}{2}I;\frac{t^2(x^2-1)}{4}\right)$$

as our generating matrix function. But this matrix function differs only trivially from (17).

As applications, we now obtained many new and known generating matrix relations for the Gegenbauer matrix polynomials in the following:

$$\rho^n C_n^A\left(\frac{x-y}{\rho}\right)=\sum_{k=0}^{n}\frac{((1-n)I-2A)_k}{k!}C_{n-k}^A(x)y^k, \tag{18}$$

$$\rho^{-2A-nI}C_n^A\left(\frac{x-y}{\rho}\right)=\sum_{k=0}^{\infty}\frac{(k+n)!}{k!n!}C_{n+k}^A(x)y^k, \tag{19}$$

which is given in [8].

$$\rho^{-2A-nI}C_n^A\left(\frac{1-xy}{\rho}\right)=\sum_{k=0}^{\infty}\frac{(2A+kI)_n}{n!}C_k^A(x)y^k, \tag{20}$$

and

$$e^{xy}\,{}_0F_1\left(-;A+\frac{1}{2}I;\frac{y^2(x^2-1)}{4}\right)=\sum_{k=0}^{\infty}[(2A)_k]^{-1}C_k^A(x)y^k. \tag{21}$$

3 Group-theoretic Method for Modified Gegenbauer Matrix Polynomials

Here, we consider the modified Gegenbauer matrix polynomials $C_n^{A+nI}(x)$ which satisfy the following matrix differential equation:

$$\begin{aligned}&(1-x^2)\frac{d^2}{dx^2}C_n^{A+nI}(x)-x(2A+(2n+1)I)\frac{d}{dx}C_n^{A+nI}(x)\\&+n(2A+3nI)C_n^{A+nI}(x)=\mathbf{0},n\geq 0.\end{aligned} \tag{22}$$

By using the following differential matrix recurrence relations

$$(1-x^2)\frac{d}{dx}C_n^{A+nI}(x)=(2A+(3n-1)I)C_{n-1}^{A+nI}(x)+nxC_n^{A+nI}(x), \tag{23}$$

and

$$(1-x^2)\frac{d}{dx}C_n^{A+nI}(x)=(2A+3nI)xC_n^{A+nI}(x)-(n+1)C_{n+1}^{A+nI}(x). \tag{24}$$

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $y\frac{\partial}{\partial y}$, and $C_n^{A+nI}(x)$ by $C_n^{A+nI}(x,y)$ in (22) we obtain the following a matrix partial differential equation:

$$\begin{aligned}&(1-x^2)\frac{\partial^2}{\partial x^2}C_n^{A+nI}(x,y)-(2A+(2n+1)I)x\frac{\partial}{\partial x}C_n^{A+nI}(x,y)\\&+y\frac{\partial}{\partial y}(2A+3y\frac{\partial}{\partial y}I)C_n^{A+nI}(x,y)=\mathbf{0}.\end{aligned} \tag{25}$$

Thus we see that $C_n^{A+nI}(x,y) = C_n^{A+nI}(x)y^n$ is a solution of the matrix partial differential equation Eq. (3.3), since $C_n^{A+nI}(x)$ is a solution of the matrix differential equation Eq. (22). We can rewrite (24) in the following form:

$$\begin{aligned}&(1-x^2)\frac{\partial^2}{\partial\, x^2}C_n^{A+nI}(x,y)+3y^2\frac{\partial^2}{\partial\, y^2}C_n^{A+nI}(x,y)-(2A+I)x\frac{\partial}{\partial x}C_n^{A+nI}(x,y)\\&-2xy\frac{\partial}{\partial y\partial x}C_n^{A+nI}(x,y)+(2A+3I)y\frac{\partial}{\partial y}C_n^{A+nI}(x,y)=\mathbf{0}.\end{aligned} \tag{26}$$

We define the differential operators $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ as follows

$$\mathbb{A}=y\frac{\partial}{\partial y}I,$$

$$\mathbb{B}=\frac{x^2-1}{y}\frac{\partial}{\partial x}I-x\frac{\partial}{\partial y}I,$$

and

$$\mathbb{C}=(x^2-1)y\frac{\partial}{\partial x}I+3xy^2\frac{\partial}{\partial y}I+2xyA.$$

Next, we determine the following partial differential operators with the aid of $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ the differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ such that

$$\mathbb{A}\Big[C_n^{A+nI}(x)y^n\Big] = nC_n^{A+nI}(x)y^n,$$

$$\mathbb{B}\Big[C_n^{A+nI}(x)y^n\Big] = -(2A+(3n-1)I)C_{n-1}^{A+nI}(x)y^{n-1},$$

and

$$\mathbb{C}\Big[C_n^{A+nI}(x)y^n\Big] = (n+1)C_{n+1}^{A+nI}(x)y^{n+1},$$

where differential operators $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ satisfy the commutator relations

$$[\mathbb{A},\mathbb{B}] = -\mathbb{B}, \quad [\mathbb{A},\mathbb{C}] = \mathbb{C}, \quad [\mathbb{B},\mathbb{C}] = -2\mathbb{A} - 2A\mathbb{I}. \tag{27}$$

Nota that: The set of linear combinations of the differential operators $\mathbb{I}$, $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ forms a Lie algebra.

It can be easily shown that the partial differential operators in (24) $\mathbb{L}$ given by

$$\mathbb{L} = (1-x^2)\frac{\partial^2}{\partial x^2}I + 3y^2\frac{\partial^2}{\partial y^2}I - (2A+I)x\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y\partial x}I + (2A+3I)y\frac{\partial}{\partial y}.$$

The second order differential operator $\mathbb{L}$ satisfies the differential operators identity as follows

$$(1-x^2)\mathbb{L} = \mathbb{B}\mathbb{C} + (2A+3\mathbb{A})(\mathbb{A}+\mathbb{I}). \tag{28}$$

It can be easily verified that $(1-x^2)\mathbb{L}$ commutes with each of the differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$,

$$[(1-x^2)\mathbb{L},\mathbb{A}] = [(1-x^2)\mathbb{L},\mathbb{B}] = [(1-x^2)\mathbb{L},\mathbb{C}] = \mathbf{0}. \tag{29}$$

The extended forms of transformation groups generated by differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are given by

$$e^{a\mathbb{A}}f(x,y,A) = f\Big(x, e^a y, A\Big), \tag{30}$$

$$e^{b\mathbb{B}}f(x,y,A) = f\Bigg(\frac{xy-b}{\sqrt{y^2-2bxy+b^2}}, \sqrt{y^2-2bxy+b^2}, A\Bigg), \tag{31}$$

and

$$e^{c\mathbb{C}}f(x,y,A) = \Big(c^2y^2-2cxy+1\Big)^{-A} f\Bigg(\frac{x-cy}{\sqrt{c^2y^2-2cxy+1}}, \frac{y}{\sqrt{c^2y^2-2cxy+1}}, A\Bigg), \tag{32}$$

where a, b and c are arbitrary constants and $f(x,y,A)$ is an arbitrary matrix function.

From the above relations the $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ commute operators, we get

$$\begin{aligned} e^{c\mathbb{C}}e^{b\mathbb{B}}e^{a\mathbb{A}}f(x,y,A) = f\Bigg(&\frac{y(x-cy)-b(c^2y^2-2cxy+1)}{\sqrt{c^2y^2-2cxy+1}\sqrt{b^2(c^2y^2-2cxy+1)-2by(x-cy)+y^2}} \\ &, e^a\frac{\sqrt{b^2(c^2y^2-2cxy+1)-2by(x-cy)+y^2}}{(c^2y^2-2cxy+1)^{\frac{3}{2}}}, A\Bigg). \end{aligned} \tag{33}$$

3.1 Generating Matrix Functions for Modified Gegenbauer Matrix Polynomials

From (26), $C_n^{A+nI}(x,y) = C_n^{A+nI}(x)y^n$ is a solution of the system

$$\mathbb{L}C_n^{A+nI}(x,y) = \mathbf{0} \quad \text{and} \quad (\mathbb{A} - n\mathbb{I})C_n^{A+nI}(x,y) = \mathbf{0}.$$

From (29) we easily get

$$e^{c\mathbb{C}}e^{b\mathbb{B}}e^{a\mathbb{A}}(1-x^2)\mathbb{L}[C_n^{A+nI}(x)y^n] = (1-x^2)\mathbb{L}e^{c\mathbb{C}}e^{b\mathbb{B}}e^{a\mathbb{A}}[C_n^{A+nI}(x)y^n].$$

Therefore the transform $e^{c\mathbb{C}}e^{b\mathbb{B}}e^{a\mathbb{A}}[C_n^{A+nI}(x)y^n]$ is annulled by $(1-x^2)\mathbb{L}$.

If we choose $a = 0$ and $C_n^{A+nI}(x,y) = C_n^{A+nI}(x)y^n$ in (33), we get

$$\begin{aligned} &e^{c\mathbb{C}}e^{b\mathbb{B}}[C_n^{A+nI}(x)y^n] \\ &= \left(b^2(c^2y^2 - 2cxy + 1) - 2by(x - cy) + y^2\right)^{\frac{1}{2}n}\left(c^2y^2 - 2cxy + 1\right)^{-(A+\frac{3}{2}nI)} \\ &\times C_n^{A+nI}\left(\frac{y(x-cy) - b(c^2y^2 - 2cxy + 1)}{\sqrt{c^2y^2 - 2cxy + 1}\sqrt{b^2(c^2y^2 - 2cxy + 1) - 2by(x - cy) + y^2}}\right). \end{aligned} \tag{34}$$

On the other hand we get

$$e^{c\mathbb{C}}e^{b\mathbb{B}}[C_n^{A+nI}(x)y^n] = \sum_{m=0}^{\infty}\frac{c^m}{m!}\sum_{k=0}^{\infty}\frac{b^k}{k!}(n-k+1)_m((1-3n)I - 2A)_k y^{n-k+m}C_{n-k+m}^{A+nI}(x). \tag{35}$$

Equating (34) and (35), we get

$$\begin{aligned} &\left(b^2(c^2y^2 - 2cxy + 1) - 2by(x - cy) + y^2\right)^{\frac{1}{2}n}\left(c^2y^2 - 2cxy + 1\right)^{-(A+\frac{3}{2}nI)} \\ &\times C_n^{A+nI}\left(\frac{y(x-cy) - b(c^2y^2 - 2cxy + 1)}{\sqrt{c^2y^2 - 2cxy + 1}\sqrt{b^2(c^2y^2 - 2cxy + 1) - 2by(x - cy) + y^2}}\right) \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{n}\frac{c^m b^k}{m!k!}(n-k+1)_m((1-3n)I - 2A)_k y^{n-k+m}C_{n-k+m}^{A+nI}(x). \end{aligned} \tag{36}$$

Here, we obtain some interesting results as the particular case of generating matrix relations (36).

Putting $b = 1$, $c = 0$ and writing $y = t$ in (36) we get of generating matrix relations

$$\begin{aligned} &\left(1 - 2xt + t^2\right)^{\frac{1}{2}n} C_n^{A+nI}\left(\frac{xt - 1}{\sqrt{1 - 2xt + t^2}}\right) \\ &= \sum_{k=0}^{n}\frac{1}{k!}((1-3n)I - 2A)_k t^{n-k}C_{n-k}^{A+nI}(x). \end{aligned} \tag{37}$$

Letting $b = 0$, $c = 1$ and $y = t$ in (36) we obtain

$$\begin{aligned} &\left(t^2 - 2xt + 1\right)^{-(A+\frac{3}{2}nI)} C_n^{A+nI}\left(\frac{x - t}{\sqrt{t^2 - 2xt + 1}}\right) \\ &= \sum_{m=0}^{\infty}\frac{1}{m!}(n+1)_m t^m C_{n+m}^{A+nI}(x). \end{aligned} \tag{38}$$

Putting $b = -\frac{1}{b}$, $c = 1$ and substituting $y = t$ in (36), we get

$$\begin{aligned}&\left(\frac{1}{b^2}(t^2-2xt+1)+\frac{2}{b}t(x-t)+t^2\right)^{\frac{1}{2}n}\left(t^2-2xt+1\right)^{-(A+\frac{3}{2}nI)}\\&\times C_n^{A+nI}\left(\frac{t(x-t)+\frac{1}{b}(t^2-2xt+1)}{\sqrt{t^2-2xt+1}\sqrt{\frac{1}{b^2}(t^2-2xt+1)+\frac{2}{b}t(x-t)+t^2}}\right)\\&=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{c^m(-\frac{1}{b})^k}{m!k!}(n-k+1)_m((1-3n)I-2A)_k t^{n-k+m}C_{n-k+m}^{A+nI}(x).\end{aligned} \tag{39}$$

Now replacing A by $A - nI$ and $t = \frac{1}{t}$ in (37) we get

$$\left(1-2xt+t^2\right)^{\frac{1}{2}n}C_n^A\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right)=\sum_{k=0}^{n}\frac{1}{k!}((1-n)I-2A)_k t^k C_{n-k}^A(x). \tag{40}$$

Again on replacing A by $A - nI$ in (38) we get

$$\left(t^2-2xt+1\right)^{-(A+nI)}C_n^A\left(\frac{x-t}{\sqrt{t^2-2xt+1}}\right)=\sum_{m=0}^{\infty}\frac{1}{m!}(n+1)_m t^m C_{n+m}^A(x). \tag{41}$$

3.2 Generating Matrix Functions for Modified Gegenbauer Matrix Polynomials $C_{n+r}^{A-nI}(x)$

Here, we consider the following operator $\mathbb{D}$:

$$\mathbb{D}=(x^2-1)y\frac{\partial}{\partial x}I-2xy^2\frac{\partial}{\partial y}I+xy(2A-I), \tag{42}$$

such that

$$\mathbb{D}[C_{n+r}^{A-nI}(x)y^n]=\frac{1}{2}(n+r+1)((1+n-r)I-2A)((1+n)I-A)^{-1}C_{n+r+1}^{A-(n+1)I}(x)y^{n+1}. \tag{43}$$

The extended form of group generated by $\mathbb{D}$ is given as follows:

$$\begin{aligned}e^{d\mathbb{D}}f(x,y)=&\left(d^2y^2(x^2-1)+2dxy+1\right)^{A-\frac{1}{2}I}\\&\times f\left(x+dy(x^2-1),\frac{y}{\sqrt{d^2y^2(x^2-1)+2dxy+1}}\right),\end{aligned} \tag{44}$$

where d is an arbitrary constant. Using (44), we obtain

$$e^{d\mathbb{D}}[C_{n+r}^{A-nI}(x)y^n]=y^n\left(d^2y^2(x^2-1)+2dxy+1\right)^{A-(n+\frac{1}{2})I}C_{n+r}^{A-nI}\left(x+dy(x^2-1)\right). \tag{45}$$

By using (43), we obtain

$$\begin{aligned}e^{d\mathbb{D}}[C_{n+r}^{A-nI}(x)y^n]&=\sum_{k=0}^{\infty}\frac{d^k}{k!}\mathbb{D}^k[C_{n+r}^{A-nI}(x)y^n]\\&=\sum_{k=0}^{\infty}\frac{d^k}{k!}\frac{1}{2^k}(n+r+1)_k((1+n-r)I-2A)_k\left(((1+n)I-A)_k\right)^{-1}C_{n+r+k}^{A-(n+k)I}(x)y^{n+k}.\end{aligned} \tag{46}$$

Equating (45) and (46) then putting $t = \frac{1}{2}dy$, we get

$$\begin{aligned}&\Big(4t^2(x^2-1)+4xt+1\Big)^{A-(n+\frac{1}{2})I} C_{n+r}^{A-nI}\Big(x+2t(x^2-1)\Big)\\&=\sum_{k=0}^{\infty}\frac{1}{k!}(n+r+1)_k((1+n-r)I-2A)_k\Big(((1+n)I-A)_k\Big)^{-1}C_{n+r+k}^{A-(n+k)I}(x)t^k.\end{aligned}\tag{47}$$

Putting $r = 0$ in (47), we get

$$\begin{aligned}&\Big(4t^2(x^2-1)+4xt+1\Big)^{A-(n+\frac{1}{2})I} C_{n}^{A-nI}\Big(x+2t(x^2-1)\Big)\\&=\sum_{k=0}^{\infty}\frac{1}{k!}(n+1)_k((1+n)I-2A)_k\Big(((1+n)I-A)_k\Big)^{-1}C_{n+k}^{A-(n+k)I}(x)t^k.\end{aligned}\tag{48}$$

Putting $r = 0$ and replacing A by $A + nI$ in (47), we get

$$\begin{aligned}&\Big(4t^2(x^2-1)+4xt+1\Big)^{A-\frac{1}{2}I} C_{n}^{A}\Big(x+2t(x^2-1)\Big)\\&=\sum_{k=0}^{\infty}\frac{1}{k!}(n+1)_k((1-n)I-2A)_k\Big((I-A)_k\Big)^{-1}C_{n+k}^{A-kI}(x)t^k.\end{aligned}\tag{49}$$

Putting $n = 0$ in (49), we get

$$\begin{aligned}&\Big(4t^2(x^2-1)+4xt+1\Big)^{A-\frac{1}{2}I} C_{n}^{A}\Big(x+2t(x^2-1)\Big)\\&=\sum_{k=0}^{\infty}(I-2A)_k\Big((I-A)_k\Big)^{-1}C_{k}^{A-kI}(x)t^k.\end{aligned}\tag{50}$$

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