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Necessary Condition for Vector-Valued Model Spaces to be Invariant Under Conjugation

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Abstract — The S^*-invariant subspaces of the Hardy-Hilbert space $H^2(E)$ (where E is finite dimensional Hilbert space of dimension greater than 1) on the unit disc is well known. In this study, we examine that, if Ω is a conjugation on E, and Θ an inner function, then there exist model spaces which are not invariant for the conjugation $C_\Omega : L^2(E) \longrightarrow L^2(E)$. Under what necessary condition the model spaces is mapped onto itself is under consideration.

Keywords — *Inner function, model spaces, conjugation.*

1. Introduction and Preliminaries

Let $\mathbb{D}$ denote the open unit disc and $\mathbb{T}$ the unit circle in the complex plane $\mathbb{C}$. Throughout the paper E will denote a fixed Hilbert space, of finite dimension d, and $\mathcal{L}(E)$ the algebra of bounded linear operators on E, which may be identified with $d \times d$ matrices. $\mathcal{L}(E)$ is Hilbert space endowed with Hilbert-Schmidt norm. $H^2(E)$ is the Hardy-Hilbert of E-valued analytic functions on $\mathbb{D}$ whose coefficients are square summable, which is a closed subspace of $L^2(E)$.

The space $L^2(E)$ is defined, as usual, by

$$L^2(E) = \left\{ f : \mathbb{T} \longrightarrow E : f(e^{it}) = \sum_{-\infty}^{\infty} a_n e^{int},\ a_n \in E\ , \sum_{-\infty}^{\infty} \|a_n\|_E^2 < \infty \right\}.$$

The inner product on $L^2(E)$ is defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle f(e^{it}), g(e^{it}) \rangle_E dt, \tag{1}$$

and $L^2(E)$ can be orthogonally decompose as

$$L^2(E) = H^2_-(E) \oplus H^2(E),$$

where $H^2_-(E)$ is the orthogonal complement of $H^2(E)$ in $L^2(E)$ with inner product defined in (1). For $f \in H^2(E)$, $f(z)$ and $f(e^{it})$ determine each other.

It is important to note that, if $dimE = 1$ (i.e $E = \mathbb{C}$) then $L^2(E)$ consists of the scalar valued functions and is denoted by $L^2(\mathbb{T})$, and all the results become trivial in that case.

By viewing $\mathcal{L}(E)$ as Hilbert space (endowed with the Hilbert Schmidt norm), one can also consider the space $L^2(\mathcal{L}(E))$, which may be identified with the matrices whose entries are from $L^2(\mathbb{T})$.

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Alternately, we may view $L^2(\mathcal{L}(E))$ also as a space of square summable Fourier series with coefficients in $\mathcal{L}(E)$.

The space $H^2(\mathcal{L}(E))$ is a closed subspace of $L^2(\mathcal{L}(E))$ whose Fourier coefficients corresponding to negative indices vanishes. We have an orthogonal decomposition

$$L^2(\mathcal{L}(E)) = [zH^2(\mathcal{L}(E))]^* \oplus H^2(\mathcal{L}(E)).$$

The unilateral shift (see [6]) $S : H^2(E) \longrightarrow H^2(E)$ is defined by $Sf = zf$, and its adjoint S^* (backward shift) is given by the formula;

$$S^* f = \frac{f - f(0)}{z}.$$

After gathering the facts in preliminaries, we will present main results in the next section. An effort has been made to make the paper self-contained.

2. Formulation and Basic Results

Definition 2.1. An inner function is an element $\Theta \in H^2(\mathcal{L}(E))$ whose boundary values are almost everywhere unitary operators in $\mathcal{L}(E)$.

Definition 2.2. A conjugation is a conjugate-linear operator $C : \mathcal{H} \longrightarrow \mathcal{H}$ that satisfies the conditions

1. C is isometric: $\langle Cf, Cg\rangle = \langle g, f\rangle \ \forall f, g \in \mathcal{H}$,
2. C is involutive: $C^2 = I$.

Model space associated to an inner function Θ, is denoted by K_Θ, and is defined by

$$K_\Theta = H^2(E) \ominus \Theta H^2(E).$$

Just like the Beurling-type subspace $\Theta H^2(E)$ constitute nontrivial invariant subspace for the unilateral shift S, the subspace K_Θ plays an analogous role for the backward shift S^*.

For a given inner function Θ, and Ω a conjugation on E, the map $C_\Omega : L^2(E) \longrightarrow L^2(E)$, defined by

$$(Cf)(e^{it}) = \Theta(e^{it})\overline{e^{it}}\Omega f(e^{it})$$

is a conjugation. It is worth noting that C_Ω does not preserve the model spaces in general. However this is true for $dim E = 1$ (see [6]) . Under what condition the model spaces is invariant under the conjugation C_Ω, this we will study in the next section.

3. Main Results

Example 3.1. If $\Theta(e^{it})^* \neq \Omega\Theta(e^{it})\Omega$ then $C_\Omega K_\Theta \nsubseteq K_\Theta$.

Let $E = \mathbb{C}^2$ and

$$\Theta(z) = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \in H^2(\mathcal{L}(\mathbb{C}^2)),$$

then

$$\Theta H^2(\mathbb{C}^2) = \left\{ \begin{pmatrix} zf \\ z^2 g \end{pmatrix} : f, g \in H^2 \right\},$$

and the model space associated to Θ is

$$K_\Theta = [\Theta H^2(\mathbb{C}^2)]^\perp = \left\{ \begin{pmatrix} f_0 \\ g_0 + g_1 z \end{pmatrix} : f_0, g_0, g_1 \in \mathbb{C} \right\}.$$

Let $\Omega : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be defined by

$$\Omega \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \overline{a_2} \\ \overline{a_1} \end{pmatrix}$$

is a conjugation.

Now consider

$$\Theta(e^{it})^* \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \overline{z} & 0 \\ 0 & \overline{z}^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1\overline{z} \\ a_2\overline{z}^2 \end{pmatrix} \tag{2}$$

and

$$\Omega\Theta\Omega \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \Omega \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} \overline{a_2} \\ \overline{a_1} \end{pmatrix} = \Omega \begin{pmatrix} \overline{a_2} z \\ \overline{a_1} z^2 \end{pmatrix} = \begin{pmatrix} a_1\overline{z}^2 \\ a_2\overline{z} \end{pmatrix} \neq \Theta(e^{it})^* \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{3}$$

Now

$$C_\Omega \begin{pmatrix} 0 \\ z \end{pmatrix} = \overline{z} \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix} \Omega \begin{pmatrix} 0 \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} \overline{z} \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{z} \\ 0 \end{pmatrix} \notin K_\Theta.$$

Theorem 3.2. If Ω is a conjugation on E. Suppose that for the inner function Θ, we have $\Theta(e^{it})^* = \Omega\Theta(e^{it})\Omega$. Then $C_\Omega K_\Theta = K_\Theta$.

PROOF. Let $f \in K_\Theta$ and $h \in H^2(E)$, then

$$\begin{aligned} \langle C_\Omega f, \Omega(zh)\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \langle \Theta(e^{it})e^{-it}\Omega f(e^{it}), e^{-it}\Omega h(e^{it})\rangle dt = \frac{1}{2\pi}\int_0^{2\pi} \langle \Omega\Theta(e^{it})^*\Omega f(e^{it}), \Omega h(e^{it})\rangle dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle \Omega\Theta(e^{it})^* f(e^{it}), \Omega h(e^{it})\rangle dt = \frac{1}{2\pi}\int_0^{2\pi} \langle h(e^{it}), \Theta(e^{it})^* f(e^{it})\rangle dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle \Theta(e^{it})h(e^{it}), f(e^{it})\rangle dt = \langle \Theta h, f\rangle = 0. \end{aligned}$$

This proves that $C_\Omega f \perp H^2_-(E)$. Next we will prove that $C_\Omega f \perp \Theta H^2(E)$. For this consider

$$\begin{aligned} \langle C_\Omega f, \Theta z^n x\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \langle \Theta(e^{it})e^{-it}\Omega f(e^{it}), \Theta(e^{it})e^{int}x\rangle dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle e^{-it}\Omega f(e^{it}), e^{int}x\rangle dt = \frac{1}{2\pi}\int_0^{2\pi} \langle \Omega f(e^{it}), e^{i(n+1)t}x\rangle dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \langle \Omega e^{i(n+1)t}x, f(e^{it})\rangle dt = \langle \Omega z^{n+1}x, f\rangle \\ &= 0. \end{aligned}$$

Here we have used the fact that $\Omega z^{n+1}x \in H^2_-(E)$. This shows that $C_\Omega K_\Theta \subset K_\Theta$ and this combined with $C_\Omega^2 = I$ follows that $C_\Omega K_\Theta = K_\Theta$. □

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