



ON TWO TYPES ALMOST (α, F_d) -CONTRACTIONS ON QUASI METRIC SPACE

HATICE ASLAN HANÇER

ABSTRACT. In this paper, first we introduce two new types almost contractions on quasi metric space named as almost (α, F_d) -contraction of type (x) and of type (y) . Then, taking into account both left and right completeness of quasi metric space, we present some fixed point results for these contractions. We also provide some illustrative and comparative examples.

1. INTRODUCTION AND PRELIMINARIES

Fundamentally, fixed point theory divides into three major subject which are topological, discrete and metric. Especially, it has been intensively improving on the metric case because of useful to applications. In general, metrical fixed point theory is related to contractive type mappings and it has been developed either taking into account the new type contractions or playing the structure of the space such as fuzzy metric space, quasi metric space, metric like space etc. A quasi metric space plays a crucial role in some fields of theoretical computer service, asymmetric functional analysis and approximation theory. Now, we will recall some basic concepts of quasi metric space.

In quasi metric spaces there are many different types of Cauchyness, yielding even more notions of completeness. Another difference comes from the fact that, in contrast to the metric case, in a quasi metric space a convergent sequence could not be Cauchy (see [3] for examples confirming this situation).

Let X be nonempty set and $d : X \times X \rightarrow \mathbb{R}^+$ be a function. Consider the following conditions on d , for all $x, y, z \in X$:

- (qm1) $d(x, x) = 0$,
- (qm2) $d(x, y) \leq d(x, z) + d(z, y)$,
- (qm3) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$,
- (qm4) $d(x, y) = 0 \Rightarrow x = y$.

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If the function d satisfies conditions (qm1) and (qm2) then d is said to be a quasi-pseudo metric on X . Further, if a quasi-pseudo metric d satisfies condition (qm3), then d is said to be a quasi metric on X , and if a quasi metric d satisfies condition (qm4), then d is said to be a T_1 -quasi metric on X . In this case, the pair (X, d) is said to be a quasi-pseudo (resp. a quasi, a T_1 -quasi) metric space. It is clear that every metric space is a T_1 -quasi metric space.

Let (X, d) be a quasi-pseudo metric space. Given a point $x_0 \in X$ and a real constant $\varepsilon > 0$, the set

$$B_d(x_0, \varepsilon) = \{y \in X : d(x_0, y) < \varepsilon\}$$

is called open ball with center x_0 and radius ε . Each quasi-pseudo metric d on X generates a topology τ_d on X which has a base the family of open balls $\{B_d(x, \varepsilon) : x \in X \text{ and } \varepsilon > 0\}$. If d is a quasi metric on X , then τ_d is a T_0 topology, and if d is a T_1 -quasi metric, then τ_d is a T_1 topology on X .

If d is a quasi-pseudo metric on X , then the function d^{-1} defined by

$$d^{-1}(x, y) = d(y, x)$$

is a quasi-pseudo metric on X and

$$d^s(x, y) = \max \{d(x, y), d^{-1}(x, y)\}$$

is a quasi metric. If d is a quasi metric, then d^{-1} is also a quasi metric, and d^s is a metric on X .

The convergence of a sequence $\{x_n\}$ to x with respect to τ_d called d -convergence and denoted by $x_n \xrightarrow{d} x$, is defined

$$x_n \xrightarrow{d} x \Leftrightarrow d(x, x_n) \rightarrow 0.$$

Similarly, the convergence of a sequence $\{x_n\}$ to x with respect to $\tau_{d^{-1}}$ called d^{-1} -convergence and denoted by $x_n \xrightarrow{d^{-1}} x$, is defined

$$x_n \xrightarrow{d^{-1}} x \Leftrightarrow d(x_n, x) \rightarrow 0.$$

Finally, the convergence of a sequence $\{x_n\}$ to x with respect to τ_{d^s} called d^s -convergence and denoted by $x_n \xrightarrow{d^s} x$, is defined

$$x_n \xrightarrow{d^s} x \Leftrightarrow d^s(x_n, x) \rightarrow 0.$$

It is clear that $x_n \xrightarrow{d^s} x \Leftrightarrow x_n \xrightarrow{d} x$ and $x_n \xrightarrow{d^{-1}} x$. More and detailed information about some important properties of quasi metric spaces and their topological structures can be found in [10, 16, 17, 18].

Definition 1 ([25]). *Let (X, d) be a quasi metric space. A sequence $\{x_n\}$ in X is called*

- left K -Cauchy (or forward Cauchy) if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, k, n \geq k \geq n_0, d(x_k, x_n) < \varepsilon,$$

- right K -Cauchy (or backward Cauchy) if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, k, n \geq k \geq n_0, d(x_n, x_k) < \varepsilon,$$

- d^s -Cauchy if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, k \geq n_0, d(x_n, x_k) < \varepsilon.$$

If a sequence is left K -Cauchy with respect to d , then it is right K -Cauchy with respect to d^{-1} . A sequence is d^s -Cauchy if and only if it is both left K -Cauchy and right K -Cauchy. Let $\{x_n\}$ be a sequence in a quasi metric space (X, d) such that

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty,$$

then it is left K -Cauchy sequence (see [10]).

It is well known that a metric space is said to be complete if every Cauchy sequence is convergent. The completeness of a quasi metric space, however, can not be uniquely defined. Taking into account the convergence and the Cauchyness of sequences in a quasi metric space, one obtains several notions of completeness, most of them being already available in the literature (see [1, 2, 9, 10, 17, 25]) with different notations. It can be found a detailed classification, some important properties and relations for completeness of quasi metric spaces in [3].

Definition 2. Let (X, d) be a quasi metric space. Then (X, d) is said to be

- left (right) K -complete if every left (right) K -Cauchy sequence is d -convergent,
- left (right) M -complete if every left (right) K -Cauchy sequence is d^{-1} -convergent,
- left (right) Smyth complete if every left (right) K -Cauchy sequence is d^s -convergent.

Remark 3. It is clear that a quasi metric space (X, d) is left M -complete if and only if (X, d^{-1}) is right K -complete. Also, a quasi metric space (X, d) is right M -complete if and only if (X, d^{-1}) is left K -complete.

Remark 4. If a quasi metric space is left Smyth complete, then it is also left K -complete.

We will consider the sequential continuity of a mapping T in our results.

Definition 5 ([28]). Let X be a nonempty set, d and ρ be two quasi metrics on X and $T : X \rightarrow X$ be a mapping. Then T is said to be sequentially d - ρ -continuous at $x \in X$, if for all sequence $\{x_n\}$ in X such that $d(x, x_n) \rightarrow 0$ implies $\rho(Tx, Tx_n) \rightarrow 0$. If T is sequentially d - ρ -continuous at all points of X , then T is said to be sequentially d - ρ -continuous on X .

On the other hand, α -admissibility and F -contractivity of a mapping are popular concepts in recent metrical fixed point theory. The concept of α -admissibility of a mapping on a nonempty set has been introduced by Samet [27]. Let X be a nonempty set, T be a self mapping of X and $\alpha : X \times X \rightarrow [0, \infty)$ be a function. Then T is said to be α -admissible if

$$\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1.$$

Using α -admissibility of a mapping, Samet, et al [27] provided some general fixed point results including many known theorems on complete metric spaces. We say that α has (B) property on a metric space (X, d) whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $d(x_n, x) \rightarrow 0$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

The concept of F -contraction was introduced by Wardowski [29]. Let \mathcal{F} be the family of all functions $F : (0, \infty) \rightarrow \mathbb{R}$ satisfying the following conditions:

- (F1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$,
- (F2) For each sequence $\{a_n\}$ of positive numbers $\lim_{n \rightarrow \infty} a_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(a_n) = -\infty$,
- (F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

Many authors have extended fixed point results on metric space by considering the family \mathcal{F} . For instance, in [4, 5, 6, 11, 26] be found some fixed point results for single valued and multivalued mappings on metric space [22]. Various fixed point results for α -admissible mapping on complete metric space can be found in [7, 12, 14, 15, 19, 20].

Definition 6 ([22]). *Let (X, d) be a metric space and $T : X \rightarrow X$ be a mapping. Then T is said to be an almost F -contraction if $F \in \mathcal{F}$ and there exist $\tau > 0$ and $L \geq 0$ such that*

$$\forall x, y \in X [d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y) + Ld(y, Tx))]. \quad (1)$$

In order to check the almost F -contractiveness of a mapping T , it is necessary to check both (1) and

$$\forall x, y \in X [d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y) + Ld(x, Ty))]. \quad (2)$$

The aim of this paper is to present some new fixed point results on quasi metric space. To do this considering both almost and F -contractiveness of a self mapping on a quasi metric space, we introduce two type almost (α, F_d) -contractions. Then by α -admissibility of a mapping, we obtaine some fixed point results on some kind of complete quasi metric space.

We can find some recent fixed point results for single valued and multivalued mappings on quasi metric spaces in [8, 2, 13, 21, 23, 24]

2. FIXED POINT RESULT

Let (X, d) be a quasi metric space, $T : X \rightarrow X$ be a mapping and $\alpha : X \times X \rightarrow [0, \infty)$ be a function. We will consider the following set to introduce a quasi metric version of F -contractivity of a mapping:

$$T_\alpha = \{(x, y) \in X \times X : \alpha(x, y) \geq 1 \text{ and } d(Tx, Ty) > 0\}.$$

As a slight different from metric space, we say that α has (B_d) (resp. $(B_{d^{-1}})$) property on a quasi metric space (X, d) whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $d(x, x_n) \rightarrow 0$ (resp. $d(x_n, x) \rightarrow 0$), then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Similarly, we say that α has (C_d) (resp. $(C_{d^{-1}})$) property on a quasi metric space (X, d) whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \in \mathbb{N}$ and $d(x, x_n) \rightarrow 0$ (resp. $d(x_n, x) \rightarrow 0$), then $\alpha(x, x_n) \geq 1$ for all $n \in \mathbb{N}$.

Definition 7. Let (X, d) be a quasi metric space. $T : X \rightarrow X$ be a mapping, $\alpha : X \times X \rightarrow [0, \infty)$ and $F \in \mathcal{F}$ be two functions. Then T is said to be an almost (α, F_d) -contraction of type (y) if there exists $\tau > 0$ and $L \geq 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y) + Ld(y, Tx)) \tag{3}$$

for all $(x, y) \in T_\alpha$.

Definition 8. Let (X, d) be a quasi metric space. $T : X \rightarrow X$ be a mapping, $\alpha : X \times X \rightarrow [0, \infty)$ and $F \in \mathcal{F}$ be two functions. Then T is said to be an almost (α, F_d) -contraction of type (x) if there exists $\tau > 0$ and $L \geq 0$ such that

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y) + Ld(x, Ty)) \tag{4}$$

for all $(x, y) \in T_\alpha$.

The following example shows the differences between the above two concepts.

Example 9. Let $X = [0, \frac{1}{4}] \cup \{1\}$ and $d(x, y) = \max\{y - x, 0\}$ for all $x, y \in X$. Then (X, d) is a quasi metric space. Consider the mappings $T : X \rightarrow X, Tx = x^2, \alpha : X \times X \rightarrow [0, \infty), \alpha(x, y) = 1$. Now for $x = 0$ and $y = 1$, then we have

$$\begin{aligned} d(Tx, Ty) &= d(T0, T1) = 1, \\ d(x, y) &= d(0, 1) = 1, \end{aligned}$$

and

$$d(y, Tx) = d(1, 0) = 0.$$

Therefore we can not find $\tau > 0, L \geq 0$ and $F \in \mathcal{F}$ satisfying (3). Thus T is not almost (α, F_d) -contraction of type (y) . However it is almost (α, F_d) -contraction of type (x) with $\tau = \ln \frac{8}{5}, L = 1$ and $F(t) = \ln t$. To see this we will consider the following cases: First note that

$$\begin{aligned} T_\alpha &= \{(x, y) \in X \times X : \alpha(x, y) \geq 1 \text{ and } d(Tx, Ty) > 0\} \\ &= \{(x, y) \in X \times X : x < y\}. \end{aligned}$$

Case 1: Let $(x, y) \in T_\alpha$ and $y = 1$, then

$$\begin{aligned} \tau + F(d(Tx, Ty)) &= \ln \frac{8}{5} + \ln(1 - x^2) \\ &= \ln \frac{8}{5} + \ln[(1 + x)(1 - x)] \\ &\leq \ln \frac{8}{5} + \ln\left[\frac{5}{4}(1 - x)\right] \\ &= \ln[2(1 - x)] \\ &= \ln[(1 - x) + (1 - x)] \\ &= F(d(x, y) + Ld(x, Ty)). \end{aligned}$$

Case 2: Let $(x, y) \in T_\alpha$ and $y \leq \frac{1}{4}$, then

$$\begin{aligned} \tau + F(d(Tx, Ty)) &= \ln \frac{8}{5} + \ln(y^2 - x^2) \\ &= \ln \frac{8}{5} + \ln[(y + x)(y - x)] \\ &\leq \ln \frac{8}{5} + \ln\left[\frac{1}{2}(y - x)\right] \\ &= \ln\left[\frac{4}{5}(y - x)\right] \\ &\leq \ln(y - x) \\ &\leq F(d(x, y) + Ld(x, Ty)). \end{aligned}$$

Therefore (4) is satisfied and so almost (α, F_d) -contraction of type (x) .

Example 10. Let $X = [0, 1]$ and $d(x, y) = \max\{y - x, 0\}$ for all $x, y \in X$. Then (X, d) is quasi metric space. Define a map $T : X \rightarrow X$, $Tx = x$, $\alpha : X \times X \rightarrow [0, \infty)$, $\alpha(x, y) = 1$ for $x, y \in X$. Then, by the similar way we see that, T is not almost (α, F_d) -contraction of type (y) but it is almost (α, F_d) -contraction of type (x) .

Now we present our main results.

Theorem 11. Let (X, d) be a left K -complete T_1 -quasi metric space, $\alpha : X \times X \rightarrow [0, \infty)$ be a function and $T : X \rightarrow X$ be an α -admissible and almost (α, F_d) -contraction of type (y) . Suppose there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has a fixed point in X provided that one of the following conditions holds:

- (i) (X, τ_d) is Hausdorff space and T is sequentially d - d continuous;
- (ii) T is sequentially d - d^{-1} continuous;
- (iii) (X, d) is left Smyth complete and α has (B_d) or $(B_{d^{-1}})$ property.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is α -admissible, then $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Now, let $d_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. If there exists $k \in \mathbb{N}$ with $d_k = d(x_k, x_{k+1}) = 0$, then x_k is a fixed point of T , since d is T_1 -quasi metric.

Suppose $d_n > 0$ for all $n \in \mathbb{N}$. In this case $(x_n, x_{n+1}) \in T_\alpha$ for all $n \in \mathbb{N}$. Since T is almost (α, F_d) -contraction of type (y) , we get

$$\begin{aligned} F(d_n) &= F(d(x_n, x_{n+1})) \\ &= F(d(Tx_{n-1}, Tx_n)) \\ &\leq F(d(x_{n-1}, x_n) + Ld(x_n, Tx_{n-1})) - \tau \\ &= F(d(x_{n-1}, x_n)) - \tau. \end{aligned}$$

Therefore we obtain that

$$F(d_n) \leq F(d_{n-1}) - \tau \leq F(d_{n-2}) - 2\tau \leq \dots \leq F(d_0) - n\tau. \quad (5)$$

From (5), we get $\lim_{n \rightarrow \infty} F(d_n) = -\infty$. Thus, from (F2), we have

$$\lim_{n \rightarrow \infty} d_n = 0.$$

From (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} d_n^k F(d_n) = 0.$$

By (5), the following holds for all $n \in \mathbb{N}$

$$d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k \tau \leq 0. \quad (6)$$

Letting $n \rightarrow \infty$ in (6), we obtain that

$$\lim_{n \rightarrow \infty} nd_n^k = 0. \quad (7)$$

From (7), there exists $n_1 \in \{1, 2, 3, \dots\}$ such that $nd_n^k \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$d_n \leq \frac{1}{n^{\frac{1}{k}}}. \quad (8)$$

Therefore $\sum_{n=1}^{\infty} d_n < \infty$. Now let $m, n \in \mathbb{N}$ with $m > n \geq n_1$, then we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &= d_n + d_{n+1} + \dots + d_{m-1} \\ &\leq \sum_{k=n}^{\infty} d_k. \end{aligned}$$

Since $\sum_{k=1}^{\infty} d_k$ is convergent, then we get $\{x_n\}$ is left K -Cauchy sequence in the quasi metric space (X, d) . Since (X, d) left K -complete, there exists $z \in X$ such that $\{x_n\}$ is d -converges to z , that is, $d(z, x_n) \rightarrow 0$ as $n \rightarrow \infty$.

First, suppose (i) holds. Thus we have $d(Tz, Tx_n) \rightarrow 0$, that is, $d(Tz, x_{n+1}) \rightarrow 0$. Since (X, d) is Hausdorff we get $z = Tz$.

Second, suppose (ii) holds. Then we have

$$d(z, Tz) \leq d(z, Tx_n) + d(Tx_n, Tz)$$

$$= d(z, x_{n+1}) + d(Tx_n, Tz).$$

Since T is sequentially d - d^{-1} continuous, we get $d(Tx_n, Tz) \rightarrow 0$, that is, $d(z, Tz) = 0$. Then z is a fixed point of T , since d is T_1 -quasi metric.

Third, suppose (iii) holds. Note that in this case since X is left Smyth complete, then $d^s(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, from (F2) and (3), it easy to conclude that

$$d(Tx, Ty) < d(x, y) + Ld(y, Tx)$$

for all $(x, y) \in T_\alpha$. Therefore, for all $x, y \in X$ with $\alpha(x, y) \geq 1$, we obtain

$$d(Tx, Ty) \leq d(x, y) + Ld(y, Tx). \quad (9)$$

Since α has (B_d) or $(B_{d^{-1}})$ property, then $\alpha(x_n, z) \geq 1$ for all $n \in X$. By (9) we get

$$\begin{aligned} d(z, Tz) &\leq d(z, x_{n+1}) + d(Tx_n, Tz) \\ &\leq d(z, x_{n+1}) + d(x_n, z) + Ld(z, x_{n+1}). \end{aligned} \quad (10)$$

Letting $n \rightarrow \infty$ in (10), we obtain that $d(z, Tz) = 0$, then $z = Tz$. \square

Theorem 12. *Let (X, d) be a right M -complete T_1 -quasi metric space, $\alpha : X \times X \rightarrow [0, \infty)$ be a function and $T : X \rightarrow X$ be an α -admissible and almost (α, F_d) -contraction of type (x) . Suppose there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$. Then T has a fixed point in X provided that one of the following conditions holds:*

- (i) $(X, \tau_{d^{-1}})$ is Hausdorff space and T is sequentially d^{-1} - d^{-1} continuous;
- (ii) T is sequentially d^{-1} - d continuous;
- (iii) (X, d) is right Smyth complete and α has (C_d) or $(C_{d^{-1}})$ property.

Proof. As in Theorem 11, we get the mentioned $\{x_n\}$ is right K -Cauchy sequence in the quasi metric space (X, d) . Since (X, d) right M -complete, there exists $z \in X$ such that $\{x_n\}$ is d^{-1} -converges to z , that is, $d(x_n, z) \rightarrow 0$ as $n \rightarrow \infty$.

First, suppose (i) holds. Thus we have $d(Tx_n, Tz) \rightarrow 0$, that is, $d(x_{n+1}, Tz) \rightarrow 0$. Since $(X, \tau_{d^{-1}})$ is Hausdorff we get $z = Tz$.

Second, suppose (ii) holds. Then we have

$$\begin{aligned} d(Tz, z) &\leq d(Tz, Tx_n) + d(Tx_n, z) \\ &= d(Tz, Tx_n) + d(x_{n+1}, z). \end{aligned}$$

Since T is sequentially d^{-1} - d continuous, we get $d(Tz, Tx_n) \rightarrow 0$, that is, $d(Tz, z) = 0$. Then z is a fixed point of T , since d is T_1 -quasi metric.

Third, suppose (iii) holds. Note that in this case since X is right Smyth complete, then $d^s(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, from (F2) and (4) it easy to conclude that

$$d(Tx, Ty) < d(x, y) + Ld(x, Ty)$$

for all $(x, y) \in T_\alpha$. Therefore, for all $x, y \in X$ with $\alpha(x, y) \geq 1$ we obtain

$$d(Tx, Ty) \leq d(x, y) + Ld(x, Ty). \quad (11)$$

Since α has (C_d) or $(C_{d^{-1}})$ property then $\alpha(z, x_n) \geq 1$ for all $n \in \mathbb{N}$. By (11) we get

$$\begin{aligned} d(Tz, z) &\leq d(Tz, Tx_n) + d(x_{n+1}, z) \\ &\leq d(z, x_n) + Ld(z, x_{n+1}) + d(x_{n+1}, z). \end{aligned} \tag{12}$$

Letting $n \rightarrow \infty$ in (12), we obtain that $d(Tz, z) = 0$, then $z = Tz$. \square

Theorem 13. *Let (X, d) be a right K -complete T_1 -quasi metric space, $\alpha : X \times X \rightarrow [0, \infty)$ be a function and $T : X \rightarrow X$ be an α -admissible and almost (α, F_d) -contraction of type (x) . Suppose there exists $x_0 \in X$ such that $\alpha(Tx_0, x_0) \geq 1$. Then T has a fixed point in X provided that one of the following conditions holds:*

- (i) (X, τ_d) is Hausdorff space and T is sequentially d - d continuous;
- (ii) T is sequentially d - d^{-1} continuous;
- (iii) (X, d) is right Smyth complete and α has (C_d) or $(C_{d^{-1}})$ property.

Proof. Let $x_0 \in X$ be such that $\alpha(Tx_0, x_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since T is α -admissible, then $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \in \mathbb{N}$. Now, let $d_n = d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$. If there exists $k \in \mathbb{N}$ with $d_k = d(x_{k+1}, x_k) = 0$, then x_k is a fixed point of T , since d is T_1 -quasi metric. Suppose $d_n > 0$ for all $n \in \mathbb{N}$. In this case $(x_{n+1}, x_n) \in T_\alpha$ for all $n \in \mathbb{N}$. Since T is almost (α, F_d) -contraction of type (x) , we get

$$\begin{aligned} F(d_n) &= F(d(x_{n+1}, x_n)) \\ &= F(d(Tx_n, Tx_{n-1})) \\ &\leq F(d(x_n, x_{n-1}) + Ld(x_n, Tx_{n-1})) - \tau \\ &= F(d(x_n, x_{n-1})) - \tau. \end{aligned}$$

Therefore, we obtain that

$$F(d_n) \leq F(d_{n-1}) - \tau \leq F(d_{n-2}) - 2\tau \leq \dots \leq F(d_0) - n\tau. \tag{13}$$

From (13), we get $\lim_{n \rightarrow \infty} F(d_n) = -\infty$. Thus, from (F2), we have

$$\lim_{n \rightarrow \infty} d_n = 0.$$

From (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} d_n^k F(d_n) = 0.$$

By (13), the following holds for all $n \in \mathbb{N}$

$$d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k \tau \leq 0. \tag{14}$$

Letting $n \rightarrow \infty$ in (14), we obtain that

$$\lim_{n \rightarrow \infty} n d_n^k = 0. \tag{15}$$

From (15), there exists $n_1 \in \{1, 2, 3, \dots\}$ such that $nd_n^k \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$d_n \leq \frac{1}{n^{\frac{1}{k}}}. \quad (16)$$

Therefore $\sum_{n=1}^{\infty} d_n < \infty$. Now let $m, n \in \mathbb{N}$ with $m > n \geq n_1$, then we get

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \dots + d(x_{n+1}, x_n) \\ &\leq d_m + d_{m-1} + \dots + d_{n+1} \\ &\leq \sum_{k=n}^{\infty} d_k. \end{aligned}$$

Since $\sum_{k=1}^{\infty} d_k$ is convergent, then we get $\{x_n\}$ is right K -Cauchy sequence in the quasi metric space (X, d) . Since (X, d) right K -complete, there exists $z \in X$ such that $\{x_n\}$ is d -converges to z , that is, $d(z, x_n) \rightarrow 0$ as $n \rightarrow \infty$.

First, suppose (i) holds. Thus we have $d(Tz, Tx_n) \rightarrow 0$, that is, $d(x_{n+1}, Tz) \rightarrow 0$. Since (X, τ_d) is Hausdorff we get $z = Tz$.

Second, suppose (ii) holds. Then we have

$$d(z, Tz) \leq d(z, x_{n+1}) + d(Tx_n, Tz).$$

Since T is sequentially d - d^{-1} continuous, we get $d(Tx_n, Tz) \rightarrow 0$, that is, $d(z, Tz) = 0$. Then z is a fixed point of T , since d is T_1 -quasi metric.

Third, suppose (iii) holds. Note that in this case since X is right Smyth complete, then $d^s(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, from (F2) and (4) it easy to conclude that

$$d(Tx, Ty) < d(x, y) + Ld(x, Ty)$$

for all $(x, y) \in T_\alpha$. Therefore, for all $x, y \in X$ with $\alpha(x, y) \geq 1$ we obtain that

$$d(Tx, Ty) \leq d(x, y) + Ld(x, Ty). \quad (17)$$

Since α has (C_d) or $(C_{d^{-1}})$ property then $\alpha(z, x_n) \geq 1$ for all $n \in \mathbb{N}$. By (17) we get

$$\begin{aligned} d(Tz, z) &\leq d(Tz, Tx_n) + d(x_{n+1}, z) \\ &\leq d(z, x_n) + Ld(z, x_{n+1}) + d(x_{n+1}, z). \end{aligned} \quad (18)$$

Letting $n \rightarrow \infty$ in (18), we obtain that $d(Tz, z) = 0$, then $z = Tz$. \square

Theorem 14. Let (X, d) be a left M -complete T_1 -quasi metric space, $\alpha : X \times X \rightarrow [0, \infty)$ be a function and $T : X \rightarrow X$ be an α -admissible and almost (α, F_d) -contraction of type (y) . Suppose there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Then T has a fixed point in X provided that one of the following conditions holds:

- (i) $(X, \tau_{d^{-1}})$ is Hausdorff space and T is sequentially d^{-1} - d^{-1} continuous;
- (ii) T is sequentially d^{-1} - d continuous;
- (iii) (X, d) is left Smyth complete and α has (B_d) or $(B_{d^{-1}})$ property.

Proof. As in Theorem 13, we get the mentioned $\{x_n\}$ is left K -Cauchy sequence in the quasi metric space (X, d) . Since (X, d) left M -complete, there exists $z \in X$ such that $\{x_n\}$ is d^{-1} -converges to z , that is, $d(x_n, z) \rightarrow 0$ as $n \rightarrow \infty$.

First, suppose (i) holds. Thus we have $d(Tx_n, Tz) \rightarrow 0$, that is, $d(x_{n+1}, Tz) \rightarrow 0$. Since $(X, \tau_{d^{-1}})$ is Hausdorff we get $z = Tz$.

Second, suppose (ii) holds. Then we have

$$d(Tz, z) \leq d(Tz, Tx_n) + d(x_{n+1}, z).$$

Since T is sequentially d^{-1} - d continuous, we get $d(Tz, Tx_n) \rightarrow 0$, that is, $d(Tz, z) = 0$. Then z is a fixed point of T , since d is T_1 -quasi metric.

Third, suppose (iii) holds. Note that in this case since X is left Smyth complete, then $d^s(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, from (F2) and (4) it easy to conclude that

$$d(Tx, Ty) < d(x, y) + Ld(y, Tx)$$

for all $(x, y) \in T_\alpha$. Therefore, for all $x, y \in X$ with $\alpha(x, y) \geq 1$ we obtain that

$$d(Tx, Ty) \leq d(x, y) + Ld(y, Tx). \quad (19)$$

Since α has (B_d) or $(B_{d^{-1}})$ property then $\alpha(x_n, z) \geq 1$ for all $n \in \mathbb{N}$. By (19) we get

$$\begin{aligned} d(z, Tz) &\leq d(z, x_{n+1}) + d(Tx_n, Tz) \\ &\leq d(z, x_{n+1}) + d(x_n, z) + Ld(z, x_{n+1}). \end{aligned} \quad (20)$$

Letting $n \rightarrow \infty$ in (20), we obtain that $d(z, Tz) = 0$, then $z = Tz$. \square

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Current address: Hatice Aslan Hançer: Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey.

E-mail address: haticeaslanhancer@gmail.com

ORCID Address: <https://orcid.org/0000-0001-5928-9599>