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MONOTONICITY RESULTS AND ASSOCIATED INEQUALITIES FOR K-GAMMA FUNCTION

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ABSTRACT

This study is inspired by the work of Neumann in 2011. In the study, we establish some double inequalities involving the ratio $\frac{\Gamma_k(n+s)}{\Gamma_k(n+k)}$, where Γ_k is the k-analogue of Euler's gamma function. Some monotonicity results involving k-gamma function are found. By the aid of these results, some inequalities such as $[\Gamma_k(k+\sigma)]^{\frac{k}{\sigma}} \leq \sum_{i=1}^{n}[\Gamma_k(k+x_i)]^{\frac{k}{x_i}}$ for $x_i > 0,\ 1 \leq i \leq n$, where $\sigma = x_1 + x_2 + \cdots + x_n$ are valid.

Keywords: Gamma function, k-Gamma function, Monotonicity, Inequality

1. INTRODUCTION

Euler's gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

Gautschi in [1] obtains bounds for the ratio of two classical gamma functions as follows:

$$e^{(s-1)\psi(n+1)} \leq \frac{\Gamma(n+s)}{\Gamma(n+1)} \leq n^{s-1} \tag{1}$$

for $0 \leq s \leq 1$ and $n \in Z^+$, where ψ denotes digamma function that is defined by the logarithmic derivative of classical gamma function, i.e. $\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. In [2], Neumann uses the properties of logarithmically convex functions in order to establish some inequalities for this function. Firstly, he gives the following definition.

Definition 1. *The function ϕ is defined by*

$$\phi \equiv \phi(a,b,x) = \left[\frac{f(a+x)}{f(b+x)}\right]^{\frac{1}{a-b}}, \qquad a+x,\ b+x \in D,\ a \neq b \tag{2}$$

where $f: D \to \mathbb{R}^+$, D is subinterval of $\mathbb{R}$ and f is log-convex.
Then author obtains the following results.

Proposition 2. *The function ϕ increases with an increase in either a or b.*

Proposition 3. *If the function f is continuously differentiable on D, the following double sided inequalities*

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$$(a-b)\frac{f'(b+x)}{f(b+x)} \le \ln\frac{f(a+x)}{f(b+x)} \le (a-b)\frac{f'(a+x)}{f(a+x)} \quad (3)$$

holds true for $a+x, b+x \in D, a \neq b$ and using $f(x) = \Gamma(x)$ in (3) yields

$$(a-b)\psi(b+x) \le \ln\frac{\Gamma(a+x)}{\Gamma(b+x)} \le (a-b)\psi(a+x). \quad (4)$$

for $a+x, b+x \in \mathbb{R}^+$, $a \neq b$.

The inequalities (4) are generalizations of the inequalities (1). Author in [2] also shows the following lemma.

Lemma 4. *Let $g: \mathbb{R}^+ \to \mathbb{R}$, and let $x_i > 0$, $1 \le i \le n$. If the function $\frac{g(x)}{x}$ is increasing on $\mathbb{R}^+$, then*

$$\sum_{i=1}^{n} g(x_i) \le g(\sigma) \quad (5)$$

where $\sigma = x_1 + x_2 + \cdots x_n$.

Inequality (5) is reversed if $\frac{g(x)}{x}$ is decreasing on $\mathbb{R}^+$.

Diaz and Pariguan define Pochhammer k-symbol and k-gamma function in [3]:

Definition 5. *Let $x \in \mathbb{C}$, $k \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. The Pochhammer k-symbol is given by*

$$(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k).$$

k-Analogue of gamma function is defined by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}$$

for $x \in \mathbb{C}\backslash k\mathbb{Z}^-$ and $k > 0$. Its integral representation is given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt$$

for $x \in \mathbb{C}$, $Re(x) > 0$.

They obtain several results that are generalizations of the classical gamma function. The next proposition is some of their results that will be used later in this paper.

Proposition 6. *The k-gamma function $\Gamma_k(x)$ satisfies the following properties:*

i. $\Gamma_k(x+k) = x\Gamma_k(x)$, (6)

ii. $\Gamma_k(k)=1,$ (7)

iii. $\Gamma_k(x)$ *is logarithmically convex for* $x \in \mathbb{R}$, (8)

iv. $$\frac{1}{\Gamma_k(x)} = xk^{-\frac{x}{k}} e^{\frac{x}{k}\gamma} \prod_{n=1}^{\infty} \left(\left(1+\frac{x}{nk}\right) e^{-\frac{x}{nk}} \right) \tag{9}$$

where γ *denotes the Eular's constant defined as* $\gamma = \lim_{n\to\infty} \left(1+\cdots+\frac{1}{n}+\ln n\right)$,

v. $$\Gamma_s(x) = \left(\frac{s}{k}\right)^{\frac{x}{s}-1} \Gamma_k\left(\frac{kx}{s}\right). \tag{10}$$

By using the equation (9), Krasniqi in [4] obtains the following series representations of k-digamma function and k-polygamma function as

$$\psi_k(x) = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{nk(x+nk)} \tag{11}$$

and

$$\psi_k^{(n)}(x) = (-1)^{n+1} n! \sum_{i=1}^{\infty} \frac{1}{(x+ik)^{n+1}} \tag{12}$$

respectively for $n = 1,2, ...$ The author also proves the following lemma and theorem in [4].

Lemma 7. *The function* $\psi_k'(x)$ *is strictly monotonic on* $(0,\infty)$.

Theorem 8. *Let* $x > 0, y \geq 0$ *be real numbers and* $k \geq 1$. *Then the function*

$$f(x) = \frac{[\Gamma_k(x+y+1)/\Gamma_k(y+1)]^{\frac{1}{x}}}{x+y+1}$$

is strictly decreasing on $(0,\infty)$.

Our motivation for this study is to give k-generalizations of the results obtained by Gautschi [1] and Neumann [2]. Then we give some monotonicity properties and obtain some inequalities for summations and multiplications of k-gamma function.

2. MAIN RESULTS

Firstly, we will prove k-generalization of Gautschi inequalities (1).

Theorem 9. *Let* n *be a positive integer,* k *be a positive real number and* $0 \leq s < k$. *Then we have*

$$e^{(s-k)\psi_k(n+k)} \leq \frac{\Gamma_k(n+s)}{\Gamma_k(n+k)} \leq n^{\frac{s-k}{k}}. \tag{13}$$

Proof. Let us define the function f by

$$f(s)=\frac{1}{k-s}\ln\frac{\Gamma_k(n+s)}{\Gamma_k(n+k)}$$

for $0\leq s<k$. By using the equation (6), we have $f(0)=\frac{1}{k}\ln\frac{\Gamma_k(n)}{\Gamma_k(n+k)}=-\frac{\ln n}{k}$ and using L'Hopital's rule leads us to

$$\lim_{s\to k} f(s)=-\lim_{s\to k}\psi_k(\mathrm{n}+\mathrm{s})=-\psi_k(n+k).$$

It is sufficient to show that the function f is monotonically decreasing. By differentiation of f with respect to s, we get

$$(k-s)f'(s)=f(s)+\psi_k(n+s).$$

Let us consider

$$\phi(s)=(k-s)\left[f(s)+\psi_k(n+s)\right].$$

Then, $\phi(k)=0$ and $\phi(0)=k\left(\psi_k(n)-\frac{\ln n}{k}\right)$. Using logarithmic derivative of the equation (10) leads us to

$$\psi_k(x)=\frac{\ln k}{k}+\frac{1}{k}\psi\left(\frac{x}{k}\right).$$

Then we obtain $\phi(0)=\psi\left(\frac{n}{k}\right)-\ln\left(\frac{n}{k}\right)$. Since $\psi(x)-\ln x$ is negative on $(0,\infty)$, we get $\phi(0)<0$ and since $\phi'(s)=(k-s)\psi_k'(n+s)$, we have $\psi_k'(n+s)>0$, from equation (12) and Lemma 7. It follows that $\phi(s)<0$. Hence for $0<s<k$, we obtain $f'(s)=\frac{f(s)+\psi_k(n+s)}{k-s}<0$. Thus we get

$$-\psi_k(n+k)\leq f(s)\leq-\frac{\ln n}{k}.$$

This completes the proof of the theorem.

Before we obtain the monotonicity result on k-gamma function, we need the following lemma.

Lemma 10. *For $x>0$, $k>0$ and $r=1,2,\ldots$ we have*

$$\psi_k(x+k)=\frac{\log k-\gamma}{k}-\sum_{n=1}^{\infty}\left[\frac{1}{x+nk}-\frac{1}{nk}\right]\tag{14}$$

and

$$\psi_k^{(r)}(x+k)=(-1)^{r+1}r!\sum_{n=1}^{\infty}\frac{1}{(x+nk)^{r+1}}.\tag{15}$$

Proof. From (9), we have

$$\ln\Gamma_k(x)+\ln x=\frac{x}{k}\ln k-\frac{x}{k}\gamma-\sum_{n=1}^{\infty}\left(\ln\left(1+\frac{x}{nk}\right)-\frac{x}{nk}\right).$$

Then by using the recurrence formula (6), we have

$$\ln\Gamma_k(x+k)=\frac{x}{k}\ln k-\frac{x}{k}\gamma-\sum_{n=1}^{\infty}\left(\ln\left(1+\frac{x}{nk}\right)-\frac{x}{nk}\right). \tag{16}$$

By differentiating the equation (16), we get the equation (14) as desired. The equation (15) can be obtained from mathematical induction.

Now we can give the following theorem.

Theorem 11. *The function* $\left[\Gamma_k(x+k)\right]^{\frac{1}{x}}$ *is logarithmically concave and increasing for* $x>-k$.

Proof. Let $f(x)=\dfrac{\ln\Gamma_k(x+k)}{x}$. It is sufficient to show that the first derivative of f is positive and the second derivative of f is negative. So from the first derivative of f and the equations (14) and (16) we have

$$\begin{aligned}f'(x)&=\frac{1}{x}\psi_k(x+k)-\frac{1}{x^2}\ln\Gamma_k(x+k)\\&=\frac{1}{x}\left[\frac{\ln k}{k}-\frac{\gamma}{k}-\sum_{n=1}^{\infty}\left[\frac{1}{x+nk}-\frac{1}{nk}\right]\right]-\frac{1}{x^2}\left[\frac{x\ln k}{k}-\frac{x\gamma}{k}-\sum_{n=1}^{\infty}\left[\ln\left(1+\frac{x}{nk}\right)-\frac{x}{nk}\right]\right]\\&=\frac{1}{x^2}\sum_{n=1}^{\infty}\left[-\frac{x}{x+nk}+\frac{x}{nk}-\frac{x}{nk}+\ln\left(1+\frac{x}{nk}\right)\right].\end{aligned}$$

By using the logarithmic expansion (see for example [5])

$$\ln\left(1+\frac{x}{nk}\right)=\sum_{i=1}^{\infty}\frac{1}{i}\left(\frac{x}{x+nk}\right)^i,$$

we obtain

$$f'(x)=\frac{1}{x^2}\sum_{n=1}^{\infty}\sum_{i=2}^{\infty}\frac{1}{i}\left(\frac{x}{x+nk}\right)^i>0.$$

Hence f is an increasing function.

We want to note that since Γ_k is log-convex function, this result can be proved by using Definition 1, letting $f(x)=\Gamma_k(x),\ a=x,\ b=0,\ x=k$,

$$\phi(x,k,0)=\left[\frac{\Gamma_k(x+k)}{\Gamma_k(k)}\right]^{\frac{1}{x}}.$$

Then the proof completes by using Proposition 2.

Now if we take second derivative of f, we get

$$f''(x)=\frac{1}{x}\psi_k'(x+k)-\frac{2}{x^2}\psi_k(x+k)+\frac{2}{x^3}\ln\Gamma_k(x+k)$$
$$=\frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{(x+nk)^2}-\frac{2}{x^2}\left[\frac{\ln k}{k}-\frac{\gamma}{k}-\sum_{n=1}^{\infty}\left[\frac{1}{x+nk}-\frac{1}{nk}\right]\right]+$$
$$+\frac{2}{x^3}\sum_{n=1}^{\infty}\left[\left(\frac{x}{x+nk}\right)^2+\frac{x}{x+nk}-\ln\left(1+\frac{x}{nk}\right)\right]$$
$$=-\frac{2}{x^3}\sum_{n=1}^{\infty}\sum_{i=3}^{\infty}\frac{1}{i}\left(1+\frac{x}{nk}\right)^i<0.$$

Hence by using the equations (14), (15) and (16), we obtain the result, as desired.

In the following theorem, we will give another monotonicity result on k-gamma function.

Theorem 12. *The function* $\frac{\Gamma_k(x+k)^{\frac{k}{x}}}{x}$ *is decreasing on* $(0,\infty)$.

Proof. Let us define the function $f(x)=\frac{\Gamma_k(x+k)^{\frac{k}{x}}}{x}$. Then,

$$\ln f(x)=\frac{k}{x}\ln\Gamma_k(x+k)-\ln x.$$

By differentiating both sides and then multiplying by x^2, we get

$$x^2\frac{f'(x)}{f(x)}=-k\ln\Gamma_k(x+k)+xk\psi_k(x+k)-x.$$

Now let us denote $h(x)=x^2\frac{f'(x)}{f(x)}$. Then,

$$\frac{1}{x}h'(x)=k\psi_k'(x+k)-\frac{1}{x}=k\sum_{i=0}^{\infty}\frac{1}{(x+ik)^2}-\frac{1}{x}$$
$$\leq k\int_0^{\infty}\frac{dt}{(x+tk)^2}-\frac{1}{x}=k\frac{1}{kx}-\frac{1}{x}=0.$$

Since k is positive number, the function h is decreasing. Then for $x>0$, we have $h(x)\leq h(0)=0$. So we obtain $f'(x)\leq 0$ as desired.

Theorem 13. *The following inequalities*

$$\left[\Gamma_k\left(k+\frac{\sigma}{n}\right)\right]^n \le \prod_{i=1}^{n}\Gamma_k\left(k+x_i\right) \le \Gamma_k\left(k+\sigma\right) \tag{17}$$

and

$$\left[\Gamma_k\left(k+\sigma\right)\right]^{\frac{k}{\sigma}} \le \sum_{i=1}^{n}\left[\Gamma_k\left(k+x_i\right)\right]^{\frac{k}{x_i}} \tag{18}$$

are valid for $x_i > 0$, $1 \le i \le n$, where $\sigma = x_1 + x_2 + \ldots x_n$.

Proof. Since $\Gamma_k\left(x\right)$ is a logarithmically convex function, we have

$$\begin{aligned}\Gamma_k\left(k+\frac{\sigma}{n}\right) &= \Gamma_k\left(\frac{k}{n}+\frac{x_1}{n}+\frac{k}{n}+\frac{x_2}{n}+\ldots\frac{k}{n}+\frac{x_n}{n}\right)\\ &\le \Gamma_k\left(k+x_1\right)^{1/n}\ldots\Gamma_k\left(k+x_n\right)^{1/n}\\ &= \prod_{i=1}^{n}\Gamma_k\left(k+x_i\right)^{1/n}.\end{aligned}$$

Hence we obtain the left side of the inequality (17). For the right side of the first inequality, we consider the fact that for $x > -k$, the function $\Gamma_k\left(x+k\right)^{1/x}$ is increasing by Theorem 11. So for $x > -k$, the function

$$\ln\Gamma_k\left(x+k\right)^{1/x} = \frac{\ln\Gamma_k\left(x+k\right)}{x}$$

is also an increasing function. Let us take $\ln\Gamma_k\left(x+k\right)$ instead of $g(x)$ in lemma 4. We get

$$\sum_{i=1}^{n}\log\Gamma_k\left(x_i+k\right) \le \log\Gamma_k\left(\sigma\right),$$

since $\frac{g(x)}{x}$ is increasing for $x > -k$. Then we can write

$$\prod_{i=1}^{n}\Gamma_k\left(x_i+k\right) \le \Gamma_k\left(\sigma+k\right).$$

In Theorem 12, we see that the function $f\left(x\right) = \frac{\Gamma_k\left(x+k\right)^{\frac{k}{x}}}{x}$ is decreasing for $x > 0$. By Lemma 4, we obtain the inequality (18).

Corollary 14. *For non-negative real numbers x, k and positive integer n, the inequalities*

$$n^{-nx} \le \frac{\Gamma_k\left(x+k\right)^n}{\Gamma_k\left(nx+k\right)} \le 1 \tag{19}$$

and

$$\sqrt[n]{\frac{k}{n^k}} \le \Gamma_k\left(k+\frac{k}{n}\right) \le k^{\frac{1}{n}} \tag{20}$$

hold true.

Proof. From the inequalities (17) and (18), we have

$$\prod_{i=1}^{n}\Gamma_k\left(k+x_i\right) \le \Gamma_k\left(k+\sigma\right) \le \left[\sum_{i=1}^{n}\left[\Gamma_k\left(k+x_i\right)\right]^{\frac{k}{x_i}}\right]^{\frac{\sigma}{k}}$$

for $x_i \ge 0$ and $1 \le i \le n$. If we take $x_1 = x_2 = \cdots x_n = x > 0,$ the last double-sided inequalities become

$$[\Gamma_k\left(x+k\right)]^n \le \Gamma_k\left(k+nx\right) \le n^{nx}[\Gamma_k\left(x+k\right)]^n.$$

Hence we obtain the inequality (19). Letting $x = \frac{k}{n}$ in the inequality (19), we get the equation (20).

At last, we want to note that all the results in this work tend to the ones in [2] as $k \to 1$.

3. CONCLUSIONS

In this study we give some inequalities for the ratios of k-analogue of gamma function, which generalize the result obtained by Gautschi. Then we find some monotonicity results for k-gamma function. By the aid of these results, we also get some inequalities involving the k-gamma function. The researchers interested in this field can generalize and find different results by using these monotonicity properties and inequalities.

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