



PRIME IDEALS OF NEARNESS SEMIRINGS

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ABSTRACT. The aim of this paper is to introduce the concept of prime (semiprime) ideals of nearness semiring theory and to introduce some properties of such ideals.

1. INTRODUCTION

In 1982, Pawlak introduced the concept of rough set, which is useful for modeling incompleteness and imprecision in information systems. A subset of a universe in the rough set theory, which is an extension of the set theory, is described by lower and upper approximations. An equivalence relation is a basic notion of the Pawlak rough set model. Iwinski has given an algebraic approach to rough sets [8]. Afterwards, rough subgroups were introduced by Biswas and Nanda [1]. The notion of a rough ideal in a semigroup was introduced by Kuroki [9]. Since then, the subject has been investigated in many papers ([2], [3], [10], [23]).

In 2002, Peters introduced near set theory, which is a generalization of rough set theory (see [17] and [18]). In this theory, Peters defined a indiscernibility relation that depends on the features of objects in order to define their nearness [21]. In his latest work, he took into consideration generalized approach theory in the work of the nearness of non-empty sets which are similar to each other [19], [20], [22].

In 2012, firstly İnan and Öztürk investigated the concept of nearness groups [5, 6] as well as and other algebraic approaches of near sets in [7], [11], [12], [13], [14], [15].

Recently, Öztürk [16] established nearness semiring theory which is a generalization of semiring theory (see [4]) and analyzed some properties of nearness semirings and ideals.

In this paper, the concept of prime (semiprime) ideals of nearness semiring theory is introduced and some properties of such ideals are given.

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2. PRELIMINARIES

For an object $x \in X$, an object description is specified via a tuple of function values $\Phi(x)$. Assume that $B \subseteq \mathcal{F}$ is a set of functions representing properties of sample objects $X \subseteq \mathcal{O}$. Take $\varphi_i \in B$, where $\varphi_i : \mathcal{O} \rightarrow \mathbb{R}$. The functions standing for object properties supply a basis for an object description $\Phi : \mathcal{O} \rightarrow \mathbb{R}^L$, $\Phi(x) = (\varphi_1(x), \varphi_2(x), \dots, \varphi_L(x))$ a vector holding measurements (returned values) linked with each functional value $\varphi_i(x)$, with description length $|\Phi| = L$, where L is a positive integer ([19]).

Sample objects $X \subseteq \mathcal{O}$ are near each other if and only if the objects have similar descriptions. The selection of functions $\varphi_i \in B$ used to specify an object of interest is very significant to consider. Recall that each φ provides a description of an object. So, let Δ_{φ_i} denote $\Delta_{\varphi_i} = |\varphi_i(x') - \varphi_i(x)|$, where $x', x \in \mathcal{O}$. Peters investigated the difference φ that leads to a description of the indiscernibility relation “ \sim_B ” [19].

Definition 1. ([19]) Let $x, x' \in \mathcal{O}, B \subseteq \mathcal{F}$.

$$\sim_B = \{(x, x') \in \mathcal{O} \times \mathcal{O} \mid \Delta_{\varphi_i} = 0 \text{ for all } \varphi_i \in B\}$$

is called the indiscernibility relation on \mathcal{O} , where description length $i \leq |\Phi|$.

The basic idea in the near set approach to object recognition is to compare object descriptions. Sets of objects X, X' are considered near each other, if the sets contain objects with at least partial matching descriptions.

Definition 2. ([19]) Let $X, X' \subseteq \mathcal{O}, B \subseteq \mathcal{F}$. Set X is called near X' if there exist $x \in X, x' \in X', \varphi_i \in B$ such that $x \sim_{\varphi_i} x'$.

<i>Symbol</i>	<i>Interpretation</i>
B	$B \subseteq \mathcal{F}$, set of probe functions,
r	$\binom{ B }{r}$, i.e. , $ B $ probe functions $\varphi_i \in B$ taken r at a time,
B_r	$r \leq B $ probe functions in B ,
\sim_{B_r}	indiscernibility relation defined using B_r ,
$[x]_{B_r}$	$[x]_{B_r} = \{x' \in \mathcal{O} \mid x \sim_{B_r} x'\}$, near equivalence class,
\mathcal{O} / \sim_{B_r}	$\mathcal{O} / \sim_{B_r} = \{[x]_{B_r} \mid x \in \mathcal{O}\} = \xi_{\mathcal{O}, B_r}$, quotient set,
$N_r(B)$	$N_r(B) = \{\xi_{\mathcal{O}, B_r} \mid B_r \subseteq B\}$, set of partitions,
ν_{N_r}	$\nu_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \rightarrow [0, 1]$, overlap function,
$N_r(B)_* X$	$N_r(B)_* X = \bigcup_{[x]_{B_r} \subseteq X} [x]_{B_r}$, lower approximation,
$N_r(B)^* X$	$N_r(B)^* X = \bigcup_{[x]_{B_r} \cap X \neq \emptyset} [x]_{B_r}$, upper approximation,
$Bnd_{N_r(B)}(X)$	$N_r(B)^* X \setminus N_r(B)_* X = \{x \in N_r(B)^* X \mid x \notin N_r(B)_* X\}$.

Table 1 : Symbols of Nearness Approximation Space

A nearness approximation space is a tuple $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$ where the approximation space is defined with a set of perceived objects \mathcal{O} , set of probe functions \mathcal{F} representing object features, \sim_{B_r} indiscernibility relation B_r defined relative to $B_r \subseteq B \subseteq \mathcal{F}$, collection of partitions (families of neighborhoods) $N_r(B)$, and overlap function ν_{N_r} ([19]).

In [15], since $\nu_{N_r} : \wp(\mathcal{O}) \times \wp(\mathcal{O}) \rightarrow [0, 1]$ is not needed, which is overlap function, is not needed when algebraic structures are studied on the nearness approximation space $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r, \nu_{N_r})$, the following definition was given.

Definition 3. ([15]) *Let \mathcal{O} be a set of perceived objects, \mathcal{F} be a set of the probe functions, \sim_{B_r} be an indiscernibility relation, and $N_r(B)$ be a collection of partitions. Then, $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ is called a weak nearness approximation space.*

Theorem 1. ([15]) *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ be a weak nearness approximation space and $X, Y \subset \mathcal{O}$, then the following statements hold;*

- i) $N_r(B)_* X \subseteq X \subseteq N_r(B)^* X$,
- ii) $N_r(B)^* (X \cup Y) = (N_r(B)^* X) \cup (N_r(B)^* Y)$,
- iii) $N_r(B)_* (X \cap Y) = (N_r(B)_* X) \cap (N_r(B)_* Y)$,
- iv) $X \subseteq Y$ implies $N_r(B)_* X \subseteq N_r(B)_* Y$,
- v) $X \subseteq Y$ implies $N_r(B)^* X \subseteq N_r(B)^* Y$,
- vi) $N_r(B)_* (X \cup Y) \supseteq (N_r(B)_* X) \cup (N_r(B)_* Y)$,
- vii) $N_r(B)^* (X \cap Y) \subseteq (N_r(B)^* X) \cap (N_r(B)^* Y)$.

Definition 4. ([16]) *Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r)$ be a weak nearness approximation space and $S \subset \mathcal{O}$. S is called a semiring on \mathcal{O} if the following properties are satisfied:*

- NSR_1) $(S, +)$ is an abelian monoid on \mathcal{O} with identity element 0,
- NSR_2) (S, \cdot) is a monoid on \mathcal{O} with identity element 1,
- NSR_3) For all $x, y, z \in S$,

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z) \text{ and } (x + y) \cdot z = (x \cdot z) + (y \cdot z)$$

- hold in $N_r(B)^* S$,
- NSR_4) For all $x \in S$,

$$0 \cdot x = 0 = x \cdot 0$$

- hold in $N_r(B)^* S$,
- NSR_5) $1 \neq 0$.

Lemma 1. ([16]) *Let $(S, +, \cdot)$ be a nearness semiring. If \sim_{B_r} is a congruence indiscernibility relation on S , then $[x]_{B_r} + [y]_{B_r} \subseteq [x+y]_{B_r}$ and $[x]_{B_r} \cdot [y]_{B_r} \subseteq [x \cdot y]_{B_r}$ for all $x, y \in S$.*

Definition 5. ([16]) *Let $(S, +, \cdot)$ be a nearness semiring, $B_r \subseteq \mathcal{F}$, where $r \leq |B|$ and $B \subseteq \mathcal{F}$, \sim_{B_r} be an indiscernibility relation on weak nearness approximation space \mathcal{O} . Then, \sim_{B_r} is called a complete congruence indiscernibility relation on nearness semiring S if $[x]_{B_r} + [y]_{B_r} = [x+y]_{B_r}$ and $[x]_{B_r} \cdot [y]_{B_r} = [x \cdot y]_{B_r}$ for all $x, y \in S$.*

Let $(S, +, \cdot)$ be a nearness semiring. Let $X + Y = \{x + y \mid x \in X \text{ and } y \in Y\}$ and $X \cdot Y = \left\{ \sum_{\text{finite}} x_i \cdot y_i \mid x_i \in X \text{ and } y_i \in Y \right\}$, where subsets X and Y of S .

Definition 6. ([16]) Let $(S, +, \cdot)$ be a nearness semiring, and A be a subsemiring of S , where $A \neq S$.

- i) A is called a right (left) ideals of S if $A \cdot S \subseteq N_r(B)^* A$ ($S \cdot A \subseteq N_r(B)^* A$).
- ii) A is called an upper-near right (left) ideals of S if $(N_r(B)^* A) \cdot S \subseteq N_r(B)^* A$ ($S \cdot (N_r(B)^* A) \subseteq N_r(B)^* A$).

Theorem 2. ([16]) Let $(S, +, \cdot)$ be a nearness semiring. The following properties hold:

- i) if $\emptyset \neq A \subseteq S$, $A + A \subseteq A$ and $A \cdot A \subseteq A$, then A is an upper-near right (left) ideal of S .
- ii) if A is a right (left) ideal of S , and $N_r(B)^* (N_r(B)^* A) = N_r(B)^* A$, then A is an upper-near right (left) ideal of S .

Theorem 3. ([16]) Let $(S, +, \cdot)$ be a nearness semiring, $\{A_i \mid i \in I\}$ be a set of ideals of S , where an arbitrary index set I .

- i) If $N_r(B)^* \left(\bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} N_r(B)^* A_i$, then $\bigcap_{i \in I} A_i$ is a ideal of S .
- ii) $\bigcup_{i \in I} A_i$ is a ideal of S .

For other notions and definitions not mentioned in this paper, the readers are referred to [4], [19], [20], [7], [15], and [16].

3. PRIME IDEALS OF NEARNESS SEMIRINGS

Definition 7. Let S be a nearness semiring and P be an ideal of S . P is called a prime (resp. semiprime) ideal of S if $A_1 \cdot A_2 \subseteq N_r(B)^* P$ ($A^2 = A \cdot A \subseteq N_r(B)^* P$) implies $A_1 \subseteq P$ or $A_2 \subseteq P$ (resp. $A \subseteq P$) for any ideals A_1 and A_2 of S (resp. for any ideal A of S).

Definition 8. Let S be a nearness semiring and P be an ideal of S . P is called an upper-near prime (resp. semiprime) ideal of S if $(N_r(B)^* A_1) \cdot (N_r(B)^* A_2) \subseteq N_r(B)^* P$ (resp. $(N_r(B)^* A) \cdot (N_r(B)^* A) \subseteq N_r(B)^* P$) implies $N_r(B)^* A_1 \subseteq P$ or $N_r(B)^* A_2 \subseteq P$ (resp. $N_r(B)^* A \subseteq P$) for any ideals A_1 and A_2 of S (resp. for any ideal A of S).

Theorem 4. Let S be a nearness semiring, A_1, A_2 and P are ideals of S such that $N_r(B)^* (N_r(B)^* A_1) = N_r(B)^* A_1$, $N_r(B)^* (N_r(B)^* A_2) = N_r(B)^* A_2$ and $N_r(B)^* (N_r(B)^* P) = N_r(B)^* P$, respectively. If P is a prime ideals and $(N_r(B)^* A_1) \cdot (N_r(B)^* A_2) \subseteq N_r(B)^* P$, then P is an upper-near prime ideal of S .

Proof. Since P is a prime ideal of S such that $N_r(B)^* (N_r(B)^* P) = N_r(B)^* P$, P is an upper-near ideal of S by Theorem 2.(ii). Suppose that $(N_r(B)^* A_1) \cdot$

$(N_r(B)^*A_2) \subseteq N_r(B)^*P$ such that $N_r(B)^*A_1 \not\subseteq P$ or $N_r(B)^*A_2 \not\subseteq P$. Then, there exists an element $x \in N_r(B)^*A_1$ such that $x \notin P$ and $y \in N_r(B)^*A_2$ such that $y \notin P$. From here, $[x]_{B_r} \cap A \neq \emptyset$ and $[y]_{B_r} \cap B \neq \emptyset \Rightarrow a_1 \in [x]_{B_r}, a_1 \in A_1$ and $a_2 \in [y]_{B_r}, a_2 \in A_2 \Rightarrow x \sim_{B_r} a_1, a_2 \in A_1$ and $y \sim_{B_r} a_2, a_2 \in A_2$. Since \sim_{B_r} is a congruence indiscernibility relation on S , and A_1 and A_2 are ideals of S , we have $xy \sim_{B_r} a_1a_2, a_1a_2 \in (N_r(B)^*A_1) \cap (N_r(B)^*A_2)$. Thus, $[xy]_{B_r} \cap ((N_r(B)^*A_1) \cap (N_r(B)^*A_2)) \neq \emptyset \Rightarrow xy \in N_r(B)^*((N_r(B)^*A_1) \cap (N_r(B)^*A_2)) \Rightarrow xy \in N_r(B)^*(N_r(B)^*A_1)$ and $xy \in N_r(B)^*(N_r(B)^*A_2)$ by Theorem 1.(vii). From hypothesis, we get $xy \in N_r(B)^*A_1$ and $xy \in N_r(B)^*A_2 \Rightarrow (xy)^2 = (xy)(xy) \in (N_r(B)^*A_1) \cdot (N_r(B)^*A_2)$, and so $(xy)^2 \in N_r(B)^*P$. Since P is a prime ideal, $xy \in P$ which is a contradiction. Hence, either $N_r(B)^*A_1 \subseteq P$ or $N_r(B)^*A_2 \subseteq P$. \square

We give the following theorem without the proof that is similar to the above proof.

Theorem 5. *Let S be a nearness semiring, A and P are ideals of S such that $N_r(B)^*(N_r(B)^*A) = N_r(B)^*A$ and $N_r(B)^*(N_r(B)^*P) = N_r(B)^*P$, respectively. If P is a semiprime ideals and $(N_r(B)^*A) \cdot (N_r(B)^*A) \subseteq N_r(B)^*P$, then P is an upper-near semiprime ideal of S .*

Let A be a non-empty subset of nearness semiring S and $s \in S$. Let $s \cdot A = \{ \sum_{finite} sa_i \mid a_i \in A \}$.

Lemma 2. *Let S be a nearness semiring. Then $a \cdot S$ is a right ideal of S for any $a \in S$.*

Proof. Let $x, y \in a \cdot S$. In this case, $x = \sum_{i=1} as_i ; s_i \in S$ and $y = \sum_{i=1} as'_i ; s'_i \in S$. Thus, $x + y = \sum_{i=1} as_i + \sum_{i=1} as'_i = \sum_{i=1} a\hat{s}_i = a \sum_{i=1} \hat{s}_i \in a \cdot (N_r(B)^*S)$ for all $\hat{s} \in S$. There exists $z \in N_r(B)^*S$ such that $x + y = az$ for any $a \in S, z \in N_r(B)^*S$. Then $[z]_{B_r} \cap S \neq \emptyset \Rightarrow c \in [z]_{B_r}, c \in S \Rightarrow z \sim_{B_r} c, c \in S$. Since \sim_{B_r} is a congruence indiscernibility relation on S , we get $az \sim_{B_r} ac, c \in S \Rightarrow ac \in [az]_{B_r}$ and $ac \in a \cdot S \Rightarrow [az]_{B_r} \cap (a \cdot S) \neq \emptyset$, so we obtain $x + y = az \in N_r(B)^*(a \cdot S)$, namely, $a \cdot S + a \cdot S \subseteq N_r(B)^*(a \cdot S)$.

Now let $x \in a \cdot S, s \in S$. Thus, $x = \sum_{i=1} as_i ; s_i \in S$. Therefore, $xs = (\sum_{i=1} as_i)s = \sum_{i=1} (as_i)s = \sum_{i=1} a(s_i s) = a \sum_{i=1} s_i s \in a \cdot (N_r(B)^*S)$, and so there exists $c \in N_r(B)^*S$ such that $xs = ac$ for all $s \in S, c \in N_r(B)^*S$. Therefore $[c]_{B_r} \cap S \neq \emptyset \Rightarrow z \in [c]_{B_r}, z \in S \Rightarrow c \sim_{B_r} z, z \in S$. Since \sim_{B_r} is a congruence indiscernibility relation on S , we get that $ac \sim_{B_r} az, z \in S \Rightarrow az \in [ac]_{B_r}$ and $az \in a \cdot S \Rightarrow [ac]_{B_r} \cap (a \cdot S) \neq \emptyset$, so we have $xs = ac \in N_r(B)^*(a \cdot S)$. Thus, we obtain $(a \cdot S) \cdot S \subseteq N_r(B)^*(a \cdot S)$. \square

In general, the intersection of ideals of the nearness semiring S is not an ideal, as shown in the following.

Example 1. Let $\mathcal{O} = \{0, 1, a, b, c, d, e, f, g, h, i, j, k, n\}$ be a set of perceptual objects where

$$\begin{aligned} 0 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\ c &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, d = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \\ g &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, h = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, j = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \\ k &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, l = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, m = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, n = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

for $U = \{ [a_{ij}]_{2 \times 2} \mid a_{ij} \in \mathbb{Z}_2 \}$, $r = 1$, $B = \{\varphi_1, \varphi_2, \varphi_3\} \subseteq \mathcal{F}$ be a set of probe functions, $S = \{a, b, d, e, h\} \subset \mathcal{O}$, $A_1 = \{a, h\} \subseteq S$ and $A_2 = \{b, h\} \subseteq S$. Values of the probe functions

$$\begin{aligned} \varphi_1 : \mathcal{O} &\rightarrow V_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}, \\ \varphi_2 : \mathcal{O} &\rightarrow V_2 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6\}, \\ \varphi_3 : \mathcal{O} &\rightarrow V_3 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} \end{aligned}$$

are given in Table 2.

	0	1	a	b	c	d	e	f	g	h	i	j	k	n
φ_1	α_1	α_2	α_1	α_3	α_1	α_3	α_4	α_3	α_4	α_3	α_1	α_4	α_5	α_5
φ_2	α_3	α_3	α_4	α_3	α_1	α_1	α_4	α_3	α_4	α_4	α_3	α_4	α_6	α_6
φ_3	α_3	α_3	α_1	α_1	α_4	α_4	α_6	α_1	α_3	α_6	α_3	α_6	α_5	α_6

Table 2

Let us now determine the near equivalence classes according to the indiscernibility relation \sim_{B_r} of elements in \mathcal{O} :

$$\begin{aligned} [0]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(0) = \alpha_1\} = \{0, a, c, i\} \\ &= [a]_{\varphi_1} = [c]_{\varphi_1} = [i]_{\varphi_1}, \\ [1]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(1) = \alpha_2\} = \{1\}, \\ [b]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(b) = \alpha_3\} = \{b, d, f, h\} \\ &= [d]_{\varphi_1} = [f]_{\varphi_1} = [h]_{\varphi_1}, \\ [e]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(e) = \alpha_4\} = \{e, g, j\}, \\ &= [g]_{\varphi_1} = [j]_{\varphi_1}, \\ [k]_{\varphi_1} &= \{x \in \mathcal{O} \mid \varphi_1(x) = \varphi_1(k) = \alpha_5\} = \{k, n\} \\ &= [n]_{\varphi_1}. \end{aligned}$$

Then, we get $\xi_{\varphi_1} = \{[0]_{\varphi_1}, [1]_{\varphi_1}, [b]_{\varphi_1}, [e]_{\varphi_1}, [k]_{\varphi_1}\}$.

$$\begin{aligned} [0]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(0) = \alpha_3\} = \{0, 1, b, f, i\} \\ &= [1]_{\varphi_2} = [b]_{\varphi_2} = [f]_{\varphi_2} = [i]_{\varphi_2}, \\ [a]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(a) = \alpha_4\} = \{a, e, g, h, j\} \\ &= [e]_{\varphi_2} = [g]_{\varphi_2} = [h]_{\varphi_2} = [j]_{\varphi_2}, \\ [c]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(\gamma) = \alpha_1\} = \{c, d\} \\ &= [d]_{\varphi_2}, \\ [k]_{\varphi_2} &= \{x \in \mathcal{O} \mid \varphi_2(x) = \varphi_2(k) = \alpha_6\} = \{k, n\}, \\ &= [n]_{\varphi_2}. \end{aligned}$$

Thus, we have $\xi_{\varphi_2} = \{[0]_{\varphi_2}, [a]_{\varphi_2}, [c]_{\varphi_2}, [k]_{\varphi_2}\}$.

$$\begin{aligned} [0]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(0) = \alpha_3\} = \{0, 1, g, i\} \\ &= [1]_{\varphi_3} = [g]_{\varphi_3} = [i]_{\varphi_3}, \\ [a]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(a) = \alpha_1\} = \{a, b, f\} \\ &= [b]_{\varphi_3} = [f]_{\varphi_3}, \\ [c]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(c) = \alpha_4\} = \{c, d\} \\ &= [d]_{\varphi_3}, \\ [e]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(e) = \alpha_6\} = \{e, h, j, n\} \\ &= [h]_{\varphi_2} = [j]_{\varphi_3} = [n]_{\varphi_3}, \\ [k]_{\varphi_3} &= \{x \in \mathcal{O} \mid \varphi_3(x) = \varphi_3(k) = \alpha_5\} = \{k\}. \end{aligned}$$

Hence, we obtain $\xi_{\varphi_3} = \{[0]_{\varphi_3}, [a]_{\varphi_3}, [c]_{\varphi_3}, [e]_{\varphi_3}, [k]_{\varphi_3}\}$. Therefore, for $r = 1$, a set of partitions of \mathcal{O} is $N_r(B) = \{\xi_{\varphi_1}, \xi_{\varphi_2}, \xi_{\varphi_3}\}$. Then we can write

$$\begin{aligned} N_1(B)^* S &= \bigcup_{[x]_{\varphi_i} \cap S \neq \emptyset} [x]_{\varphi_i} \\ &= [0]_{\varphi_1} \cup [a]_{\varphi_1} \cup [e]_{\varphi_1} \cup [a]_{\varphi_2} \cup [c]_{\varphi_2} \cup [a]_{\varphi_3} \cup [c]_{\varphi_3} \cup [e]_{\varphi_3} \\ &= \{0, a, b, c, d, e, f, g, h, i, j, n\}. \end{aligned}$$

Considering the following table of operation:

+	a	b	d	e	h
a	0	e	h	b	d
b	e	0	i	b	f
d	h	i	0	j	a
e	b	a	j	0	i
h	d	j	a	i	0

In that case, $(S, +)$ is an abelian monoid on \mathcal{O} with identity element 0. Considering the following table of operation:

·	a	b	d	e	h
a	a	0	d	a	h
b	b	0	c	0	d
d	0	a	0	a	0
e	e	0	g	e	n
h	a	e	0	e	0

Then (S, \cdot) is a semigroup on \mathcal{O} . Also, $(S, +, \cdot)$ satisfies conditions (NSR_3) , (NSR_4) and (NSR_5) . Therefore, $(S, +, \cdot)$ is a semiring on the weak nearness approximation space \mathcal{O} by Definition 4, i.e., $(S, +, \cdot)$ is a Γ -nearness semiring. Moreover,

$$\begin{aligned}
 N_1(B)^* A_1 &= \bigcup_{[x]_{\varphi_i} \cap A_1 \neq \emptyset} [x]_{\varphi_i} \\
 &= \{0, a, b, c, d, f, g, h, i, j, n\}.
 \end{aligned}$$

Considering the above table of operations, A_1 is an ideal of S .

$$\begin{aligned}
 N_1(B)^* A_2 &= \bigcup_{[x]_{\varphi_i} \cap A_2 \neq \emptyset} [x]_{\varphi_i} \\
 &= \{0, a, b, c, d, e, f, g, h, i, j\}.
 \end{aligned}$$

Similarly, A_2 is an ideal of S . Also,

$$\begin{aligned}
 N_1(B)^* (A_1 \cap A_2) &= \bigcup_{[x]_{\varphi_i} \cap (A_1 \cap A_2) \neq \emptyset} [x]_{\varphi_i} \\
 &= \{a, b, d, e, f, g, h, j, n\}.
 \end{aligned}$$

In this case, let $h \in A_1 \cap A_2 = \{h\}$, and so $h + h = 0 \notin N_1(B)^* (A_1 \cap A_2)$. Therefore, $A_1 \cap A_2$ is not an ideal of S by Definition 6. Furthermore, considering

the following tables of operations:

+	0	a	b	c	d	e	f	g	h	i	j	n
0	0	a	b	c	d	e	f	g	h	i	j	n
a	a	0	e	1	h	b	k	m	d	j	i	l
b	b	e	0	f	i	a	c	l	j	d	h	m
c	c	1	f	0	g	k	b	d	m	l	n	j
d	d	h	i	g	0	j	l	c	a	b	e	k
e	e	b	a	k	j	0	1	n	i	h	d	g
f	f	k	c	b	l	1	0	i	n	g	m	h
g	g	m	l	d	l	n	i	0	1	f	k	e
h	h	d	j	m	a	i	n	1	0	e	b	f
i	i	j	d	l	b	h	f	f	e	0	a	1
j	j	i	h	n	e	d	m	k	b	a	0	c
n	n	l	m	j	k	g	h	e	f	1	c	0

and

·	0	a	b	c	d	e	f	g	h	i	j	n
0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	a	0	0	d	a	0	d	h	d	h	h
b	0	b	0	0	c	b	0	c	f	c	f	f
c	0	0	b	c	0	b	f	c	0	b	b	f
d	0	0	a	d	0	a	h	d	0	a	a	h
e	0	e	0	0	g	e	0	g	n	g	n	n
f	0	b	b	c	c	0	f	0	f	f	c	0
g	0	0	b	g	0	e	n	g	0	e	e	n
h	0	a	e	g	0	e	n	g	0	h	e	n
i	0	b	a	d	c	e	h	g	f	1	k	n
j	0	e	a	d	g	b	h	c	n	m	l	f
n	0	e	e	g	g	0	n	0	n	n	g	0

(NSR₁), (NSR₂), (NSR₃), (NSR₄), and (NSR₅) properties have to hold in $N_r(B)^*S$ for all elements of S . However, sum or multiplication of elements in $N_r(B)^*S$ may not always belong to $N_r(B)^*S$ (or \mathcal{O}). For instance, $d + f = l \notin \mathcal{O}$ for $d, f \in N_r(B)^*S$, $a + c = 1 \notin N_r(B)^*S$ for $a, c \in N_r(B)^*S$, $j \cdot j = l \notin N_r(B)^*S$ for $j \in N_r(B)^*S$.

Theorem 6. Let S be a nearness semiring and $\{P_i \mid i \in I\}$ be a set of prime (resp. semiprime) ideals of S where an arbitrary index set I .

i) If $N_r(B)^* \left(\bigcap_{i \in I} P_i \right) = \bigcap_{i \in I} N_r(B)^* P_i$, then $\bigcap_{i \in I} P_i$ is a prime (resp. semiprime) ideal of S .

ii) $\bigcup_{i \in I} P_i$ is a prime (resp. semiprime) ideal of S .

Proof. i) From Theorem 3.(i), we get $\bigcap_{i \in I} P_i$ is an ideal of S . Let $A_1 \cdot A_2 \subseteq N_r(B)^* \left(\bigcap_{i \in I} P_i \right)$ for any two ideals A_1 and A_2 of S . Then, $A_1 \cdot A_2 \subseteq \bigcap_{i \in I} N_r(B)^* P_i$ by hypothesis, and hence $A_1 \cdot A_2 \subseteq N_r(B)^* P_i$ for all $i \in I$. Since P_i are prime ideals of S for all $i \in I$, $A_1 \subseteq P_i$ or $A_2 \subseteq P_i$ for all $i \in I$. In this case, $A_1 \subseteq \bigcap_{i \in I} P_i$ or $A_2 \subseteq \bigcap_{i \in I} P_i$.

ii) $\bigcup_{i \in I} P_i$ is an ideal of S by Theorem 3.(ii). Let $A_1 \cdot A_2 \subseteq N_r(B)^* \left(\bigcup_{i \in I} P_i \right)$ for any ideals A_1 and A_2 of S . Then, $A_1 \cdot A_2 \subseteq \bigcup_{i \in I} N_r(B)^* P_i$ by Theorem 1.(ii). There is at least one $i_0 \in I$ such that $A_1 \cdot A_2 \subseteq N_r(B)^* P_{i_0}$. Since P_{i_0} are prime ideals of S for $i_0 \in I$, $A_1 \subseteq P_{i_0}$ or $A_2 \subseteq P_{i_0}$ for $i_0 \in I$. Therefore, $A_1 \subseteq \bigcup_{i \in I} P_i$ or $A_2 \subseteq \bigcup_{i \in I} P_i$. \square

Theorem 7. Let S be a nearness semiring and $a, b \in S$. If P is a prime right ideal of S such that $N_r(B)^*(N_r(B)^*P) = N_r(B)^*P$, then $a \cdot S \cdot b \subseteq N_r(B)^*P$ implies $a \in P$ or $b \in P$.

Proof. Let $a \cdot S \cdot b \subseteq N_r(B)^*P$. In this case, we have $(a \cdot S \cdot b) \cdot S \subseteq (N_r(B)^*P) \cdot S \subseteq N_r(B)^*P$ by Theorem 2.(ii). Therefore, by Lemma 2 $a \cdot S$ and $b \cdot S$ are right ideals of S , and since P is prime right ideal of S , $a \cdot S \subseteq P$ or $b \cdot S \subseteq P$. There exists $e \in N_r(B)^*S$ such that $a = ea$ for all $a \in S$. Therefore, either $a \in P$ or $b \in P$. \square

We give the following theorem without the proof.

Theorem 8. Let S be a nearness semiring and $a \in S$. If P is a semiprime right ideal of S such that $N_r(B)^*(N_r(B)^*P) = N_r(B)^*P$, then $a \cdot S \cdot a \subseteq N_r(B)^*P$ implies $a \in P$.

Theorem 9. Let S be a nearness semiring, P be a right ideal of S such that $N_r(B)^*(N_r(B)^*P) = N_r(B)^*P$ and $a, b \in S$. If $a \cdot S \cdot b \subseteq N_r(B)^*P$ implies $a \in P$ or $b \in P$, then P is a prime right ideal of S .

Proof. Let A_1 and A_2 be any two right ideals of S such that $A_1 \cdot A_2 \subseteq N_r(B)^*P$ and $A_1 \not\subseteq P$. Thus, there exists an element $a_1 \in A_1$ such that $a_1 \notin P$. For any $a_2 \in A_2$, we have $a_1 \cdot S \cdot a_2 = (a_1 \cdot S) \cdot a_2 \subseteq (N_r(B)^*A_1) \cdot a_2$. On the other hand, let $x \in (N_r(B)^*A_1) \cdot a_2$ such that $x = \sum_{i=1}^n x_i a_2$; $x_i \in N_r(B)^*A_1$, $a_2 \in A_2$, $1 \leq i \leq n$. Then $x_i \in N_r(B)^*A_1 \Rightarrow [x_i]_{B_r} \cap A_1 \neq \emptyset \Rightarrow c \in [x_i]_{B_r}$, $c \in A_1 \Rightarrow x_i \sim_{B_r} c$, $c \in A_1$, $1 \leq i \leq n$. Since \sim_{B_r} is a congruence indiscernibility relation on S , we get that $x_i a_2 \sim_{B_r} c a_2$, $c a_2 \in A_1 \cdot A_2 \subseteq N_r(B)^*P$, $1 \leq i \leq n$. Since P is a right ideal of S such that $N_r(B)^*(N_r(B)^*P) = N_r(B)^*P$, we get that $\sum_{i=1}^n x_i a_2 \sim_{B_r} \sum_{i=1}^n c a_2$, $\sum_{i=1}^n c a_2 \in N_r(B)^*P$, $1 \leq i \leq n$. Hence, we have $\sum_{i=1}^n c a_2 \in [\sum_{i=1}^n x_i a_2]_{B_r}$ and $\sum_{i=1}^n c a_2 \in$

$N_r(B)^*P$. Therefore, $[\sum_{i=1}^n a_i b]_{B_r} \cap (N_r(B)^*P) \neq \emptyset \Rightarrow [x]_{B_r} \cap (N_r(B)^*P) \neq \emptyset$, so we obtain $x \in N_r(B)^*(N_r(B)^*P) = N_r(B)^*P$, namely, $a_1 \cdot S \cdot a_2 \subseteq N_r(B)^*P$. By hypothesis, we have $a_2 \in P$, and so $A_2 \subseteq P$. This completes the proof. \square

Theorem 10. *Let S be a nearness semiring, P be a right ideal of S such that $N_r(B)^*(N_r(B)^*P) = N_r(B)^*P$, and $a \in S$. If $a \cdot S \cdot a \subseteq N_r(B)^*P$ implies $a \in P$, then P is a semiprime right ideal of S .*

Definition 9. *Let S be a nearness semiring, A_1, A_2 and P be ideals of S . P is called an irreducible (resp. a strongly irreducible) ideal of S if $A_1 \cap A_2 = N_r(B)^*P$ (resp. $A_1 \cap A_2 \subseteq N_r(B)^*P$) implies $A_1 = P$ or $A_2 = P$ (resp. $A_1 \subseteq P$ or $A_2 \subseteq P$).*

Theorem 11. *Let S be a nearness semiring and $\{A_i \mid i \in I\}$ be a set of ideals of S where an arbitrary index set I . If $N_r(B)^* \left(\bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} N_r(B)^*A_i$, then every strongly irreducible and semiprime ideal of S is a prime ideal of S .*

Proof. Let $N_r(B)^* \left(\bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} N_r(B)^*A_i$ for all ideals A_i of S , and P be a strongly irreducible and semiprime ideal of S . Let $A_1 \cdot A_2 \subseteq N_r(B)^*P$ for any ideals A_1 and A_2 of S . Then, $A_1 \cap A_2$ is an ideal of S by Theorem 3.(i). Therefore, $(A_1 \cap A_2)^2 = (A_1 \cap A_2) \cdot (A_1 \cap A_2) \subseteq A_1 \cdot A_2 \subseteq N_r(B)^*P \Rightarrow (A_1 \cap A_2)^2 \subseteq N_r(B)^*P$. Since P is a semiprime ideal of S , we get that $A_1 \cap A_2 \subseteq P$. Thus, $A_1 \cap A_2 \subseteq N_r(B)^*P$ by Theorem 1.(i). We get $A_1 \subseteq P$ or $A_2 \subseteq P$, for P is a strongly irreducible ideal of S . \square

4. CONCLUSION

We have introduced the concept of prime (semiprime) ideals of semiring on weak nearness approximation spaces and we have given some properties of such ideals. One can investigate others properties of nearness semiring. Also, this paper will contribute to the application in several algebraic structures such as prime (semiprime, maximal, etc.) ideals of ring, gamma ring, gamma semiring, and etc. on weak nearness approximation spaces.

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