

An investigation on homomorphisms and subhyperalgebras of Σ -hyperalgebras

S. Rasouli^{*}, D. Heidari[†] and B. Davvaz[‡]

Abstract

In this paper, we introduce the notion of Σ -hyperalgebras for an arbitrary signature Σ and provide some examples. Then we extend the notions of several kinds of homomorphisms and study their properties. Also, we study subhyperalgebras of a Σ -hyperalgebra \mathfrak{A} , $Sub(\mathfrak{A})$, under algebraic closure operators S , H and I . Finally, we introduce the notions of closed, invertible, ultraclosed and conjugable subhyperalgebras and investigate their connections to each other.

2000 AMS Classification: 20N20, 08A05, 08A30, 03C05.

Keywords: universal algebra, hyperalgebra, homomorphism, subhyperalgebra.

Received 13 : 02 : 2012 : Accepted 13 : 10 : 2013 Doi : 10.15672/HJMS.2014437521

1. Introduction

The theory of hypergroups was introduced by F. Marty in 1934, (see [22]). Hyperalgebras (or multialgebras) are particular relational systems which generalize the concept of universal algebras. The hyperstructure theory and its applications so far, have been investigated by many mathematicians in various fields, for example in graphs and hypergraphs theory [4], in categories theory [12, 25] and in n-ary hyperalgebras [8, 21]. Recent book [7] contains wealthy applications. There are also applications to the following subjects: geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, etc.

There are several types of homomorphisms have been considered since the first papers on hypergroups (for instance, by M. Dresher, O. Ore [11], M. Krasner [18], J. Kuntzmann [19]) and later by M. Koskas [17]. However, the first explicit construction of hypergroup

^{*}Department of Mathematics, Persian Gulf University, Bushehr, 75169, IRAN
Email: srasouli@pgu.ac.ir

[†]Department of Mathematics, Najafabad Branch Islamic Azad University Najafabad, Isfahan, Iran

Email:dheidari82@gmail.com

[‡]Department of Mathematics, Yazd University, Yazd, IRAN
Email:davvaz@yazduni.ac.ir

homomorphisms was given by P. Corsini [2]. A unified theory of various types of homomorphisms was given by J. Jantosciak [16]. Some other types of homomorphisms and connections among them were studied by V. Leoreanu [20]. There are more than 10 types of hypergroup homomorphisms. A detailed presentation of all these homomorphisms, connections between them and various examples can be found in [3].

Also, in the hypergroup theory there are several kinds of subhypergroups. Among the mathematicians who studied this topic, we mention F. Marty [22], M. Drescher, O. Ore [11], M. Krasner [18] who analyzed closed and invertible subhypergroups. Later Y. Sureau [32] has studied ultraclosed, invertible and conjugable subhypergroups.

On the other hand, in 1962 G. Grätzer began to study of multialgebras [12] and it follows by H. E. Pickett [26] who investigated homomorphisms and subalgebras of multialgebras. Then H. Höft and P. E. Howard [14] and G. E. Hansoul [15] studied multialgebras and T. Vougiouklis [33] discussed about the representations of hyperstructures. Furthermore, C. Pelea studied the fundamental relation of multialgebras [23] and investigated them on the universal algebras point of view in [24, 25]. Thus it seems adequate to undertake a study of multialgebras in the spirit of universal algebras.

In this article, we introduce the notion of Σ -hyperalgebra, \mathfrak{A} , for an arbitrary signature Σ . Then we generalize notions of various types of homomorphisms and subhyperalgebras of \mathfrak{A} and we study their properties.

2. A brief excursion into hyperalgebras

This section contains a survey of the basic elements of universal algebras and hyperalgebras which will be used in the next sections. In fact, we explain what is meant by a hyperalgebra and then give several examples of familiar hyperalgebras. These examples show that different hyperalgebras may have several common properties. This observation provides a motivation for the study of Σ -hyperalgebras.

In the sequel A is a fixed nonempty set, $\mathcal{P}^*(A)$ is the family of all nonempty subsets of A , w is the set of positive integers and for $n \in w$ we denote the set of n -tuples over A , by A^n . Also, by $B \subseteq_w A$ we mean that B is a finite subset of A .

Let A be a nonempty set. A family \mathcal{C} of subsets of A that is closed under the intersection of arbitrary subfamilies is called a closed-set system over A . If \mathcal{C} is also closed under the union of subfamilies that are directed under inclusion, then it is called an algebraic closed-set system. An algebraic closed set system \mathcal{C} forms an algebraic lattice $(\mathcal{C}, \cap, \vee)$ under set-theoretic inclusion. The closure operator associated with a given closed set system \mathcal{C} is denoted by $Cl_{\mathcal{C}}$.

An n -ary hyperoperation (or function) on A is a function σ from A^n to $\mathcal{P}^*(A)$; n is the arity (or rank) of σ . A finitary hyperoperation is an n -ary hyperoperation, for some n . The image of (a_1, \dots, a_n) under an n -ary hyperoperation σ is denoted by $\sigma(a_1, \dots, a_n)$. A hyperoperation σ on A is called a nullary hyperoperation (or constant) if its arity is zero; in fact, a nullary hyperoperation on A is just an element of $\mathcal{P}^*(A)$; i.e. a nonvoid subset of A . A hyperoperation σ on A is unary, binary or ternary if its arity is 1, 2 or 3, respectively.

A signature or language type is a set Σ together with a mapping $\rho : \Sigma \rightarrow w$. The elements of Σ are called hyperoperation symbols. For each $\sigma \in \Sigma$, $\rho(\sigma)$ is called the arity or rank of σ . In the sequel, for each $n \in w$, $\Sigma_n = \{\sigma \mid \rho(\sigma) = n\}$.

Let Σ be a signature. A Σ -hyperalgebraic structure is an ordered couple $\mathfrak{A} = (A, (\sigma^{\mathfrak{A}} : \sigma \in \Sigma))$, where A is a nonempty set and $\sigma^{\mathfrak{A}}$ is a function from $A^{\rho(\sigma)}$ to $\mathcal{P}^*(A)$, for all $\sigma \in \Sigma$. The set A is called the universe (or underlying set) of \mathfrak{A} and the $\sigma^{\mathfrak{A}}$'s are called the fundamental hyperoperations of \mathfrak{A} . In the following, we prefer to write just σ for $\sigma^{\mathfrak{A}}$ if this convention creates an ambiguity which seldom causes a problem.

In this paper we shall use the following abbreviated notations: the sequence x_i, \dots, x_j will be denoted by x_i^j . For $j < i$ is the empty symbol. In this convention

$$\sigma^{\mathfrak{A}}(x_1, \dots, x_i, y_{i+1}, \dots, y_j, z_{j+1}, \dots, z_{\rho(\sigma)})$$

will be written as $\sigma^{\mathfrak{A}}(x_1^i, y_{i+1}^j, z_{j+1}^{\rho(\sigma)})$. In this case when $y_{i+1} = \dots = y_j = y$ the last expression will be written in the form $\sigma^{\mathfrak{A}}(x_1^i, y^{(j-i)}, z_{j+1}^{\rho(\sigma)})$. Similarly, for subsets $B_1^{\rho(\sigma)}$ of A we define

$$\sigma^{\mathfrak{A}}(B_1^{\rho(\sigma)}) = \cup\{\sigma^{\mathfrak{A}}(b_1^{\rho(\sigma)})|b_i \in B_i, \forall i \in I_{\rho(\sigma)}\}.$$

Also, for each $i \in I_{\rho(\sigma)}$ and $a_1^{\rho(\sigma)} \in A$ we define

$$\sigma_i^{\mathfrak{A}}(a_1^{\rho(\sigma)}) = \{z \in A : a_i \in \sigma^{\mathfrak{A}}(a_1^{i-1}, z, a_{i+1}^{\rho(\sigma)})\}$$

and for each $i \in I_{\rho(\sigma)}$ and $B_1^{\rho(\sigma)} \subseteq A$ we define

$$\sigma_i^{\mathfrak{A}}(B_1^{\rho(\sigma)}) = \cup\{\sigma_i^{\mathfrak{A}}(b_1^{\rho(\sigma)})|b_i \in B_i\}.$$

A hyperalgebra \mathfrak{A} is unary if all hyperoperations are unary and it is mono-unary if it has just one unary hyperoperation. \mathfrak{A} is a hypergroupoid if it has just one binary hyperoperation σ . According to [9], Σ -hyperalgebra \mathfrak{A} is called an n -ary hypergroupoid if $\Sigma = \{\sigma\}$ and $\rho(\sigma) = n$. If σ is a binary hyperoperation, we write $a\sigma b$ for the image of (a, b) under σ . A hyperalgebra \mathfrak{A} is finite if $|A|$ is finite. Let \mathfrak{A} be a Σ -hyperalgebra. Hyperoperation $\sigma^{\mathfrak{A}}$ is called trivial hyperoperation if for any $(a_1^{\rho(\sigma)}) \in A^{\rho(\sigma)}$, we have $|\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)})| = 1$. \mathfrak{A} is called a trivial Σ -hyperalgebra if for any $\sigma \in \Sigma$, $\sigma^{\mathfrak{A}}$ is trivial.

Let $\mathfrak{A} = (A, (\sigma^{\mathfrak{A}} : \sigma \in \Sigma))$ be a Σ -hyperalgebra and $\sigma \in \Sigma$. Now, we extend an n -ary hyperoperation $\sigma^{\mathfrak{A}}$ to an n -ary operation $\sigma^{\mathfrak{A}\mathfrak{A}}$ on $\mathcal{P}^*(A)$ by setting for all $(A_1^{\rho(\sigma)}) \in \mathcal{P}^*(A)^{\rho(\sigma)}$

$$\sigma^{\mathfrak{A}\mathfrak{A}}(A_1^{\rho(\sigma)}) = \bigcup\{\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)})|a_i \in A_i, i \in I_{\rho(\sigma)}\}.$$

It is easy to see that $\mathfrak{A}\mathfrak{A} = (\mathcal{P}^*(A), < \sigma^{\mathfrak{A}\mathfrak{A}} : \sigma \in \Sigma >)$ is a Σ -algebra.

2.1. Definition. Let \mathfrak{A} be a Σ -hyperalgebra, $\sigma \in \Sigma$ and $j < i < \rho(\sigma)$. We say that σ is weakly (i, j) -associative if for each $a_1^{2\rho(\sigma)-1} \in A$ we have

$$\sigma^{\mathfrak{A}}(a_1^{i-1}, \sigma^{\mathfrak{A}}(a_i^{\rho(\sigma)+i-1}, a_{\rho(\sigma)+i}^{2\rho(\sigma)-1})) \cap \sigma^{\mathfrak{A}}(a_1^{j-1}, \sigma^{\mathfrak{A}}(a_j^{\rho(\sigma)+j-1}, a_{\rho(\sigma)+j}^{2\rho(\sigma)-1})) \neq \emptyset,$$

and (i, j) -associative if

$$\sigma^{\mathfrak{A}}(a_1^{i-1}, \sigma^{\mathfrak{A}}(a_i^{\rho(\sigma)+i-1}, a_{\rho(\sigma)+i}^{2\rho(\sigma)-1})) = \sigma^{\mathfrak{A}}(a_1^{j-1}, \sigma^{\mathfrak{A}}(a_j^{\rho(\sigma)+j-1}, a_{\rho(\sigma)+j}^{2\rho(\sigma)-1})).$$

We say that σ is (weakly) associative if it is (weakly) (i, j) -associative for each $i, j \in I_{\rho(\sigma)}$.

Definition 2.1 is a generalization of associative binary hyperoperation. An n -ary hypergroupoid \mathfrak{A} is called an n -ary semihypergroup (n -ary H_v semigroup) if σ is (weakly) associative.

2.2. Examples. (i) [9] Let $A = \{a, b, c\}$ with the hyperoperation \circ defined by the following table

\circ	a	b	c
a	a	b	c
b	b	$\{a, c\}$	$\{a, b\}$
c	c	$\{a, b\}$	$\{a, b\}$

Now, we define the ternary hyperoperation σ on A by $\sigma(x, y, z) = x \circ y \circ z$, for each $x, y, z \in A$. One can see that $\mathfrak{A} = (A, \sigma)$ is a 3-ary semihypergroup.

(ii) Let $A = \{a, b, c\}$ with the hyperoperation \circ defined by the following table

\circ	a	b	c
a	a	b	c
b	b	c	$\{a, b\}$
c	c	b	$\{a, b\}$

It is easy to check that $\mathfrak{A} = (A, \circ)$ is a H_v semigroup but it is not a semihypergroup.

2.3. Definition. [8, 9] Let \mathfrak{A} be a Σ -hyperalgebra, $\sigma \in \Sigma$ and $B \subseteq A$. We say that B has the reproduction axiom with respect to σ if for each $b_1^{i-1}, b, b_{i+1}^{\rho(\sigma)} \in B$ and $1 \leq i \leq \rho(\sigma)$ the relation

$$(2.1) \quad b \in \sigma_i^{\mathfrak{A}}(b_1^{i-1}, x, b_{i+1}^{\rho(\sigma)})$$

has a solution $x \in B$. Observe that condition (1) can be reformulated as follows

$$(2.2) \quad B = \sigma^{\mathfrak{A}}(b_1^{i-1}, B, b_{i+1}^{\rho(\sigma)-i}).$$

An n -ary semihypergroup (H_v semigroup) \mathfrak{A} is called an n -ary hypergroup (H_v group) if A satisfies the reproduction axiom. An n -ary hypergroupoid which satisfies the reproduction axiom is called n -ary hyperquasigroup.

2.4. Examples. (1) [8] Let $A = \{a, b, c\}$ be a set with a 3-ary hyperoperation σ as follows:

$$\begin{array}{lll} \sigma(a, a, a) = a & \sigma(b, b, a) = \{a, c\} & \sigma(c, a, a) = c \\ \sigma(a, a, b) = b & \sigma(b, b, b) = \{b, c\} & \sigma(c, a, b) = \{b, c\} \\ \sigma(a, a, c) = c & \sigma(b, b, c) = A & \sigma(c, a, c) = \{a, b\} \\ \sigma(a, b, a) = b & \sigma(b, a, a) = b & \sigma(c, b, a) = \{b, c\} \\ \sigma(a, b, b) = \{a, c\} & \sigma(b, a, b) = \{a, c\} & \sigma(c, b, b) = A \\ \sigma(a, b, c) = \{b, c\} & \sigma(b, a, c) = \{b, c\} & \sigma(c, b, c) = A \\ \sigma(a, c, a) = c & \sigma(b, c, a) = \{b, c\} & \sigma(c, c, a) = \{a, b\} \\ \sigma(a, c, b) = \{b, c\} & \sigma(b, c, b) = A & \sigma(c, c, b) = A \\ \sigma(a, c, c) = \{a, b\} & \sigma(b, c, c) = A & \sigma(c, c, c) = \{b, c\} \end{array}$$

One can see that $\mathfrak{A} = \{A, \sigma\}$ is a 3-ary hypergroup.

(2) [9] Let $B = \{a, b, c\}$. We define hyperoperation σ as follows

$$\sigma(x, y, z) = \begin{cases} x & x = y = z \\ b & x \neq y \neq z \\ z & x = y, x \neq z, x \neq b \\ \{a, c\} & x = y = b, z \neq b \end{cases}$$

One can see that $\mathfrak{B} = \{B, \sigma\}$ is a 3-ary hypergroup.

2.5. Definition. Let \mathfrak{A} be a Σ -hyperalgebra, $\sigma \in \Sigma$ and $\{i, j\} \subseteq I_{\rho(\sigma)}$. We say that σ is $\{i, j\}$ -commutative if for each $a_1^{\rho(\sigma)} \in A$ we have

$$\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)}) = \sigma^{\mathfrak{A}}(a_1^{t-1}, a_s, a_{t+1}^{s-1}, a_t, a_{s+1}^{\rho(\sigma)})$$

where $t = \min\{i, j\}$ and $s = \max\{i, j\}$. We say that σ is commutative if it is $\{i, j\}$ -commutative for each $i, j \in I_{\rho(\sigma)}$.

Obviously, the 3-ary hypergroup $\mathfrak{A} = (A, \sigma)$ in Example 2.4 is commutative, too.

2.6. Definition. [9] Let \mathfrak{A} be a Σ -hyperalgebra and $\sigma \in \Sigma$. We say that \mathfrak{A} has a weak neutral element with respect to σ if there exists an element $e \in A$ such that

$$x \in \sigma^{\mathfrak{A}}(e^{(i-1)}, x, e^{(\rho(\sigma)-i)})$$

holds for all $x \in A$ and $i \in I_{\rho(\sigma)}$. If for all $x \in A$ and $i \in I_{\rho(\sigma)}$, we have

$$x = \sigma^{\mathfrak{A}}(e^{(i-1)}, x, e^{(\rho(\sigma)-i)})$$

then “ e ” is called a neutral element. The set of weak neutral elements is denoted by $WN_{\mathfrak{A}}$ and the set of neutral elements is denoted by $N_{\mathfrak{A}}$. Clearly, $N_{\mathfrak{A}} \subseteq WN_{\mathfrak{A}}$.

An n -ary semihypergroup \mathfrak{A} is called an n -ary W-hypermonoid if $WN_{\mathfrak{A}} \neq \emptyset$ and it is called an n -ary hypermonoid if $N_{\mathfrak{A}} \neq \emptyset$. Obviously, if \mathfrak{A} is a 2-ary hypermonoid, then $|N_{\mathfrak{A}}| = 1$.

2.7. Definition. Let \mathfrak{A} be a Σ -hyperalgebra and $\sigma \in \Sigma$. We say that $\sigma^{\mathfrak{A}}$ satisfies the weak idempotent property if $x \in \sigma(\rho(x))$, for each $x \in A$. Also, we say that $\sigma^{\mathfrak{A}}$ has the idempotent property if $x = \sigma(\rho(x))$.

An n -ary commutative semihypergroup $\mathfrak{A} = (A, \sigma)$ is called an n -ary hypersemilattice if “ \circ ” satisfies the weak idempotent property and \mathfrak{A} is called an n -ary semihyperlattice if “ \circ ” satisfies the idempotent property.

2.8. Examples. 1) Let $A = \{a, b, c, d\}$ with the hyperoperation \circ defined by the following table

\circ	a	b	c	d
a	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{b\}$	$\{a, b\}$	$\{d\}$	$\{c, d\}$
c	$\{c\}$	$\{d\}$	$\{a, c\}$	$\{b, d\}$
d	$\{d\}$	$\{c, d\}$	$\{b, d\}$	A

One can see that $\mathfrak{A} = (A, \circ)$ is a 2-ary hypersemilattice. Also, if we define the ternary hyperoperation \circ' on A by $\circ'(x, y, z) = x \circ y \circ z$ for all $x, y, z \in A$, then one can see easily that $\mathfrak{A}' = (A, \circ')$ is a 3-ary hypersemilattice.

2) Let $B = \{a, b, c, d\}$ with the hyperoperation \circ defined by the following table

\circ	a	b	c	d
a	$\{a\}$	$\{b, c, d\}$	$\{c, d\}$	$\{d\}$
b	$\{b, c, d\}$	$\{b\}$	$\{c, d\}$	$\{d\}$
c	$\{c, d\}$	$\{c, d\}$	$\{c\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

One can see that $\mathfrak{B} = (B, \circ)$ is a 2-ary semihyperlattice. Also, if we define the ternary hyperoperation \circ' on B by $\circ'(x, y, z) = x \circ y \circ z$ for all $x, y, z \in B$, then one can see easily that $\mathfrak{B}' = (B, \circ')$ is a 3-ary semihyperlattice.

2.9. Example. Consider Examples 2.8. One can see that \mathfrak{A}' is a 3-ary W-hypermonoid and \mathfrak{B}' is a 3-ary hypermonoid.

2.10. Examples. i) A commutative hypermonoid $\mathfrak{A} = (A, \circ, e)$ is called canonical hypergroup if

- every element has a unique inverse, which means that for all $x \in A$, there exists a unique $x^{-1} \in A$, such that $e \in x \circ x^{-1}$,
- it is reversible, which means that if $x \in y \circ z$, then $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

ii) A Krasner hyperring is a hyperalgebraic structure $\mathfrak{R} = (R, +, \cdot)$ which $(R, +, 0)$ is a canonical hypergroup, (R, \cdot) is a semiring and the multiplication, \cdot , is distributive with respect to the hyperoperation $+$.

One can find definitions of hyperlattices [27, 28], hyper-MV algebras [29, 31], hyper-BCK algebras [30] and etc.

3. Homomorphisms

In the universal algebra theory, the concepts of congruence, quotient algebra and homomorphism are closely related. In this section, we give some ideas about homomorphisms of hyperalgebras and we state connection between them.

3.1. Definition. Let \mathfrak{A} and \mathfrak{B} be Σ -algebras. A map $f : A \rightarrow B$ is a homomorphism (resp. dual homomorphism), in symbols $f : \mathfrak{A} \rightarrow \mathfrak{B}$, if, for all $\sigma \in \Sigma$ and all $a_1^{\rho(\sigma)} \in A$,

$$f(\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)})) \subseteq \sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)}))$$

$$(resp. f(\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)})) \supseteq \sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)})))$$

Also, f is called a good homomorphism if

$$f(\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)})) = \sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)}))$$

$Hom(\mathfrak{A}, \mathfrak{B})$ will denote the set of all homomorphisms and $Hom_G(\mathfrak{A}, \mathfrak{B})$ will denote the set of all good homomorphisms from \mathfrak{A} to \mathfrak{B} . $f : A \rightarrow B$ is called an isomorphism if $f \in Hom(\mathfrak{A}, \mathfrak{B})$ and $f^{-1} \in Hom(\mathfrak{B}, \mathfrak{A})$, too.

3.2. Example. Let $M = \{0, a, b, 1\}$. Consider the following tables:

\oplus	0	a	b	1
0	{0}	{a}	{b}	{1}
a	{a}	{b}	{1}	{1}
b	{b}	{1}	{1}	{1}
1	{1}	{1}	{1}	{1}

$*$	0	a	b	1
0	1	b	a	0

It is easy to see that $\mathfrak{M} = (M, \oplus, *, 0)$ is a trivial hyper-MV algebra which is totally ordered as an MV-algebra. Also, let $HS_3 = (S_3, \oplus, *, 0)$ is the hyper-MV algebra where $S_3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $x \oplus y := [0, \min(1, x + y)] \cap S_3$ and $x^* = 1 - x$. Now, let $f : \mathfrak{M} \rightarrow HS_3$ be a map that $f(0) = 0$, $f(a) = \frac{1}{3}$, $f(b) = \frac{2}{3}$ and $f(1) = 1$. It is easy to check that f is a hyper-MV algebra homomorphism. For example $f(a \oplus a) = f(\{b\}) = \{\frac{2}{3}\}$ and $f(a) \oplus f(a) = \frac{1}{3} \oplus_{HS_3} \frac{1}{3} = \{0, \frac{1}{3}, \frac{2}{3}\}$ so $f(a \oplus a) \subsetneq f(a) \oplus f(a)$. Also, if we define $g : HS_3 \rightarrow \mathfrak{M}$ such that $g(0) = 0$, $g(\frac{1}{3}) = a$, $g(\frac{2}{3}) = b$ and $g(1) = 1$, then g is a dual homomorphism that is not a homomorphism, since $g(\frac{1}{3} \oplus_{HS_3} \frac{1}{3}) = \{0, a, b\}$ but $g(\frac{1}{3}) \oplus g(\frac{1}{3}) = \{b\}$.

3.3. Theorem. A bijective homomorphism of Σ -hyperalgebras is an isomorphism if and only if it is good.

Proof. Let $\mathfrak{A}, \mathfrak{B}$ be Σ -hyperalgebras, $f : A \rightarrow B$ is a bijection, $\sigma \in \Sigma$ and $a_1^{\rho(\sigma)} \in A$. First, suppose that f is an isomorphism and $b \in \sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)}))$. So $f^{-1}(b) \in f^{-1}(\sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)})))$ and since $f^{-1} \in Hom(\mathfrak{B}, \mathfrak{A})$ we obtain that $b \in f(\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)}))$ and it implies that $f \in Hom_G(\mathfrak{A}, \mathfrak{B})$. Conversely, suppose that f is a bijective good homomorphism. Let $\sigma \in \Sigma$ and $b_1^{\rho(\sigma)} \in B$. Since, f is onto, there are $a_1^{\rho(\sigma)} \in A$, such that $f(a_i) = b_i$, for each $i \in I_{\rho(\sigma)}$. Now, we have

$$\begin{aligned} f^{-1}(\sigma^{\mathfrak{B}}(b_1^{\rho(\sigma)})) &= f^{-1}(\sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)}))) \\ &= f^{-1}(f(\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)}))) \\ &= \sigma^{\mathfrak{A}}(f^{-1}(b_1), \dots, f^{-1}(b_{\rho(\sigma)})). \end{aligned}$$

□

If there is a good isomorphism between \mathfrak{A} and \mathfrak{B} , then we write $\mathfrak{A} \cong \mathfrak{B}$. Clearly, \cong is an equivalence relation on the set of all Σ -hyperalgebras.

3.4. Definition. Let $\mathfrak{A}, \mathfrak{B}$ be two Σ -hyperalgebras. A map $f : A \rightarrow B$, is called

- a *very good homomorphism* if it is good and for each $\sigma \in \Sigma$, $a_1^{\rho(\sigma)} \in A$ and $i \in I_{\rho(\sigma)}$, we have

$$f(\sigma_i^{\mathfrak{A}}(a_1^{\rho(\sigma)})) = \sigma_i^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)})).$$
- a *2-homomorphism* if for each $\sigma \in \Sigma$ and $a_1^{\rho(\sigma)} \in A$, we have

$$f^{-1}(\sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)}))) = f^{-1}(f(\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)}))).$$
- an *almost strong homomorphism* if for each $\sigma \in \Sigma$ and $a_1^{\rho(\sigma)} \in A$, we have

$$f^{-1}(\sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)}))) = \sigma^{\mathfrak{A}}(f^{-1}(f(a_1)), \dots, f^{-1}(f(a_{\rho(\sigma)}))).$$

3.5. Theorem. *Let $\mathfrak{A}, \mathfrak{B}$ be two Σ -hyperalgebras. If $f : \mathfrak{A} \rightarrow \mathfrak{B}$ is a good homomorphism, then f is a 2-homomorphism. Furthermore, if f is a very good homomorphism, then f is an almost strong homomorphism.*

Proof. Let $\sigma \in \Sigma$ and $a_1^{\rho(\sigma)} \in A$. First let $f \in \text{Hom}_G(\mathfrak{A}, \mathfrak{B})$. Let $a \in f^{-1}(f(\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)})))$, so $f(a) \in f(\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)}))$ and it implies that there is $a' \in \sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)})$ such that $f(a) = f(a')$. Since f is good we obtain that $a \in f^{-1}(\sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)})))$. Conversely, assume that $a \in f^{-1}(\sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)})))$, so $f(a) \in \sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)}))$ and it shows that f is a 2-homomorphism.

Now, suppose that f is a very good homomorphism. Since $f \in \text{Hom}(\mathfrak{A}, \mathfrak{B})$ we obtain that $f(\sigma^{\mathfrak{A}}(f^{-1}(f(a_1)), \dots, f^{-1}(f(a_{\rho(\sigma)})))) \subseteq \sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)}))$. It shows that

$$\sigma^{\mathfrak{A}}(f^{-1}(f(a_1)), \dots, f^{-1}(f(a_{\rho(\sigma)}))) \subseteq f^{-1}(\sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)}))).$$

Conversely, assume that $a \in f^{-1}(\sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)})))$. Hence, $f(a) \in \sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)}))$, so for each $i \in I_{\rho(\sigma)}$ we have

$$f(a_i) \in \sigma_i^{\mathfrak{B}}(f(a_1), \dots, f(a_{i-1}), f(a), f(a_{i+1}), \dots, f(a_{\rho(\sigma)})).$$

Since, f is very good we obtain that

$$f(a_i) \in f(\sigma_i^{\mathfrak{A}}(a_1^{i-1}, a, a_{i+1}^{\rho(\sigma)})).$$

Hence, there is $a'_i \in \sigma_i^{\mathfrak{A}}(a_1^{i-1}, a, a_{i+1}^{\rho(\sigma)})$ such that $f(a_i) = f(a'_i)$. It implies that $a \in \sigma^{\mathfrak{A}}(a_1^{i-1}, f^{-1}(f(a_i)), a_{i+1}^{\rho(\sigma)})$. Since i is arbitrary, continuing the above method shows that f is an almost strong homomorphism. \square

3.6. Definition. Let $\mathfrak{A}, \mathfrak{B}$ be two Σ -hyperalgebras, $f \in \text{Hom}(\mathfrak{A}, \mathfrak{B})$, $\sigma \in \Sigma$ and $i \in I_{\rho(\sigma)}$. f is called *strong on the i -th component* respect to σ if for all $a_1^{\rho(\sigma)} \in A$, $f(a) \in \sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)}))$ implies that there exists $a'_i \in A$ such that $f(a_i) = f(a'_i)$ and $a \in \sigma^{\mathfrak{A}}(a_1^{i-1}, a'_i, a_{i+1}^{\rho(\sigma)})$. f is called *strong respect to σ* if for all $i \in I_{\rho(\sigma)}$, f is strong on the i -th component. f is called *strong* if for each $\sigma \in \Sigma$, f is strong respect to σ .

3.7. Theorem. *Any strong homomorphism is almost strong.*

Proof. Let $f \in \text{Hom}(\mathfrak{A}, \mathfrak{B})$ be a strong homomorphism, $\sigma \in \Sigma$ and $a_1^{\rho(\sigma)} \in A$. Since $f \in \text{Hom}(\mathfrak{A}, \mathfrak{B})$, similar to Theorem 3.5, we can obtain that $\sigma^{\mathfrak{A}}(f^{-1}(f(a_1)), \dots, f^{-1}(f(a_{\rho(\sigma)}))) \subseteq f^{-1}(\sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)})))$. Now, assume that $a \in f^{-1}(\sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)})))$. So $f(a) \in \sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)}))$. Since f is strong respect to σ , for each $i \in I_{\rho(\sigma)}$, there exists $a'_i \in A$ such that $f(a_i) = f(a'_i)$ and $a \in \sigma^{\mathfrak{A}}(a_1^{i-1}, a'_i, a_{i+1}^{\rho(\sigma)})$, whence $a \in \sigma^{\mathfrak{A}}(a_1^{i-1}, f^{-1}(f(a_i)), a_{i+1}^{\rho(\sigma)}) \subseteq \sigma^{\mathfrak{A}}(f^{-1}(f(a_1)), \dots, f^{-1}(f(a_{\rho(\sigma)})))$. \square

4. Subhyperalgebras

There are several important methods of constructing new hyperalgebras from given ones. Three of the most fundamental are the formation of subhyperalgebras, homomorphic images, and direct products. In this section, subhyperalgebras will occupy us.

4.1. Definition. Let \mathfrak{A} be a Σ -hyperalgebra and $B \subseteq A$. Then B is called a subhyperuniverse of \mathfrak{A} , if for all $\sigma \in \Sigma$ and $b_1^{\rho(\sigma)} \in B$ we have $\sigma^{\mathfrak{A}}(b_1^{\rho(\sigma)}) \subseteq B$. The set of all subhyperuniverses of \mathfrak{A} will be denoted by $Sub(\mathfrak{A})$.

Note that this implies $\sigma^{\mathfrak{A}} \subseteq B$ for every $\sigma \in \Sigma_0$, and that the empty set is a subhyperuniverse of \mathfrak{A} if and only if $\Sigma_0 = \emptyset$, i.e., \mathfrak{A} has no distinguished constants.

4.2. Example. Consider hypersemilattice \mathfrak{A} in Examples 2.8. It is easy to see that $B = \{a, b\}$ is a subhyperuniverse of \mathfrak{A} .

4.3. Theorem. $(A, Sub(\mathfrak{A}))$ is an algebraic closed set system for every Σ -hyperalgebra \mathfrak{A} .

Proof. Let $\mathcal{K} \subseteq Sub(\mathfrak{A})$. Let $\sigma \in \Sigma$ and $a_1^{\rho(\sigma)} \in \cap \mathcal{K}$. Then for every $K \in \mathcal{K}$, we have $a_1^{\rho(\sigma)} \in K$ and hence $\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)}) \subseteq K$. So $\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)}) \subseteq \cap \mathcal{K}$. Therefore, $\cap \mathcal{K} \in Sub(\mathfrak{A})$.

Now, assume that \mathcal{K} is a directed subset of $Sub(\mathfrak{A})$. Let $a_1^{\rho(\sigma)} \in \cup \mathcal{K}$. Since there is only a finite number of a_i and \mathcal{K} is directed, they are all contained in a single $K \in \mathcal{K}$. Hence $\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)}) \subseteq K$. Therefore, $\cup \mathcal{K} \in Sub(\mathfrak{A})$. \square

The closure operator associated with the closed set system $(A, Sub(\mathfrak{A}))$ is denoted by $Sg^{\mathfrak{A}}$. Thus $Sg^{\mathfrak{A}} : \mathcal{P}(A) \rightarrow Sub(\mathfrak{A})$ and $Sg^{\mathfrak{A}}(X) = \cap \{K \in Sub(\mathfrak{A}) : X \subseteq K\}$; this is called the subhyperuniverse generated by X . Hyperalgebra \mathfrak{A} is called finitely generated if there exists a finite subset X of A such that $Sg^{\mathfrak{A}}(X) = A$.

B is a maximal proper subhyperuniverse of \mathfrak{A} if $B \neq A$ and there does not exist a $C \in Sub(\mathfrak{A})$ such that $B \subsetneq C \subsetneq A$.

4.4. Theorem. Let \mathfrak{A} be a finitely generated Σ -algebra. Then every proper subhyperuniverse of \mathfrak{A} is included in a maximal proper one.

Proof. Let $A = Sg^{\mathfrak{A}}(X)$, for $X \subsetneq A$. Assume that B is a proper subhyperuniverse of \mathfrak{A} . Let $\mathcal{K} = \{K \in Sub(\mathfrak{A}) : B \subseteq K \subsetneq A\}$. Since \mathcal{K} contains B , it is nonempty. Suppose that $\mathcal{C} \subseteq \mathcal{K}$ be a chain. Clearly, \mathcal{C} is directed so by Theorem 4.3, $\cup \mathcal{C} \in Sub(\mathfrak{A})$. Because X is finite and \mathcal{K} is directed, one can see that $\cup \mathcal{C}$ is a proper subhyperuniverse of \mathfrak{A} . hence, by Zorn's lemma \mathcal{K} has a maximal element. \square

4.5. Theorem. (Principle of Structural Induction) Let \mathfrak{A} be a Σ -hyperalgebra generated by X . To prove that a property \mathcal{P} holds for each element of A , it suffices to show that

induction basis. \mathcal{P} holds for each element of X .

induction step. If $\sigma \in \Sigma$ and \mathcal{P} holds for each of elements $a_1^{\rho(\sigma)} \in A$, then \mathcal{P} holds for each elements of $\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)})$.

Proof. Let $P = \{x \in A : \mathcal{P} \text{ holds for } x\}$. $X \subseteq P$ and P is closed under the hyperoperations of \mathfrak{A} . Hence, $P \in Sub(\mathfrak{A})$ and it implies that $A = Sg^{\mathfrak{A}}(X) \subseteq P$. \square

Let \mathfrak{A} be a Σ -hyperalgebra and $X \subseteq A$. We define

$$E(X) = X \cup \{t \in \sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)}) : \forall \sigma \in \Sigma, a_1^{\rho(\sigma)} \in X\}.$$

Now, we define $E^n(X)$ for $n \geq 0$ by

$$\begin{aligned} E^0(X) &= X, \\ E^n(X) &= E(E^{n-1}(X)). \end{aligned}$$

4.6. Theorem. *Let \mathfrak{A} be a Σ -hyperalgebra and $X \subseteq A$. Then*

$$Sg^{\mathfrak{A}}(X) = \cup_{i=0}^{\infty} E^i(X).$$

Proof. Let $P = \{x \in Sg^{\mathfrak{A}}(X) : x \in I = \cup_{i=0}^{\infty} E^i(X)\}$. Clearly, $X \subseteq P$. Now, let $\sigma \in \Sigma$ and $x_1, \dots, x_{\rho(\sigma)} \in P$. One can see that $E^i(X) \subseteq E^{i+1}(X)$ for each $i \in \omega$, so it implies that there is a $n \in \omega$ such that $x_1, \dots, x_{\rho(\sigma)} \in E^n(X)$. Thus $\sigma^{\mathfrak{A}}(x_1, \dots, x_{\rho(\sigma)}) \subseteq E^{n+1}(X) \subseteq I$. Therefore by Theorem 4.5, we get the result. \square

4.7. Definition. Let \mathfrak{A} and \mathfrak{B} be two Σ -hyperalgebras. Then \mathfrak{B} is a subhyperalgebra of \mathfrak{A} if $B \subseteq A$ and every fundamental operation of \mathfrak{B} is the restriction of the corresponding operation of \mathfrak{A} , i.e., for each function symbol $\sigma \in \Sigma$, $\sigma^{\mathfrak{B}}$ is $\sigma^{\mathfrak{A}}$ restricted to B ; we write simply $\mathfrak{B} \leq \mathfrak{A}$. If $\mathfrak{B} \leq \mathfrak{A}$, then $B \in Sub(\mathfrak{A})$. Conversely, if $B \in Sub(\mathfrak{A})$ and $B \neq \emptyset$, then there is a unique $\mathfrak{B} \leq \mathfrak{A}$ such that B is the universe of \mathfrak{B} .

4.8. Theorem. *Let $\mathfrak{A}, \mathfrak{B}$ be Σ -hyperalgebras, $f \in Hom_G(\mathfrak{A}, \mathfrak{B})$ and $h \in Hom(\mathfrak{A}, \mathfrak{B})$. Then*

- (i) *For each $K \in Sub(\mathfrak{A})$, $f(K) \in Sub(\mathfrak{B})$.*
- (ii) *For each $L \in Sub(\mathfrak{B})$, $h^{-1}(L) \in Sub(\mathfrak{A})$, if $h^{-1}(L) \neq \emptyset$.*
- (iii) *For each $X \subseteq A$, $f(Sg^{\mathfrak{A}}(X)) \in Sg^{\mathfrak{A}}(f(X))$.*

Proof. (i) Let $\sigma \in \Sigma$ and $b_1^{\rho(\sigma)} \in f(K)$. Choose $a_1^{\rho(\sigma)} \in K$ such that $f(a_1) = b_1, \dots, f(a_{\rho(\sigma)}) = b_{\rho(\sigma)}$. Then $\sigma^{\mathfrak{B}}(b_1^{\rho(\sigma)}) = \sigma^{\mathfrak{B}}(f(a_1), \dots, f(a_{\rho(\sigma)})) = f(\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)})) \subseteq f(K)$.

(ii) Assume that $h^{-1}(L) \neq \emptyset$. Let $\sigma \in \Sigma$ and $a_1^{\rho(\sigma)} \in h^{-1}(L)$. So $f(a_1), \dots, f(a_{\rho(\sigma)}) \in L$. Then $h(\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)})) \subseteq \sigma^{\mathfrak{B}}(h(a_1), \dots, h(a_{\rho(\sigma)})) \subseteq L$. Hence, $\sigma^{\mathfrak{A}}(a_1^{\rho(\sigma)}) \subseteq h^{-1}(L)$.

(iii) $f(X) \subseteq f(Sg^{\mathfrak{A}}(X)) \in Sub(\mathfrak{B})$, by part (i). Therefore, $Sub^{\mathfrak{B}}(f(X)) \subseteq f(Sg^{\mathfrak{A}}(X))$. For the reverse inclusion, $X \subseteq f^{-1}(f(X)) \subseteq f^{-1}(Sg^{\mathfrak{A}}(f(X))) \in Sub(\mathfrak{A})$, by part (ii). Hence, $Sg^{\mathfrak{A}}(X) \subseteq f^{-1}(Sg^{\mathfrak{A}}(f(X)))$. \square

Let $Alg_H(\Sigma)$ be the class of all Σ -hyperalgebras. \leq is a partial ordering of $Alg_H(\Sigma)$. Clearly, it is reflexive and antisymmetric. If $\mathfrak{C} \leq \mathfrak{B}$ and $\mathfrak{B} \leq \mathfrak{A}$, then $C \subseteq A$ and for each σ and $c_1^{\rho(\sigma)} \in C$, we have $\sigma^{\mathfrak{C}}(c_1^{\rho(\sigma)}) = \sigma^{\mathfrak{B}}(c_1^{\rho(\sigma)}) = \sigma^{\mathfrak{A}}(c_1^{\rho(\sigma)})$. Hence, \leq is transitive.

For any class K of $Alg_H(\Sigma)$ we define

$$S(K) = \{\mathfrak{B} \in Alg_H(\Sigma) : \exists \mathfrak{A} \in K(\mathfrak{B} \leq \mathfrak{A})\}.$$

4.9. Theorem. *S is an algebraic closure operator on $Alg_H(\Sigma)$.*

Proof. Clearly, $K \subseteq S(K)$ by the reflexivity of \leq and by transitivity of \leq , we obtain that $SS(K) = (K)$. Also, $K \subseteq L$ implies $S(K) \subseteq S(L)$. Furthermore, $S(K) = \cup\{S(\mathfrak{A}) : \mathfrak{A} \in S(K)\}$ and it implies that S is algebraic closed. \square

Define the binary relation \preceq on $Alg_H(\Sigma)$ by $\mathfrak{B} \preceq \mathfrak{A}$ if there is an onto good homomorphism $f : \mathfrak{A} \rightarrow \mathfrak{B}$. Clearly, \preceq is reflexive and transitive.

For any class K of $Alg_H(\Sigma)$ we define

$$H(K) = \{\mathfrak{B} \in Alg_H(\Sigma) : \exists \mathfrak{A} \in K(\mathfrak{B} \preceq \mathfrak{A})\},$$

$$I(K) = \{\mathfrak{B} \in Alg_H(\Sigma) : \exists \mathfrak{A} \in K(\mathfrak{B} \cong \mathfrak{A})\},$$

the classes respectively of homomorphic and isomorphic images of algebras of K . Similar to Theorem 4.9, we can show that H and I are algebraic closure operators on $Alg_H(\Sigma)$.

4.10. Theorem. *For any class K of Σ -hyperalgebras, we have*

- (i) $\text{SH}(\mathbf{K}) \subseteq \text{HS}(\mathbf{K})$,
- (ii) HS is an algebraic closure operator on $\text{Alg}_{\mathbf{H}}(\Sigma)$.

Proof. (i) Let $\mathfrak{C} \in \text{SH}(\mathbf{K})$ so for some $\mathfrak{A} \in \mathbf{K}$ and onto good homomorphism $f : \mathfrak{A} \rightarrow \mathfrak{B}$ we have $\mathfrak{C} \leq \mathfrak{B}$. By Theorem 4.8, we have $f^{-1}(\mathfrak{C}) \leq \mathfrak{A}$. Since f is onto we have $\mathfrak{C} \in \text{HS}(\mathbf{K})$.

- (ii) Obviously, HS is extensive. By part (i), we have $\text{HSHS}(\mathbf{K}) \subseteq \text{HHSS}(\mathbf{K}) = \text{HS}(\mathbf{K})$ so we get that HS is idempotent. Finally, monotonicity of \mathbf{S} and \mathbf{H} imply that HS is monotonic. Also, $\mathfrak{B} \in \text{HS}(\mathbf{K})$ if and only if there is a $\mathfrak{A} \in \mathbf{K}$ such that $\mathfrak{B} \in \text{HS}(\mathfrak{A})$ so we have $\text{HS}(\mathbf{K}) \subseteq \cup\{\text{HS}(\mathfrak{A}) : \mathfrak{A} \in \mathbf{K}\}$ and it implies that HS is algebraic. □

5. Some type of subhyperalgebras

There are several kinds of subhyperalgebras. In what follows, we introduce closed, invertible, ultraclosed and conjugable subhyperalgebras and some connections among them. Let us present now the definition of these types of subhyperalgebras.

5.1. Definition. Let \mathfrak{A} be a Σ -hyperalgebra, $B \in \text{Sub}(\mathfrak{A})$, $\sigma \in \Sigma$ and $1 \leq i \leq \rho(\sigma)$. We say that B is

- (1) i -closed with respect to σ if for each $x \in A$ and $b_1^{\rho(\sigma)} \in B$, from $x \in \sigma_i^A(b_1^{\rho(\sigma)})$ it follows that $x \in B$.
- (2) i -invertible with respect to σ if for each $x, y \in A$, from $x \in \sigma^{\mathfrak{A}}(\begin{smallmatrix} (i-1) \\ B \end{smallmatrix}, y, \begin{smallmatrix} (\rho(\sigma)-i) \\ B \end{smallmatrix})$ it follows that $y \in \sigma^{\mathfrak{A}}(\begin{smallmatrix} (i-1) \\ B \end{smallmatrix}, x, \begin{smallmatrix} (\rho(\sigma)-i) \\ B \end{smallmatrix})$.
- (3) ultraclosed on the right (on the left) with respect to σ if for each $x \in A$ we have $\sigma^{\mathfrak{A}}(x, \begin{smallmatrix} (\rho(\sigma)-1) \\ B \end{smallmatrix}) \cap \sigma^{\mathfrak{A}}(x, \begin{smallmatrix} (\rho(\sigma)-2) \\ B \end{smallmatrix}, B^c) = \emptyset$ ($\sigma^{\mathfrak{A}}(\begin{smallmatrix} (\rho(\sigma)-1) \\ B \end{smallmatrix}, x) \cap \sigma^{\mathfrak{A}}(B^c, \begin{smallmatrix} (\rho(\sigma)-2) \\ B \end{smallmatrix}, x) = \emptyset$).
- (4) i -conjugable with respect to σ if it is i -closed with respect to σ and for each $x \in A$ there exist $x_1^{\rho(\sigma)-1} \in A$ such that $\sigma^{\mathfrak{A}}(x_1^{i-1}, x, x_{i+1}^{\rho(\sigma)-1}) \subseteq B$.

We say that B is closed (*resp.* invertible, conjugable) with respect to σ , if it is i -closed (*resp.* invertible, conjugable) with respect to σ , for each $1 \leq i \leq \rho(\sigma)$. Also, we say that B is closed (*resp.* invertible, conjugable) if it is closed (*resp.* invertible, conjugable) for each $\sigma \in \Sigma$. Sets of closed, invertible and conjugable subhyperalgebra of hyperalgebra \mathfrak{A} are denoted by $\text{Sub}_{cl}(\mathfrak{A})$, $\text{Sub}_{in}(\mathfrak{A})$ and $\text{Sub}_{co}(\mathfrak{A})$, respectively.

- 5.2. Examples.** (1) Consider 3-hypergroup \mathfrak{A} in Examples 2.4. Let $I = \{a\}$. Clearly, I is a subhyperalgebra of \mathfrak{A} which it is closed and invertible but it is not ultraclosed, since $\sigma(b, a, a) \cap \sigma(b, a, c) = b$, and it is not conjugable.
- (2) Consider commutative 3-ary hypergroup \mathfrak{B} in Examples 2.4. Let $J = \{b\}$. One can see that J is a subhyperalgebra of \mathfrak{B} which it is closed, invertible, ultraclosed and conjugable.

The following theorems present some connections among the above types of subhyperalgebras.

5.3. Theorem. Let \mathfrak{A} be a Σ -hyperalgebra. Then $(\text{Sub}_{cl}(\mathfrak{A}), \subseteq)$ and $(\text{Sub}_{in}(\mathfrak{A}), \subseteq)$ are algebraic closed set systems. Furthermore, $\text{Sub}_{in}(\mathfrak{A}) \subseteq \text{Sub}_{cl}(\mathfrak{A})$.

Proof. Obviously, $(\text{Sub}_{cl}(\mathfrak{A}), \subseteq)$ and $(\text{Sub}_{in}(\mathfrak{A}), \subseteq)$ are closed set systems. Also, similar to Theorem 4.3, we can conclude that $(\text{Sub}_{cl}(\mathfrak{A}), \subseteq)$ and $(\text{Sub}_{in}(\mathfrak{A}), \subseteq)$ are algebraic.

Now, suppose that $\sigma \in \Sigma$ and $1 \leq i \leq \rho(\sigma)$, $b_1^{\rho(\sigma)} \in B$ and $x \in A$ such that $x \in \sigma_i^{\mathfrak{A}}(b_1^{\rho(\sigma)})$. By hypothesis $x \in \sigma^{\mathfrak{A}}(b_1^{\rho(\sigma)}) \subseteq B$ and it holds the result. □

5.4. Lemma. Let \mathfrak{A} be a Σ -hyperalgebra, $B \in \text{Sub}(\mathfrak{A})$ and $\sigma \in \Sigma$ such that it is $(1, \rho(\sigma))$ -associative. Then B is $1(\rho(\sigma))$ -invertible if and only if $\{\sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B})\}_{x \in A}$ ($\{\sigma^{\mathfrak{A}}(\binom{(\rho(\sigma)-1)}{B}, x)\}_{x \in A}$) is a disjoint family of subsets of A .

Proof. Suppose that $\sigma \in \Sigma$, B is 1-invertible with respect to σ and $z \in \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}) \cap \sigma^{\mathfrak{A}}(y, \binom{(\rho(\sigma)-1)}{B})$, for some $x, y \in A$. Then we have

$$\begin{aligned} \sigma^{\mathfrak{A}}(z, \binom{(\rho(\sigma)-1)}{B}) &\subseteq \sigma^{\mathfrak{A}}(\sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}), \binom{(\rho(\sigma)-1)}{B}) \\ &= \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-2)}{B}, \sigma^{\mathfrak{A}}(\binom{(\rho(\sigma)}{B})) \\ &\subseteq \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}) \end{aligned}$$

By hypothesis $x \in \sigma^{\mathfrak{A}}(z, \binom{(\rho(\sigma)-1)}{B})$ and similarly we can conclude that $\sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}) \subseteq \sigma^{\mathfrak{A}}(z, \binom{(\rho(\sigma)-1)}{B})$. So $\sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}) = \sigma^{\mathfrak{A}}(z, \binom{(\rho(\sigma)-1)}{B})$. Also, we can get $\sigma^{\mathfrak{A}}(y, \binom{(\rho(\sigma)-1)}{B}) = \sigma^{\mathfrak{A}}(z, \binom{(\rho(\sigma)-1)}{B})$ and it shows that $\{\sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B})\}_{x \in A}$ is a disjoint family of subsets of A .

Conversely, suppose that $x \in \sigma^{\mathfrak{A}}(y, \binom{(\rho(\sigma)-1)}{B})$. Then $\sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}) \subseteq \sigma^{\mathfrak{A}}(z, \binom{(\rho(\sigma)-1)}{B})$ whence $\sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}) = \sigma^{\mathfrak{A}}(z, \binom{(\rho(\sigma)-1)}{B})$. Thus we have $x \in \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B})$ hence we obtain that $y \in \sigma^{\mathfrak{A}}(y, \binom{(\rho(\sigma)-1)}{B}) = \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B})$. Similarly, we can show that B is $\rho(\sigma)$ -invertible if and only if $\{\sigma^{\mathfrak{A}}(\binom{(\rho(\sigma)-1)}{B}, x)\}_{x \in A}$ is a disjoint family of subsets of A . \square

5.5. Lemma. Let \mathfrak{A} be a Σ -hyperalgebra, $\sigma \in \Sigma$ and $B \in \text{Sub}(\mathfrak{A})$ such that A, B satisfies the reproduction axiom with respect to σ and B be $\rho(\sigma)(1)$ -closed. Then $\sigma^{\mathfrak{A}}(\binom{(\rho(\sigma)-1)}{B}, B^c) = B^c$, $(\sigma^{\mathfrak{A}}(B^c, \binom{(\rho(\sigma)-1)}{B})) = B^c$.

Proof. We have

$$\begin{aligned} A &= \sigma^{\mathfrak{A}}(\binom{(\rho(\sigma)-1)}{B}, A) \\ &= \sigma^{\mathfrak{A}}(\binom{(\rho(\sigma))}{B}) \cup \sigma^{\mathfrak{A}}(\binom{(\rho(\sigma)-1)}{B}, B^c) \\ &= B \cup \sigma^{\mathfrak{A}}(\binom{(\rho(\sigma)-1)}{B}, B^c). \end{aligned}$$

On the other hand, assume that $b \in B \cap \sigma^{\mathfrak{A}}(\binom{(\rho(\sigma)-1)}{B}, B^c)$. So there exist $c \in B^c$ and $b_1^{\rho(\sigma)-1} \in B$ such that $b \in \sigma^{\mathfrak{A}}(b_1^{\rho(\sigma)-1}, c)$. Now, since B is $\rho(\sigma)$ -closed we obtain that $c \in B$. It is a contradiction. \square

5.6. Theorem. Let \mathfrak{A} be a Σ -hyperalgebra, $\sigma \in \Sigma$ and $B \in \text{Sub}(\mathfrak{A})$ such that A, B satisfies the reproduction axiom with respect to σ and B be conjugable. Then B is ultraclosed on the right (on the left).

Proof. Suppose that $x \in A$. Denote $C = \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}) \cap \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-2)}{B}, B^c)$. Since B is conjugable it follows that B is closed and there exist $x_1^{\rho(\sigma)-1} \in A$ such that

$\sigma^{\mathfrak{A}}(x_1^{\rho(\sigma)-1}, x) \subseteq B$. By Lemma 5.5, we obtain that

$$\begin{aligned} \sigma^{\mathfrak{A}}(x_1^{\rho(\sigma)-1}, C) &\subseteq \sigma^{\mathfrak{A}}(x_1^{\rho(\sigma)-1}, \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B})) \cap \sigma^{\mathfrak{A}}(x_1^{\rho(\sigma)-1}, \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-2)}{B}, B^c)) \\ &\subseteq B \cap \sigma^{\mathfrak{A}}(\sigma^{\mathfrak{A}}(x_1^{\rho(\sigma)-1}, x), \binom{(\rho(\sigma)-2)}{B}, B^c) \\ &\subseteq B \cap \sigma^{\mathfrak{A}}(B, \binom{(\rho(\sigma)-2)}{B}, B^c) \\ &= B \cap B^c = \emptyset. \end{aligned}$$

Hence $\sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}) \cap \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-2)}{B}, B^c) = \emptyset$, which means that B is ultraclosed on the right. Similarly, we can show that B is ultraclosed on the left. \square

5.7. Lemma. *Let \mathfrak{A} be a Σ -hyperalgebra, $\sigma \in \Sigma$ and B be an ultraclosed subhyperalgebra on the right (on the left) which satisfies the reproduction axiom with respect to σ . Then B is $\rho(\sigma)(1)$ -closed with respect to σ .*

Proof. Let $x \in A \setminus B$ and $b \in B$. Suppose that $b \in \sigma^{\mathfrak{A}}(b_1^{\rho(\sigma)-1}, x) \subseteq \sigma^{\mathfrak{A}}(b_1, \binom{(\rho(\sigma)-2)}{B}, B^c)$. Since B is ultraclosed on the right we have $b \notin \sigma^{\mathfrak{A}}(b_1, \binom{(\rho(\sigma)-1)}{B})$. By the reproduction axiom we conclude that $b \notin B$. It is a contradiction. \square

5.8. Theorem. *Let \mathfrak{A} be a Σ -hyperalgebra, $\sigma \in \Sigma$ such that it is $(1, \rho(\sigma))$ -associative and B be an ultraclosed subhyperalgebra on the right (on the left) which satisfies the reproduction axiom with respect to σ . Then B is $1(\rho(\sigma))$ -invertible with respect to σ .*

Proof. Suppose that B is ultraclosed on the right. Let $y \in \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B})$, for some $x, y \in A$. By associativity we have

$$\sigma^{\mathfrak{A}}(y, \binom{(\rho(\sigma)-1)}{B}) \subseteq \sigma^{\mathfrak{A}}(\sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}), \binom{(\rho(\sigma)-1)}{B}) \subseteq \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}).$$

On the other hand, By Lemma 5.7 we get that B is $\rho(\sigma)$ -closed and then we obtain Lemma 5.5 and by this lemma we have

$$\begin{aligned} \sigma^{\mathfrak{A}}(y, \binom{(\rho(\sigma)-2)}{B}, B^c) &\subseteq \sigma^{\mathfrak{A}}(\sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}), \binom{(\rho(\sigma)-2)}{B}, B^c) \\ &= \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-2)}{B}, \sigma^{\mathfrak{A}}(\binom{(\rho(\sigma)-1)}{B}, B^c)) \\ &= \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-2)}{B}, B^c) \end{aligned}$$

Thus, by the reproduction axiom we have

$$A = \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}) \cup \sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-2)}{B}, B^c) = \sigma^{\mathfrak{A}}(y, \binom{(\rho(\sigma)-1)}{B}) \cup \sigma^{\mathfrak{A}}(y, \binom{(\rho(\sigma)-2)}{B}, B^c)$$

Since, B is ultraclosed on the right we get that $\sigma^{\mathfrak{A}}(x, \binom{(\rho(\sigma)-1)}{B}) = \sigma^{\mathfrak{A}}(y, \binom{(\rho(\sigma)-1)}{B})$. It shows that $\{\sigma^{\mathfrak{A}}(x, B^{\rho(\sigma)-1})\}_{x \in A}$ is a disjoint family of subsets of A so by Lemma 5.4 we conclude that B is 1-invertible with respect to σ . Similarly, we can show that B is $\rho(\sigma)$ -invertible, if it is ultraclosed on the left. \square

Let \mathfrak{A} be a Σ -hyperalgebra and $\sigma \in \Sigma$. We denote

$$I_\sigma = \{e_1^{\rho(\sigma)} \in A \mid \exists x \in A, \text{ such that } x \in \sigma^{\mathfrak{A}}(e_1^{i_1}, x, e_i^{\rho(\sigma)-1}), \text{ for some } 1 \leq i \leq \rho(\sigma)\}.$$

5.9. Lemma. *Let \mathfrak{A} be a Σ -hyperalgebra, $\sigma \in \Sigma$ such that it is $(1, \rho(\sigma))$ -associative and B be $\rho(\sigma)(1)$ -closed with respect to σ which satisfies the reproduction axiom with respect to σ such that $I_\sigma \subseteq B$. Then B is $1(\rho(\sigma))$ -invertible.*

Proof. Let $y \in \sigma^{\mathfrak{A}}(x, \overset{(\rho(\sigma)-1)}{B})$, for some $x, y \in A$. Suppose that $x \notin \sigma^{\mathfrak{A}}(y, \overset{(\rho(\sigma)-1)}{B})$. Since, $x \in A = \sigma^{\mathfrak{A}}(y, \overset{(\rho(\sigma)-1)}{B}) \cup \sigma^{\mathfrak{A}}(y, \overset{(\rho(\sigma)-2)}{B}, B^c)$, we get $x \in \sigma^{\mathfrak{A}}(y, \overset{(\rho(\sigma)-2)}{B}, B^c)$. Now, by Lemma 5.5, we have

$$\begin{aligned} x \in \sigma^{\mathfrak{A}}(y, \overset{(\rho(\sigma)-2)}{B}, B^c) &\subseteq \sigma^{\mathfrak{A}}(\sigma^{\mathfrak{A}}(x, \overset{(\rho(\sigma)-1)}{B}), \overset{(\rho(\sigma)-2)}{B}, B^c) \\ &= \sigma^{\mathfrak{A}}(x, \overset{(\rho(\sigma)-2)}{B}, \sigma^{\mathfrak{A}}(\overset{(\rho(\sigma)-1)}{B}, B^c)) \\ &= \sigma^{\mathfrak{A}}(x, \overset{(\rho(\sigma)-2)}{B}, B^c). \end{aligned}$$

So there exist $b_1^{\rho(\sigma)-2} \in B$ and $c \in B^c$ such that $x \in \sigma^{\mathfrak{A}}(x, b_1^{\rho(\sigma)-2}, c)$. Hence $c \in I_\sigma \cap B^c$ and it is a contradiction. \square

5.10. Theorem. *Let \mathfrak{A} be a Σ -hyperalgebra, $\sigma \in \Sigma$ such that it is $(1, \rho(\sigma))$ -associative and $B \in \text{Sub}(\mathfrak{A})$ which satisfies the reproduction axiom with respect to σ . If B is 1-closed and $\rho(\sigma)$ -close with respect to σ and $I_\sigma \subseteq B$ then B is ultraclosed.*

Proof. Let B is $\rho(\sigma)$ -close with respect to σ and $I_\sigma \subseteq B$. Suppose that $z \in \sigma^{\mathfrak{A}}(x, \overset{(\rho(\sigma)-1)}{B}) \cap \sigma^{\mathfrak{A}}(x, \overset{(\rho(\sigma)-2)}{B}, B^c)$. By Lemma 5.9, associativity and Lemma 5.5, we obtain that

$$x \in \sigma^{\mathfrak{A}}(z, \overset{(\rho(\sigma)-1)}{B}) \subseteq \sigma^{\mathfrak{A}}(\sigma^{\mathfrak{A}}(x, \overset{(\rho(\sigma)-2)}{B}, B^c), \overset{(\rho(\sigma)-1)}{B}) = \sigma^{\mathfrak{A}}(x, \overset{(\rho(\sigma)-2)}{B}, B^c).$$

Thus, there exist $b_1^{\rho(\sigma)-2} \in B$ and $c \in B^c$ such that $x \in \sigma^{\mathfrak{A}}(x, b_1^{\rho(\sigma)-2}, c)$ and it implies that $c \in I_\sigma \cap B^c$. It shows that $\sigma^{\mathfrak{A}}(x, \overset{(\rho(\sigma)-1)}{B}) \cap \sigma^{\mathfrak{A}}(x, \overset{(\rho(\sigma)-2)}{B}, B^c) = \emptyset$. Hence B is ultraclosed on the right. \square

6. Conclusion

The above discussion shows that we can extend some notions of hypergroup theory to a Σ -hyperalgebra for an arbitrary signature Σ . This paper provides suitable tools for doing more research in the area of hyperstructures, such as on homomorphisms and subhyperalgebras. Also, by consideration the operator \mathbf{P} one can research on a variety of an arbitrary Σ -hyperalgebras. **Acknowledgements**

Authors are extremely grateful to the referees for giving them many valuable comments and helpful suggestions which helped to improve the presentation of this paper.

References

- [1] G. Birkhoff, *Lattice Theory*, 3rd edition. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, 1967.
- [2] P. Corsini, *Sur les homomorphismes d hypergroupes*, Rend. Sem. Univ., 52, Padova, Italy, (1974) 117-140.
- [3] P. Corsini, *Prolegomena of Hypergroup Theory*, 2nd edn. (Aviani editor, 1993).
- [4] P. Corsini, *Hypergraphs and Hypergroups*, Algebra Universalis, 35 (1996) 548-555.
- [5] P. Corsini, V. Leoreanu, *Hypergroups and binary relations*, Algebra Universalis, 43 (2000) 321-330.
- [6] P. Corsini, V. Leoreanu, *Applications of hyperstructure theory*, Kluwer, Dordrecht 2003.
- [7] B. Davvaz, V. Leoreanu, *Hypergroup Theory and Applications*, International Academic Press, 2007.
- [8] B. Davvaz, T. Vougiouklis, *n-ary hypergroups*, Iran. J. Sci. Technol. 30(A2) (2006) 165-174.
- [9] B. Davvaz, W. A. Dudek, S. Mirvakili, *Neutral elements, fundamental relations and n-ary hypergroups*, International Journal of Algebra and Computation, 19(4) (2009) 567-583.
- [10] K. Denecke, S. L. Wismath, *Universal algebra and coalgebra*, World Scientific, 2009.
- [11] M. Dresher, O. Ore, *Theory of multigroups*, Amer. J. Math., 60 (1938) 705-733.

- [12] G. Grätzer, *A representation theorem for multi-algebras*, Arch. Math., 3 (1962) 452-456.
- [13] G. Grätzer, *Universal algebra. Second edition*, Springer-Verlag 1979.
- [14] H. Höft, P. E. Howard, *Representing multi-algebras by algebras, the axiom of choice, and the axiom of dependent choice*, Algebra Universalis, 13 (1981) 69-77.
- [15] G. E. Hansoul, *A subdirect decomposition theorem for multialgebras*, Algebra universalis, 16 (1983) 275-281.
- [16] J. Jantosciak, *Homomorphisms, equivalences and reductions in hypergroups*, Riv. Mat. Pura Appl., 9 (1991) 23-47.
- [17] M. Koskas, *Equivalences et homomorphismes des demi-hypergroupes*, C. R. Acad. Sci. Paris, t. 256, (1963).
- [18] M. Krasner, *Hypergroups extramoduliformes et moduli forms*, C. R. Acad. Sci.(paris) 219 (1944) 473-476.
- [19] J. Kuntzmann, *Homomorphie nre systèmrs multiformes*, C. R. Acad. Sci. Paris, 205, (1937).
- [20] V. Leoreanu, *New results on the hypergroups homomorphisms*, J. Inform. Optim. Sci., 20(2) (1999) 287-298.
- [21] V. Leoreanu-Fotea, B. Davvaz, *n-hypergroups and binary relations*, European J. Combinatorics, 29 (2008) 1207-1218.
- [22] F. Marty, *Sur une generalisation de la notion de groupe*, in: some congrès des Mathématiciens Scandinaves, Stockholm, (1934) 45-49.
- [23] C. Pelea, *On the fundamental relation of a multialgebra*, Ital. J. Pure Appl. Mat., 10 (2001) 327-342.
- [24] C. Pelea, *Identities and multialgebras*, Ital. J. Pure Appl. Mat., 15 (2004) 83-92.
- [25] C. Pelea, I. Purdea, *Multialgebras, universal algebras and identities*, J. Aust. Math. Soc., 81 (2006) 121-139.
- [26] H. E. Pickett, *Homomorphisms and subalgebras of multialgebras*. Pacific J. Math., 21 (1967) 327-342.
- [27] S. Rasouli and B. Davvaz, *Lattices derived from hyperlattices*, Communications in Algebra, 38(8) (2010) 2720-2737.
- [28] S. Rasouli, B. Davvaz, *Construction and spectral topology on Hyperlattices*, Mediterr. J. Math., 7 (2010) 249-262.
- [29] S. Rasouli, D. Heidari, B. Davvaz, *η -Relations and Transitivity Conditions of η on Hyper-MV algebras*, J. of Multi-Valued Logics & Soft Computing, 15, (2009) 517-524.
- [30] S. Rasouli, D. Heidari, B. Davvaz, *β -Relations and Transitivity Conditions of β on Hyper-BCK algebras*, Hacettepe Journal of Mathematics and Statistics, 39 (4) (2010), 461-469.
- [31] S. Rasouli, B. Davvaz, *Homomorphism, ideals and binary relations on hyper-MV algebras*, J. of Multi-Valued Logics & Soft Computing, 17, (2011) 47-68.
- [32] Y. Sureau, *Contribution a la thorie des hypergroupes et hypergroupes operant transivement sur un ensemble*, Doctoral Thesis, (1980).
- [33] T. Vougiouklis, *Hyperstructures and their representations*, Hadronic Press, Inc., 115, Palm Harber, USA, 1994.