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# On partially $\tau$-quasinormal subgroups of finite groups

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**Abstract**

Let $H$ be a subgroup of a group $G$. We say that: (1) $H$ is $\tau$-quasinormal in $G$ if $H$ permutes with every Sylow subgroup $Q$ of $G$ such that $(|H|, |Q|) = 1$ and $(|H|, |Q^G|) \neq 1$; (2) $H$ is partially $\tau$-quasinormal in $G$ if $G$ has a normal subgroup $T$ such that $HT$ is $S$-quasinormal in $G$ and $H \cap T \leq H_{\tau G}$, where $H_{\tau G}$ is the subgroup generated by all those subgroups of $H$ which are $\tau$-quasinormal in $G$. In this paper, we find a condition under which every chief factor of $G$ below a normal subgroup $E$ of $G$ is cyclic by using the partial $\tau$-quasinormality of some subgroups.

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## 1. Introduction

All groups considered in the paper are finite. The notations and terminology in this paper are standard, as in [4] and [6]. $G$ always denotes a finite group, $\pi(G)$ denotes the set of all prime dividing $|G|$ and $F^*(G)$ is the generalized Fitting subgroup of $G$, i.e., the product of all normal quasinilpotent subgroups of $G$.

Normal subgroup plays an important role in the study of the structure of groups. Many authors are interested to extend the concept of normal subgroup. For example, a subgroup $H$ of $G$ is said to be $S$-quasinormal [7] in $G$ if $H$ permutes with every Sylow subgroup of $G$. As a generalization of $S$-quasinormality, a subgroup $H$ of $G$ is said to be $\tau$-quasinormal [11] in $G$ if $H$ permutes with every Sylow subgroup $Q$ of $G$ such that $(|H|, |Q|) = 1$ and $(|H|, |Q^G|) \neq 1$. On the other hand, Wang [17] extended normality as

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follows: a subgroup $H$ of $G$ is said to be $c$-normal in $G$ if there exists a normal subgroup $K$ of $G$ such that $HK = G$ and $H \cap K \leq H_G$, where $H_G$ is the maximal normal subgroup of $G$ contained in $H$. In the literature, many people have studied the influence of the $\tau$-quasinormality and $c$-normality on the structure of finite groups and obtained many interesting results (see [2, 5, 8, 11, 12, 17, 19]). As a development, we now introduce a new concept:

**1.1. Definition.** A subgroup $H$ of a group $G$ is said to be partially $\tau$-quasinormal in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is $S$-quasinormal in $G$ and $H \cap T \leq H_{\tau G}$, where $H_{\tau G}$ is the subgroup generated by all those subgroups of $H$ which are $\tau$-quasinormal in $G$.

Clearly, partially $\tau$-quasinormal subgroup covers both the concepts of $\tau$-quasinormal subgroup and $c$-normal subgroup. However, the following examples show that the converse is not true.

**1.2. Example.** Let $G = S_4$ be the symmetric group of degree 4.

(1) Let $H$ be a Sylow 3-subgroup of $G$ and $N$ the normal abelian 2-subgroup of $G$ of order 4. Then $HN = A_4 \trianglelefteq G$ and $H \cap N$=1. Hence $H$ is a partially $\tau$-quasinormal subgroup of $G$. But, obviously, $H$ is not $c$-normal in $G$.

(2) Let $H$=$\langle(14)\rangle$. Obviously, $HA_4$=$G$ and $H \cap A_4$=1. Hence $H$ is partially $\tau$-quasinormal in $G$. But, obviously, $H$ is not $\tau$-quasinormal in $G$.

A normal subgroup $E$ of a group $G$ is said to be hypercyclically embedded in $G$ if every chief factor of $G$ below $E$ is cyclic. The product of all normal hypercyclically embedded subgroups of $G$ is denoted by $Z_{\mathcal{U}}(G)$. In [15] and [16], Skiba gave some characterizations of normal hypercyclically embedded subgroups related to $S$-quasinormal subgroups. The main purpose of this paper is to give a new characterization by using partially $\tau$-quasinormal property of maximal subgroups of some Sylow subgroups. We obtain the following result.

**Main Theorem.** *Let $E$ be a normal subgroup of $G$. Suppose that there exists a normal subgroup $X$ of $G$ such that $F^*(E) \leq X \leq E$ and $X$ satisfies the following properties: for every non-cyclic Sylow $p$-subgroup $P$ of $X$, every maximal subgroup of $P$ not having a supersoluble supplement in $G$ is partially $\tau$-quasinormal in $G$. Then $E$ is hypercyclically embedded in $G$.*

The following theorems are the main stages in the proof of Main Theorem.

**1.3. Theorem.** Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of $P$ not having a $p$-nilpotent supplement in $G$ is partially $\tau$-quasinormal in $G$, then $G$ is soluble.

**1.4. Theorem.** Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then $G$ is $p$-nilpotent if and only if every maximal subgroup of $P$ not having a $p$-nilpotent supplement in $G$ is partially $\tau$-quasinormal in $G$.

**1.5. Theorem.** Let $E$ be a normal subgroup in $G$ and let $P$ be a Sylow $p$-subgroup of $E$, where $p$ is a prime divisor of $|E|$ with $(|E|, p-1) = 1$. Suppose that every maximal subgroup of $P$ not having a $p$-supersoluble supplement in $G$ is partially $\tau$-quasinormal in $G$. Then each chief factor of $G$ between $E$ and $O_{p'}(E)$ is cyclic.

**1.6. Theorem.** Let $E$ be a normal subgroup of a group $G$. Suppose that for each $p \in \pi(E)$, every maximal subgroup of non-cyclic Sylow $p$-subgroup $P$ of $E$ not having a $p$-supersoluble supplement in $G$ is partially $\tau$-quasinormal in $G$. Then every chief factor of $G$ below $E$ is cyclic.

## 2. Preliminaries

**2.1. Lemma** ( [3] and [7])**.** Suppose that $H$ is a subgroup of $G$ and $H$ is $S$-quasinormal in $G$. Then

(1) If $H \leq K \leq G$, then $H$ is $S$-quasinormal in $K$.

(2) If $N$ is a normal subgroup of $G$, then $HN$ is $S$-quasinormal in $G$ and $HN/N$ is $S$-quasinormal in $G/N$.

(3) If $K \leq G$, then $H \cap K$ is $S$-quasinormal in $K$.

(4) $H$ is subnormal in $G$.

(5) If $K \leq G$ and $K$ is $S$-quasinormal in $G$, then $H \cap K$ is $S$-quasinormal in $G$.

**2.2. Lemma** ([11, Lemmas 2.2 and 2.3])**.** Let $G$ be a group and $H \leq K \leq G$.

(1) If $H$ is $\tau$-quasinormal in $G$, then $H$ is $\tau$-quasinormal in $K$.

(2) Suppose that $H$ is normal in $G$ and $\pi(K/H) = \pi(K)$. If $K$ is $\tau$-quasinormal in $G$, then $K/H$ is $\tau$-quasinormal in $G/H$.

(3) Suppose that $H$ is normal in $G$. Then $EH/H$ is $\tau$-quasinormal in $G/H$ for every $\tau$-quasinormal subgroup $E$ in $G$ satisfying $(|H|, |E|) = 1$.

(4) If $H$ is $\tau$-quasinormal in $G$ and $H \leq O_p(G)$ for some prime $p$, then $H$ is $S$-quasinormal in $G$.

(5) $H_{\tau G} \leq H_{\tau K}$.

(6) Suppose that $K$ is a $p$-group and $H$ is normal in $G$. Then $K_{\tau G}/H \leq (K/H)_{\tau(G/H)}$.

(7) Suppose that $H$ is normal in $G$. Then $E_{\tau G}H/H \leq (EH/H)_{\tau(G/H)}$ for every $p$-subgroup $E$ of $G$ satisfying $(|H|, |E|) = 1$.

**2.3. Lemma.** Let $G$ be a group and $H \leq G$. Then

(1) If $H$ is partially $\tau$-quasinormal in $G$ and $H \leq K \leq G$, then $H$ is partially $\tau$-quasinormal in $K$.

(2) Suppose that $N \trianglelefteq G$ and $N \leq H$. If $H$ is a $p$-group and $H$ is partially $\tau$-quasinormal in $G$, then $H/N$ is partially $\tau$-quasinormal in $G/N$.

(3) Suppose that $H$ is a $p$-subgroup of $G$ and $N$ is a normal $p'$-subgroup of $G$. If $H$ is partially $\tau$-quasinormal in $G$, then $HN/N$ is partially $\tau$-quasinormal in $G/N$.

(4) If $H$ is partially $\tau$-quasinormal in $G$ and $H \leq K \trianglelefteq G$, then there exists $T \trianglelefteq G$ such that $HT$ is $S$-quasinormal in $G$, $H \cap T \leq H_{\tau G}$ and $HT \leq K$.

*Proof.* (1) Let $N$ be a normal subgroup of $G$ such that $HN$ is $S$-quasinormal in $G$ and $H \cap N \leq H_{\tau G}$. Then $K \cap N \trianglelefteq K$, $H(K \cap N) = HN \cap K$ is $S$-quasinormal in $K$ by Lemma 2.1(3) and $H \cap (K \cap N) = H \cap N \leq H_{\tau G} \leq H_{\tau K}$ by Lemma 2.2(5). Hence $H$ is partially $\tau$-quasinormal in $K$.

(2) Suppose that $H$ is partially $\tau$-quasinormal in $G$. Then there exists $K \trianglelefteq G$ such that $HK$ is $S$-quasinormal in $G$ and $H \cap K \leq H_{\tau G}$. This implies that $KN/N \trianglelefteq G/N$ and $(H/N)(KN/N) = HK/N$ is $S$-quasinormal in $G/N$ by Lemma 2.1(2). In view of Lemma 2.2(6), $H/N \cap KN/N = (H \cap K)N/N \leq H_{\tau G}N/N = H_{\tau G}/N \leq (H/N)_{\tau(G/N)}$. Thus $H/N$ is partially $\tau$-quasinormal in $G/N$.

(3) Suppose that $H$ is partially $\tau$-quasinormal in $G$. Then there exists $K \trianglelefteq G$ such that $HK$ is $S$-quasinormal in $G$ and $H \cap K \leq H_{\tau G}$. Clearly, $KN/N \trianglelefteq G$ and $(HN/N)(KN/N) = HKN/N$ is $S$-quasinormal in $G/N$ by Lemma 2.1(2). On the other hand, since $(|HN : H|, |HN : N|)$=1, $HN/N \cap KN/N = (HN \cap K)N/N = (H \cap K)(N \cap K)N/N = (H \cap K)N/N \leq H_{\tau G}N/N$. In view of Lemma 2.2(7), we have $H_{\tau G}N/N \leq (HN/N)_{\tau(G/N)}$. Hence $HN/N$ is partially $\tau$-quasinormal in $G/N$.

(4) Suppose that $H$ is partially $\tau$-quasinormal in $G$. Then there exists $N \trianglelefteq G$ such that $HN$ is $S$-quasinormal in $G$ and $H \cap N \leq H_{\tau G}$. Let $T = N \cap K$. Then $T \trianglelefteq G$, $HT = H(N \cap K) = HN \cap K$ is $S$-quasinormal in $G$ by Lemma 2.1(5), $HT \leq K$ and $H \cap T = H \cap N \cap K = H \cap N \leq H_{\tau G}$. □

**2.4. Lemma.** Let $G$ be a group and $p$ a prime dividing $|G|$ with $(|G|, p-1) = 1$.

(1) If $N$ is normal in $G$ of order $p$, then $N$ lies in $Z(G)$.
(2) If $G$ has cyclic Sylow $p$-subgroups, then $G$ is $p$-nilpotent.
(3) If $M \leq G$ and $|G : M| = p$, then $M \trianglelefteq G$.
(4) If $G$ is $p$-supersoluble, then $G$ is $p$-nilpotent.

*Proof.* (1), (2) and (3) can be found in [18, Theorem 2.8]. Now we only prove (4). Let $A/B$ be an arbitrary chief factor of $G$. If $G$ is $p$-supersolvable, then $A/B$ is either a cyclic group with order $p$ or a $p'$-group. If $|A/B| = p$, then $|\mathrm{Aut}(A/B)| = p - 1$. Since $G/C_G(A/B)$ is isomorphic to a subgroup of $\mathrm{Aut}((A/B)$, the order of $G/C_G(A/B)$ must divide $(|G|, p-1) = 1$, which shows that $G = C_G(A/B)$. Therefore, we have $G$ is $p$-nilpotent. □

**2.5. Lemma** ([10, Lemma 2.12])**.** Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of $P$ has a $p$-nilpotent supplement in $G$, then $G$ is $p$-nilpotent.

**2.6. Lemma** ([13, Theorem A])**.** If $P$ is an $S$-quasinormal $p$-subgroup of a group $G$ for some prime $p$, then $N_G(P) \geq O^p(G)$.

**2.7. Lemma** ([6, VI, 4.10])**.** Assume that $A$ and $B$ are two subgroups of a group $G$ and $G \neq AB$. If $AB^g = B^gA$ holds for any $g \in G$, then either $A$ or $B$ is contained in a nontrivial normal subgroup of $G$.

**2.8. Lemma** ([20, Chap.1, Theorem 7.19])**.** Let $H$ be a normal subgroup of $G$. Then $H \leq Z_{\mathscr{U}}(G)$ if and only if $H/\Phi(H) \leq Z_{\mathscr{U}}(G/\Phi(H))$.

**2.9. Lemma** ([14, Lemma 2.11])**.** Let $N$ be an elementary abelian normal subgroup of a group $G$. Assume that $N$ has a subgroup $D$ such that $1 < |D| < |N|$ and every subgroup $H$ of $N$ satisfying $|H| = |D|$ is $S$-quasinormal in $G$. Then some maximal subgroup of $N$ is normal in $G$.

**2.10. Lemma.** Let $N$ be a non-identity normal $p$-subgroup of a group $G$. If $N$ is elementary and every maximal subgroup of $N$ is partially $\tau$-quasinormal in $G$, then some maximal subgroup of $N$ is normal in $G$.

*Proof.* If $|N| = p$, then it is clear. Let $L$ be a non-identity minimal normal $p$-subgroup of $G$ contained in $N$. First we assume that $N \neq L$. By Lemma 2.3(2), the hypothesis still holds on $G/L$. Then by induction some maximal subgroup $M/L$ of $N/L$ is normal in $G/L$. Clearly, $M$ is a maximal subgroup of $N$ and $M$ is normal in $G$. Consequently the lemma follows. Now suppose that $L = N$. Let $M$ be any maximal subgroup of $N$. Then by the hypothesis, there exists $T \trianglelefteq G$ such that $MT$ is $S$-quasinormal in $G$ and $M \cap T \leq M_{\tau G}$. Suppose that $M \neq M_{\tau G}$. Then $MT \neq M$ and $T \neq 1$. If $N \leq MT$, then $N = N \cap MT = M(N \cap T)$. Hence $N \leq T$, which implies that $M = M \cap T = M_{\tau G}$, a contradiction. If $N \nleq MT$, then $M = M(T \cap N) = MT \cap N$ is $S$-quasinormal in $G$ by Lemma 2.1(5), a contradiction again. Hence $M = M_{\tau G}$. In view of Lemma 2.2(4), $M$ is $S$-quasinormal in $G$. By Lemma 2.9, some maximal subgroup of $N$ is normal in $G$. Thus the lemma holds. □

**2.11. Lemma** ([15, Theorem B])**.** Let $\mathscr{F}$ be any formation and $G$ a group. If $H \triangleleft G$ and $F^*(H) \leq Z_{\mathscr{F}}(G)$, then $H \leq Z_{\mathscr{F}}(G)$.

## 3. Proofs of Theorems

*Proof of Theorem 1.3.* Assume that this theorem is false and let $G$ be a counterexample with minimal order. We proceed the proof via the following steps.

(1) $O_p(G) = 1$.

Assume that $L = O_p(G) \neq 1$. Clearly, $P/L$ is a Sylow $p$-subgroup of $G/L$. Let $M/L$ be a maximal subgroup of $P/L$. Then $M$ is a maximal subgroup of $P$. If $M$ has a $p$-nilpotent supplement $D$ in $G$, then $M/L$ has a $p$-nilpotent supplement $DL/L$ in $G/L$. If $M$ is partially $\tau$-quasinormal in $G$, then $M/L$ is partially $\tau$-quasinormal in $G/L$ by Lemma 2.3(2). Hence $G/L$ satisfies the hypothesis of the theorem. The minimal choice of $G$ implies that $G/L$ is soluble. Consequently, $G$ is soluble. This contradiction shows that step (1) holds.

(2) $O_{p'}(G) = 1$.

Assume that $R = O_{p'}(G) \neq 1$. Then, obviously, $PR/R$ is a Sylow $p$-subgroup of $G/R$. Suppose that $M/R$ is a maximal subgroup of $PR/R$. Then there exists a maximal subgroup $P_1$ of $P$ such that $M = P_1R$. If $P_1$ has a $p$-nilpotent supplement $D$ in $G$, then $M/R$ has a $p$-nilpotent supplement $DR/R$ in $G/R$. If $P_1$ is partially $\tau$-quasinormal in $G$, then $M/R$ is partially $\tau$-quasinormal in $G/R$ by Lemma 2.3(3). The minimal choice of $G$ implies that $G/R$ is soluble. By the well known Feit-Thompson's theorem, we know that $R$ is soluble. It follows that $G$ is soluble, a contradiction.

(3) $P$ is not cyclic.

If $P$ is cyclic, then $G$ is $p$-nilpotent by Lemma 2.4, and so $G$ is soluble, a contradiction.

(4) If $N$ is a minimal normal subgroup of $G$, then $N$ is not soluble. Moreover, $G = PN$.

If $N$ is $p$-soluble, then $O_p(N) \neq 1$ or $O_{p'}(N) \neq 1$. Since $O_p(N)$ char $N \trianglelefteq G$, $O_p(N) \leq O_p(G)$. Analogously $O_{p'}(N) \leq O_{p'}(G)$. Hence $O_p(G) \neq 1$ or $O_{p'}(G) \neq 1$, which contradicts step (1) or step (2). Therefore $N$ is not soluble. Assume that $PN < G$. By Lemma 2.3(1), every maximal subgroup of $P$ not having a $p$-nilpotent supplement in $PN$ is partially $\tau$-quasinormal in $PN$. Thus $PN$ satisfies the hypothesis. By the minimal choice of $G$, $PN$ is soluble and so $N$ is soluble. This contradiction shows that $G = PN$.

(5) $G$ has a unique minimal normal subgroup $N$.

By step (4), we see that $G = PN$ for every normal subgroup $N$ of $G$. It follows that $G/N$ is soluble. Since the class of all soluble groups is closed under subdirect product, $G$ has a unique minimal normal subgroup, say $N$.

(6) The final contradiction.

If every maximal subgroup of $P$ has a $p$-nilpotent supplement in $G$, then, in view of Lemma 2.5, $G$ is $p$-nilpotent and so $G$ is soluble. This contradiction shows that we may choose a maximal subgroup $P_1$ of $P$ such that $P_1$ is partially $\tau$-quasinormal in $G$. Then there exists a normal subgroup $T$ of $G$ such that $P_1T$ is $S$-quasinormal in $G$ and $P_1 \cap T \leq (P_1)_{\tau G}$. If $T = 1$, then $P_1$ is $S$-quasinormal in $G$. In view of Lemma 2.6, $P_1 \trianglelefteq PO^p(G) = G$. By step (5), $P_1 = 1$ or $N \leq P_1$. Since $N$ is not soluble by step (4), we have that $P_1 = 1$. Consequently, $P$ is cyclic, which contradicts step (3). Hence $T \neq 1$ and $N \leq T$. It follows that $P_1 \cap N = (P_1)_{\tau G} \cap N$. For any Sylow $q$-subgroup $N_q$ of $N$ with $q \neq p$, $N_q$ is also a Sylow $q$-subgroup of $G$ by step (4). From step (2) it is easy to see that $(P_1)_{\tau G}N_q = N_q(P_1)_{\tau G}$. Then $(P_1)_{\tau G}N_q \cap N = N_q((P_1)_{\tau G} \cap N) = N_q(P_1 \cap N)$, i.e., $P_1 \cap N$ is $\tau$-quasinormal in $N$. Since $N$ is a direct product of some isomorphic non-abelian simple groups, we may assume that $N \cong N_1 \times \cdots \times N_k$. By Lemma 2.2(1), $P_1 \cap N$ is $\tau$-quasinormal in $(P_1 \cap N)N_1$. Thus $(P_1 \cap N)(N_{1q})^{n_1} \cap N_1 = (N_{1q})^{n_1}(P_1 \cap N \cap N_1) = (N_{1q})^{n_1}(P_1 \cap N_1)$ for any $n_1 \in N_1$, where $N_{1q}$ is a Sylow $q$-subgroup of $N_1$ with $q \neq p$. Since $(N_{1q})^{n_1}(P_1 \cap N_1) \neq N_1$, we have $N_1$ is not simple by Lemma 2.7, a contradiction.

*Proof of Theorem 1.4.* If $G$ is $p$-nilpotent, then $G$ has a normal Hall $p'$-subgroup $G_{p'}$. Let $P_1$ be any maximal subgroup of $P$. Then $|G : P_1G_{p'}| = p$. In view of Lemma 2.4(3), $P_1G_{p'} \trianglelefteq G$. Obviously, $P_1 \cap G_{p'} = 1$. Hence $P_1$ is partially $\tau$-quasinormal in $G$.

Now we prove the sufficient part. Assume that the assertion is false and let $G$ be a counterexample with minimal order.

(1) $G$ is soluble.

It follows directly from Theorem 1.3.

(2) $G$ has a unique minimal normal subgroup $N$ such that $G/N$ is $p$-nilpotent. Moreover, $\Phi(G) = 1$.

Let $N$ be a minimal normal subgroup of $G$. Since $G$ is solvable by step (1), $N$ is an elementary abelian subgroup. It is easy to see that $G/N$ satisfies the hypothesis of our theorem by Lemma 2.3. By the minimal choice of $G$, $G/N$ is $p$-nilpotent. Since the class of all $p$-nilpotent groups is a saturated formation, $N$ is a unique minimal normal subgroup of $G$ and $\Phi(G) = 1$.

(3) $P$ is not cyclic.

If $P$ is cyclic, $G$ is $p$-nilpotent by Lemma 2.4(2), a contradiction.

(4) $O_{p'}(G) = 1$.

(5) Every maximal subgroup of $P$ has a $p$-nilpotent supplement in $G$.

It is clear that $N \leq O_p(G)$. By $\Phi(G) = 1$, we may choose a maximal subgroup $M$ of $G$ such that $G = NM$ and $G/N \cong M$. Let $P_1$ be an arbitrary maximal subgroup of $P$. We will show $P_1$ has a $p$-nilpotent supplement in $G$. Since $N$ has the $p$-nilpotent supplement $M$ in $G$, we only need to prove $N \leq P_1$ when $P_1$ is partially $\tau$-quasinormal in $G$. Let $T$ be a normal subgroup of $G$ such that $P_1T$ is $S$-quasinormal in $G$ and $P_1 \cap T \leq (P_1)_{\tau G}$. First, we assume that $T = 1$, i.e., $P_1$ is $S$-quasinormal in $G$. In view of Lemma 2.6, $P_1 \trianglelefteq PO^p(G) = G$. By virtue of Lemma 2.4(2) and step (3), $P_1 \neq 1$. Hence $N \leq P_1$ by step (2). Now, assume that $T \neq 1$. Then $N \leq T$. It follows that $P_1 \cap N = (P_1)_{\tau G} \cap N$. For any Sylow $q$-subgroup $G_q$ of $G$ ($p \neq q$), $(P_1)_{\tau G}G_q = G_q(P_1)_{\tau G}$ in view of step (4). Then $(P_1)_{\tau G} \cap N = (P_1)_{\tau G}G_q \cap N \trianglelefteq (P_1)_{\tau G}G_q$. Obviously, $P_1 \cap N \trianglelefteq P$. Therefore $P_1 \cap N$ is normal in $G$. By the minimality of $N$, we have $P_1 \cap N = N$ or $P_1 \cap N = 1$. If the later holds, then the order of $N$ is $p$ since $P_1 \cap N$ is a maximal subgroup of $N$. Consequently, $G$ is $p$-nilpotent by step (2) and Lemma 2.4(1). This contradiction shows that $P_1 \cap N = N$ and so $N \leq P_1$.

(6) The final contradiction.

Since every maximal subgroup of $P$ has a $p$-nilpotent supplement in $G$ by step (5), we have $G$ is $p$-nilpotent by Lemma 2.5, a contradiction.

*Proof of Theorem 1.5.* Assume that this theorem is false and and consider a counterexample $(G, E)$ for which $|G||E|$ is minimal.

(1) $E$ is $p$-nilpotent.

Let $P_1$ be a maximal subgroup of $P$. If $P_1$ has a $p$-supersolvable supplement $T$ in $G$, then $P_1$ has a $p$-supersolvable supplement $T \cap E$ in $E$. Since $(|E|, p-1) = 1$, $T \cap E$ is also $p$-nilpotent by Lemma 2.4(4). If $P_1$ is partially $\tau$-quasinormal in $G$, then $P_1$ is also partially $\tau$-quasinormal in $E$ by Lemma 2.3(1). Hence every maximal subgroup of $P$ not having a $p$-nilpotent supplement in $E$ is partially $\tau$-quasinormal in $E$. In view of Theorem 1.4, $E$ is $p$-nilpotent.

(2) $P = E$.

By step (1), $O_{p'}(E)$ is the normal Hall $p'$-subgroup of $E$. Suppose that $O_{p'}(E) \neq 1$. It is easy to see that the hypothesis of the theorem holds for $(G/O_{p'}(E), E/O_{p'}(E))$. By induction, every chief factor of $G/O_{p'}(E)$ between $E/O_{p'}(E)$ and 1 is cyclic. Consequently, each chief factor of $G$ between $E$ and $O_{p'}(E)$ is cyclic. This condition shows that $O_{p'}(E) = 1$ and so $P = E$.

(3) $\Phi(P) = 1$.

Suppose that $\Phi(P) \neq 1$. By Lemma 2.3(2), it is easy to see that the hypothesis of the theorem holds for $(G/\Phi(P), P/\Phi(P))$. By the choice of $(G, E)$, every chief factor of $G/\Phi(P)$ below $P/\Phi(P)$ is cyclic. In view of Lemma 2.8, every chief factor of $G$ below $P$ is cyclic, a contradiction.

(4) Every maximal subgroup of $P$ is partially $\tau$-quasinormal in $G$.

Suppose that there is some maximal subgroup $V$ of $P$ such that $V$ has a $p$-supersolvable supplement $B$ in $G$, then $G = PB$ and $P \cap B \neq 1$. Since $P \cap B \trianglelefteq B$, we may assume that $B$ has a minimal normal subgroup $N$ contained in $P \cap B$. It is clear that $|N| = p$. Since $P$ is elementary abelian and $G = PB$, we have that $N$ is also normal in $G$. It is easy to see that the hypothesis is still true for $(G/N, P/N)$. Hence every chief factor of $G/N$ below $P/N$ is cyclic by virtue of the choice of $(G, E)$. It follows that every chief factor of $G$ below $P$ is cyclic. This contradiction shows that all maximal subgroups of $P$ are partially $\tau$-quasinormal in $G$.

(5) $P$ is not a minimal normal subgroup of $G$.

Suppose that $P$ is a minimal normal subgroup of $G$, then some maximal subgroup of $P$ is normal in $G$ by Lemma 2.10, which contradicts the minimality of $P$.

(6) If $N$ is a minimal normal subgroup of $G$ contained in $P$, then $P/N \leq Z_{\mathscr{U}}(G/N)$, $N$ is the only minimal normal subgroup of $G$ contained in $P$ and $|N| > p$.

Indeed, by Lemma 2.3(2), the hypothesis holds on $(G/N, P/N)$ for any minimal normal subgroup $N$ of $G$ contained in $P$. Hence every chief factor of $G/N$ below $P/N$ is cyclic by the choice of $(G, E) = (G, P)$. If $|N| = p$, every chief factor of $G$ below $P$ is cyclic, a contradiction. If $G$ has two minimal normal subgroups $R$ and $N$ contained in $P$, then $NR/R \leq P/R$ and from the $G$-isomorphism $NR/R \cong N$ we have $|N| = p$, a contradiction. Hence, (6) holds.

(7) The final contradiction.

Let $N$ be a minimal normal subgroup of $G$ contained in $P$ and $N_1$ any maximal subgroup of $N$. We show that $N_1$ is $S$-quasinormal in $G$. Since $P$ is an elementary abelian $p$-group, we may assume that $D$ is a complement of $N$ in $P$. Let $V = N_1D$. Obviously, $V$ is a maximal subgroup of $P$. By step (4), $V$ is partially $\tau$-quasinormal in $G$. By Lemma 2.3(4), there exist a normal subgroup $T$ of $G$ such that $VT$ is $S$-quasinormal in $G$, $V \cap T \leq V_{\tau G}$ and $VT \leq P$. In view of Lemma 2.2(4), $V_{\tau G}$ is an $S$-quasinormal subgroup of $G$. If $T = P$, then $V = V_{\tau G}$ is $S$-quasinormal in $G$ and hence $V \cap N = N_1D \cap N = N_1(D \cap N) = N_1$ is $S$-quasinormal in $G$ by Lemma 2.1(5). If $T = 1$, then $V = VT$ is $S$-quasinormal in $G$. Consequently, we have also $N_1$ is $S$-quasinormal in $G$. Now we assume that $1 < T < P$. Hence $N \leq T$ by step (6). Then, $N_1 = V \cap N = V_{\tau G} \cap N$ is $S$-quasinormal in $G$ by virtue of Lemma 2.1(5). Hence some maximal subgroup of $N$ is normal in $G$ by Lemma 2.9. Consequently, $|N| = p$. This contradicts step (6).

*Proof of Theorem 1.6.* Let $q$ be the smallest prime dividing $|E|$. In view of step (1) of the proof of Theorem 1.5, $E$ is $q$-nilpotent. Let $E_{q'}$ be the normal Hall $q'$-subgroup of $E$. If $E_{q'} = 1$, then every chief factor of $G$ below $E$ is cyclic by Theorem 1.5. Hence we may assume that $E_{q'} \neq 1$. Since $E_{q'}$ char $E \trianglelefteq G$, we see that $E_{q'} \trianglelefteq G$. By Lemma 2.3(3), the hypothesis of the theorem holds for $(G/E_{q'}, E/E_{q'})$. By induction, every chief factor of $G/E_{q'}$ below $E/E_{q'}$ is cyclic. On the other hand, $(G, E_{q'})$ also satisfies the hypothesis of the theorem in view of Lemma 2.3(1). By induction again, we have also every chief factor of $G$ below $E_{q'}$ is cyclic. Hence it follows that every chief factor of $G$ below $E$ is cyclic.

*Proof of Main Theorem.* Applying Theorem 1.6, $X$ is hypercyclically embedded in $G$. Since $F^*(E) \leq X$, we have that $F^*(E)$ is also hypercyclically embedded in $G$. By virtue of Lemma 2.11, $E$ is also hypercyclically embedded in $G$.

## 4. Some Applications

**4.1. Theorem.** Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and $E$ a normal subgroup of a group $G$ such that $G/E \in \mathscr{F}$. Suppose that for every non-cyclic Sylow subgroup $P$ of $E$, every maximal subgroup of $P$ not having a supersoluble supplement in $G$ is partially $\tau$-quasinormal in $G$. Then $G \in \mathscr{F}$.

*Proof.* Applying our Main Theorem, every chief factor of $G$ below $E$ is cyclic. Since $\mathscr{F}$ contains $\mathscr{U}$, we know $E$ is contained in the $\mathscr{F}$-hypercentre of $G$. From $G/E \in \mathscr{F}$, it follows that $G \in \mathscr{F}$. □

**4.2. Theorem.** Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and $E$ a normal subgroup of a group $G$ such that $G/E \in \mathscr{F}$. Suppose that for every non-cyclic Sylow subgroup $P$ of $F^*(E)$, every maximal subgroup of $P$ not having a supersoluble supplement in $G$ is partially $\tau$-quasinormal in $G$. Then $G \in \mathscr{F}$.

*Proof.* The proof is similar to that of Theorem 4.1. □

**4.3. Corollary** ([9, Theorem 3.4])**.** Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and $E$ a normal subgroup of a group $G$ such that $G/E \in \mathscr{F}$. If every maximal subgroup of any Sylow subgroup of $F^*(E)$ is $S$-quasinormal in $G$, then $G \in \mathscr{F}$.

**4.4. Corollary** ([19, Theorem 3.4])**.** Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and $E$ a normal subgroup of a group $G$ such that $G/E \in \mathscr{F}$. If every maximal subgroup of any Sylow subgroup of $F^*(E)$ is $c$-normal in $G$, then $G \in \mathscr{F}$.

**4.5. Corollary** ([1, Theorem 1.4])**.** Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and $E$ a soluble normal subgroup of a group $G$ such that $G/E \in \mathscr{F}$. If every maximal subgroup of any Sylow subgroup of $F(E)$ is $S$-quasinormal in $G$, then $G \in \mathscr{F}$.

**4.6. Corollary** ([8, Theorem 2])**.** Let $G$ be a group and $E$ a soluble normal subgroup of $G$ such that $G/E$ is supersolvable. If all maximal subgroups of the Sylow subgroups of $F(E)$ are $c$-normal in $G$, then $G$ is supersolvable.

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