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Hermite-Hadamard type inequalities for harmonically convex functions

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Abstract

The author introduces the concept of harmonically convex functions and establishes some Hermite-Hadamard type inequalities of these classes of functions.

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1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. For some results which generalize, improve and extend the inequalities (1.1) we refer the reader to the recent papers (see [1, 2, 3, 4, 6, 5, 7]).

The main purpose of this paper is to introduce the concept of harmonically convex functions and establish some results connected with the right-hand side of new inequalities similar to the inequality (1.1) for these classes of functions. Some applications to special means of positive real numbers are also given.

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2. Main Results

2.1. Definition. Let $I \subset \mathbb{R}\backslash \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y)+(1-t)f(x) \tag{2.1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be harmonically concave.

2.2. Example. Let $f : (0,\infty) \to \mathbb{R},\ f(x) = x$, and $g : (-\infty, 0) \to \mathbb{R},\ g(x) = x$, then f is a harmonically convex function and g is a harmonically concave function.

The following proposition is obvious from this example:

2.3. Proposition. *Let $I \subset \mathbb{R}\backslash \{0\}$ be a real interval and $f : I \to \mathbb{R}$ is a function, then ;*

- *if $I \subset (0,\infty)$ and f is convex and nondecreasing function then f is harmonically convex.*
- *if $I \subset (0,\infty)$ and f is harmonically convex and nonincreasing function then f is convex.*
- *if $I \subset (-\infty,0)$ and f is harmonically convex and nondecreasing function then f is convex.*
- *if $I \subset (-\infty,0)$ and f is convex and nonincreasing function then f is a harmonically convex.*

The following result of the Hermite-Hadamard type holds.

2.4. Theorem. *Let $f : I \subset \mathbb{R}\backslash \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{2.2}$$

The above inequalities are sharp.

Proof. Since $f : I \to \mathbb{R}$ is a harmonically convex function, we have, for all $x, y \in I$ (with $t = \frac{1}{2}$ in the inequality (2.1))

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(y)+f(x)}{2}.$$

Choosing $x = \frac{ab}{ta+(1-t)b},\ y = \frac{ab}{tb+(1-t)a}$, we get

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{f\left(\frac{ab}{tb+(1-t)a}\right)+f\left(\frac{ab}{ta+(1-t)b}\right)}{2}.$$

Further, integrating for $t \in [0,1]$, we have

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right) \\ \leq\ & \frac{1}{2}\left[\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right)dt+\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right)dt\right]. \end{aligned} \tag{2.3}$$

Since each of the integrals is equal to $\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$, we obtain the left-hand side of the inequality (2.2) from (2.3).

The proof of the second inequality follows by using (2.1) with $x = a$ and $y = b$ and integrating with respect to t over $[0, 1]$.

Now, consider the function $f : (0, \infty) \to \mathbb{R}$, $f(x) = 1$. thus

$$\begin{aligned} 1 &= f\left(\frac{xy}{tx + (1-t)y}\right) \\ &= tf(y) + (1-t)f(x) = 1 \end{aligned}$$

for all $x, y \in (0, \infty)$ and $t \in [0, 1]$. Therefore f is harmonically convex on $(0, \infty)$. We also have

$$f\left(\frac{2ab}{a+b}\right) = 1, \quad \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx = 1,$$

and

$$\frac{f(a) + f(b)}{2} = 1$$

which shows us the inequalities (2.2) are sharp. ∎

For finding some new inequalities of Hermite-Hadamard type for functions whose derivatives are harmonically convex, we need a simple lemma below.

2.5. Lemma. *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $a < b$. If $f' \in L[a, b]$ then*

$$\begin{aligned} &\frac{f(a) + f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \\ &= \frac{ab(b-a)}{2}\int_0^1 \frac{1-2t}{(tb + (1-t)a)^2} f'\left(\frac{ab}{tb + (1-t)a}\right) dt. \end{aligned} \tag{2.4}$$

Proof. Let

$$I^* = \frac{ab(b-a)}{2}\int_0^1 \frac{1-2t}{(tb + (1-t)a)^2} f'\left(\frac{ab}{tb + (1-t)a}\right) dt.$$

By integrating by part, we have

$$I^* = \frac{(2t-1)}{2} f\left(\frac{ab}{tb + (1-t)a}\right)\Bigg|_0^1 - \int_0^1 f\left(\frac{ab}{tb + (1-t)a}\right) dt.$$

Setting $x = \frac{ab}{tb+(1-t)a}$, $dx = \frac{-ab(b-a)}{(tb+(1-t)a)^2}dt = \frac{-x^2(b-a)}{ab}dt$, we obtain

$$I^* = \frac{f(a) + f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx$$

which gives the desired representation (2.4). ∎

2.6. Theorem. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a<b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}\left[\lambda_2\left|f'(a)\right|^q + \lambda_3\left|f'(b)\right|^q\right]^{\frac{1}{q}}, \tag{2.5}$$

where

$$\begin{aligned}
\lambda_1 &= \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right),\\
\lambda_2 &= \frac{-1}{b(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right),\\
\lambda_3 &= \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right)\\
&= \lambda_1 - \lambda_2.
\end{aligned}$$

Proof. From Lemma 2.5 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|\\
\leq\ &\frac{ab(b-a)}{2}\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|dt\\
\leq\ &\frac{ab(b-a)}{2}\left(\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|\frac{1-2t}{(tb+(1-t)a)^2}\right|\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence, by harmonically convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|\\
\leq\ &\frac{ab(b-a)}{2}\left(\int_0^1\frac{|1-2t|}{(tb+(1-t)a)^2}dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\frac{|1-2t|\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]}{(tb+(1-t)a)^2}dt\right)^{\frac{1}{q}}\\
\leq\ &\frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}\left[\lambda_2\left|f'(a)\right|^q+\lambda_3\left|f'(b)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

It is easily check that

$$\begin{aligned}&\int_0^1 \frac{|1-2t|}{(tb+(1-t)a)^2}dt \\ =\ & \frac{1}{ab}-\frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right),\end{aligned}$$

$$\begin{aligned}&\int_0^1 \frac{|1-2t|\,(1-t)}{(tb+(1-t)a)^2}dt \\ =\ & \frac{1}{a\,(b-a)}-\frac{3b+a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right),\end{aligned}$$

$$\begin{aligned}&\int_0^1 \frac{|1-2t|\,t}{(tb+(1-t)a)^2}dt \\ =\ & \frac{-1}{b\,(b-a)}+\frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right).\end{aligned}$$

∎

2.7. Theorem. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q > 1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\ \leq\ & \frac{ab\,(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mu_1\,|f'(a)|^q+\mu_2\,|f'(b)|^q\right)^{\frac{1}{q}},\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}\mu_1 &= \frac{\left[a^{2-2q}+b^{1-2q}\left[(b-a)(1-2q)-a\right]\right]}{2\,(b-a)^2\,(1-q)\,(1-2q)},\\ \mu_2 &= \frac{\left[b^{2-2q}-a^{1-2q}\left[(b-a)(1-2q)+b\right]\right]}{2\,(b-a)^2\,(1-q)\,(1-2q)}.\end{aligned}$$

Proof. From Lemma 2.5, Hölder's inequality and the harmonically convexity of $|f'|^q$ on $[a,b]$,we have,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\
&\leq \frac{ab\,(b-a)}{2}\left(\int_0^1|1-2t|^p\,dt\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1\frac{1}{(tb+(1-t)a)^{2q}}\left|f'\left(\frac{ab}{tb+(1-t)a}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{ab\,(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1\frac{t\,|f'(a)|^q+(1-t)\,|f'(b)|^q}{(tb+(1-t)a)^{2q}}dt\right)^{\frac{1}{q}},
\end{aligned}$$

where an easy calculation gives

$$\begin{aligned}
&\int_0^1\frac{t}{(tb+(1-t)a)^{2q}}dt \\
&= \frac{\left[a^{2-2q}+b^{1-2q}\left[(b-a)(1-2q)-a\right]\right]}{2\,(b-a)^2\,(1-q)\,(1-2q)}
\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}
&\int_0^1\frac{1-t}{(tb+(1-t)a)^{2q}}dt \\
&= \frac{\left[b^{2-2q}-a^{1-2q}\left[(b-a)(1-2q)+b\right]\right]}{2\,(b-a)^2\,(1-q)\,(1-2q)}.
\end{aligned}\tag{2.8}$$

Substituting equations (2.7) and (2.8) into the above inequality results in the inequality (2.6), which completes the proof. ∎

3. Some applications for special means

Let us recall the following special means of two nonnegative number a,b with $b>a$:

(1) The arithmetic mean

$$A=A(a,b):=\frac{a+b}{2}.$$

(2) The geometric mean

$$G=G(a,b):=\sqrt{ab}.$$

(3) The harmonic mean

$$H=H(a,b):=\frac{2ab}{a+b}.$$

(4) The Logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

(5) The p-Logarithmic mean

$$L_p = L_p(a,b) := \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \quad p \in \mathbb{R} \backslash \{-1, 0\}.$$

(6) the Identric mean

$$I = I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

3.1. Proposition. *Let $0 < a < b$. Then we have the following inequality*

$$H \leq \frac{G^2}{L} \leq A.$$

Proof. The assertion follows from the inequality (2.2) in Theorem 2.4, for $f : (0, \infty) \to \mathbb{R},\ f(x) = x.$ ∎

3.2. Proposition. *Let $0 < a < b$. Then we have the following inequality*

$$H^2 \leq G^2 \leq A(a^2, b^2).$$

Proof. The assertion follows from the inequality (2.2) in Theorem 2.4, for $f : (0, \infty) \to \mathbb{R},\ f(x) = x^2.$ ∎

3.3. Proposition. *Let $0 < a < b$ and $p \in (-1, \infty) \backslash \{0\}$. Then we have the following inequality*

$$H^{p+2} \leq G^2.L_p^p \leq A(a^{p+2}, b^{p+2}).$$

Proof. The assertion follows from the inequality (2.2) in Theorem 2.4, for $f : (0, \infty) \to \mathbb{R},\ f(x) = x^{p+2},\ p\,(-1, \infty) \backslash \{0\}.$ ∎

3.4. Proposition. *Let $0 < a < b$. Then we have the following inequality*

$$H^2 \ln H \leq G^2 \ln I \leq A\left(a^2 \ln a, b^2 \ln b\right).$$

Proof. The assertion follows from the inequality (2.2) in Theorem 2.4, for $f : (0, \infty) \to \mathbb{R},\ f(x) = x^2 \ln x.$ ∎

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