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THE INVERSE KINEMATICS OF ROLLING CONTACT OF TIMELIKE CURVES LYING ON TIMELIKE SURFACES

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Abstract. Rolling contact between two surfaces plays an important role in robotics and engineering such as spherical robots, single wheel robots, and multi-fingered robotic hands to drive a moving surface on a fixed surface. The rolling contact pairs have one, two, or three degrees of freedom (DOFs) consisting of angular velocity components. Rolling contact motion can be divided into two categories: spin-rolling motion and pure-rolling motion. Spin-rolling motion has three (DOFs), and pure-rolling motion has two (DOFs). Further, it is well known that the contact kinematics can be divided into two categories: forward kinematics and inverse kinematics. In this paper, we investigate the inverse kinematics of spin-rolling motion without sliding of one timelike surface on another timelike surface in the direction of timelike unit tangent vectors of their timelike trajectory curves by determining the desired motion and the coordinates of the contact point on each surface. We get three nonlinear algebraic equations as inputs by using curvature theory in Lorentzian geometry. These equations can be reduced as a univariate polynomial of degree six by applying the Darboux frame method. This polynomial enables us to obtain rapid and accurate numerical root approximations and to analyze the rolling rate as an output. Moreover, we obtain another outputs: the rolling direction and the compensatory spin rate.

1. Introduction

Rolling contact is used in many areas of robotics and engineering such as spherical robots, single wheel robots, and multi-fingered robotic hands to drive from one configuration (position and orientation) to another. In mechanical systems, rolling contact without sliding engenders a non-integrable kinematic constraints on the systems velocity which are called non-holonomic constraints. This non-holonomy calls for the two contact loci have equal arc lengths in a given time interval [11].

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There are two categories of kinematics of the rolling contact. The first is pure-rolling motion and the second is spin-rolling motion [9]. On the other hand, in the rolling contact, there are two geometric constraints. The first is that the unit normal vectors of the two surfaces are made to coincide at the contact point. The second is that the contact points have the same velocity. To put it another way, the two contact trajectory curves are tangent to each other and have the same rolling rate. Thus, a moving surface has spin-rolling motion or pure-rolling motion under these two geometric constraints. Further, there is another constraint for a surface to have pure-rolling motion. This constraint is explicitly demonstrated to be that the two contact trajectories have the same geodesic curvature, that is, the angular velocity component ω_3 about the direction of the unit normal vector n is zero at the contact point. Thus, the contact trajectories are not arbitrary [10]. Pure-rolling motion has 2 degrees of freedom (DOFs). It has instantaneous rotation axis passes through the contact point in all cases and this axis is parallel to the common tangent plane of two surfaces. Spin-rolling motion, which is also called twist-rolling motion, has 3 degrees of freedom (DOFs) consisting of three angular velocity components: ω_1, ω_2 about the axes T and g on the tangent plane, respectively, and ω_3 about the common normal axis n at the contact point. Its instantaneous rotation axis can be in any arbitrary direction which is the characteristic difference from pure-rolling motion [9].

The contact kinematics is given in two classifications. The first is forward kinematics and the second is inverse kinematics. The forward kinematics includes the problem of using kinematic equations as the inputs of the geometry of the two surfaces and the contact locus on each surface to compute the motion of the moving surface as the output. The inverse kinematics includes the problem of determining the control parameters that give the moving surface the desired motion as the inputs of the geometry of two surfaces and the desired angular velocity of the moving surface. These inputs are the angular velocity components ω_1, ω_2 and ω_3 [10, 11]. Since the moving surface has three rotational DOFs, three outputs are generally needed to realize the desired angular velocity. These outputs are rolling direction, rolling rate and compensatory spin rate that can be used as the inputs to design a control system for the mechanical system [11].

In traditional approaches, the inverse kinematics of rolling contact formulated the kinematics in terms of the derivatives of surface parameterization [15, 17, 18, 20, 27]. We need to solve a system of five nonlinear ordinary differential equations. Moreover, the formulation is restricted to a particular choice of coordinate frame. When we change the origin or the orientation of the frame, we have to re-establish the formulation. In proposed approach, we need to solve a system of three nonlinear ordinary differential equations. This brings with a major advantage of the proposed approach which is differed from traditional approach. Hence, for hard real-time systems, the proposed approach is more suitable than the traditional one. In the form of the system, three contact equations are engendered and these equations are simplified to a univariate polynomial of degree six, which is suitable for numerical root approximations. The polynomial formulation has three advantages. First, the method of numerical root approximations can be accurately and rapidly computed using any commercial or open-source software [4, 19]. Second, the spin velocity is revealed to consist of the induced spin velocity and the compensatory spin velocity. Third, the three outputs of the inverse kinematics are invariant with respect to a particular choice of coordinate frame.

Many researchers have extensively studied kinematics of a point contact between rigid bodies. Neimark and Fufaev [23] were the first to adopt the moving frame along the lines of curvature to derive the velocity equation of spin-rolling motion. Cai and Roth [6, 7] investigated instantaneous time-based kinematics of rigid objects in point contact, both in planar and spatial cases, and focused on two special motions, including sliding and pure-rolling motion and they aimed to measure the relative motion at the point of contact. Montana [20] derived the contact equations by examining the kinematics of the sliding-spin-rolling motion with the help of the differential geometry of the surfaces and obtained the first order differentiable kinematic equation from these equations. After, he [21] investigated the motion planning and control of the multi-fingered robotic hand. Li and Canny [16] used Montana's contact equations to investigate the existence of an admissible path between two configurations in the case of pure rolling and, if it does, then how to find it. Sarkar et al. [27] extended Montana's definition but with a different approach by obtaining the acceleration equations and they demonstrated the obvious dependence on Christoffel symbols and they simplified the derivative of the metric tensor. Marigo and Bicchi [18] obtained similar equations with Montana's contact equations using a different approach that allowed an analysis of admissibility of a pure-rolling contact. Agrachev and Sachkov [1] solved the controllability problem of a pair of pure-rolling rigid bodies. Chelouah and Chitour [8] gave two procedures to analyze the motion-planning problem when one manifold was a plane and the other was a convex surface. Cui and Dai [9] investigated the forward kinematics of spin-rolling motion without sliding by applying the Darboux frame method and then Cui [10] studied the kinematics of sliding-rolling motion of two contact surfaces. Kerr and Roth [15] studied the kinematics of rolling contact in multi-fingered hands with the help of differential equations of the first order of the surface parametrizations of the grasped object and the fingertip. Cui and Dai [11] also investigated the inverse kinematics of rolling contact by using polynomial formulation when the desired angular velocity and the coordinates of the contact point on each surface were given in Euclidean 3-space. Then, they obtained admissible rolling motion between two contact surfaces. Cui et al. [12] studied in-hand forward and inverse kinematics with rolling contact. They presented a systematic approach to the forward and inverse kinematics of in-hand manipulation. In order to show the proposed approach, they used a two-fingered planar robot hand and a three-fingered spatial robot hand. For the fundamental concepts of kinematics, see [5, 14, 22, 24].

This paper is organized as follows:
In Section 2, we give basic concepts in Lorentzian 3-space. In Section 3, we study the inverse kinematics of rolling contact motion of two timelike surfaces. We aim to study the relative motion of a timelike moving surface rotating about any axis through the contact point with the timelike fixed surface in the direction of the timelike unit tangent vectors of their timelike trajectory curves. Initially, we obtain three contact equations when the coordinates of the contact point on the fixed surface, the coordinates of the contact point on the moving surface and the desired angular velocity are given. Then, we give an example to obtain three contact nonlinear algebraic equations and simplify them into a univariate polynomial of degree six by applying the moving-frame method in Lorentzian geometry. Then, we solve this univariate polynomial formulation of the contact equations by using a numerical method. In Section 4, we give a conclusion.

2. Preliminaries

In this section, we give a brief summary of basic concepts for the reader who is not familiar with Lorentzian 3-space [3, 25, 26, 28].

Lorentzian space $\mathbb{R}^3_1$ is the real vector space

$$\langle a, b\rangle = a_1b_1 + a_2b_2 - a_3b_3$$

where $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in \mathbb{R}^3$. According to this metric, an arbitrary vector $a = (a_1, a_2, a_3)$ in $\mathbb{R}^3_1$ can have one of three Lorentzian causal characters: if $\langle a, a\rangle > 0$ or $a = 0$ then a is called a spacelike vector; if $\langle a, a\rangle < 0$ then a is called a timelike vector; if $\langle a, a\rangle = 0$ and $a \neq 0$ then a is called a null (lightlike) vector [25]. We note that a timelike vector is future pointing or past pointing if the first compound of vector is positive or negative, respectively. The norm of a vector $a \in \mathbb{R}^3_1$ is given by $||a|| = \sqrt{|\langle a, a\rangle|}$. If the vector a is a spacelike vector, then $||a||^2 = \langle a, a\rangle$; If the vector a is a timelike vector, then $||a||^2 = -\langle a, a\rangle$ [28].

Let $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ be two vectors in $\mathbb{R}^3_1$. Then, Lorentzian vector product of a and b can be defined by [28]

$$a \times b = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_1b_2 - a_2b_1)$$

Definition 1. [3, 26]

(i) *Hyperbolic angle: Let a and b be future pointing (or past pointing) timelike vectors in $\mathbb{R}^3_1$. Then there is a unique real number $\theta \geq 0$ such that $\langle a, b\rangle = -||a||||b|| \cosh\theta$, and this number is called the hyperbolic angle between the vectors a and b.*

(ii) *Central angle: Let a and b be spacelike vectors in $\mathbb{R}^3_1$ and they span a timelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that $\langle a, b\rangle = ||a||||b|| \cosh\theta$, and this number is called the central angle between the vectors a and b.*

(iii) *Spacelike angle: Let a and b be spacelike vectors in $\mathbb{R}^3_1$ and they span a spacelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that $|\langle a, b\rangle| = ||a||||b||\cos\theta$, and this number is called the spacelike angle between the vectors a and b.*

(iv) *Lorentzian timelike angle: Let a be a spacelike vector and b be a timelike vector in $\mathbb{R}^3_1$. Then there is a unique real number $\theta \geq 0$ such that $|\langle a, b\rangle| = ||a||||b||\sinh\theta$, and this number is called the Lorentzian timelike angle between the vectors a and b.*

An arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{R}^3_1$ can locally be spacelike, timelike, or null (lightlike), if all of its velocity vectors $\boldsymbol{\alpha}'(s)$ are spacelike, timelike, or null (lightlike), respectively. A surface in Lorentzian space $\mathbb{R}^3_1$ is called a spacelike, or a timelike or a lightlike surface if the normal vector of the surface is timelike, or spacelike, or lightlike vector, respectively [25].

The Lorentzian and hyperbolic unit spheres are given by

$$S^2_1 = \left\{a = (a_1, a_2, a_3) \in \mathbb{R}^3_1 : \langle a, a\rangle = 1\right\}$$

and

$$H^2_0 = \left\{a = (a_1, a_2, a_3) \in \mathbb{R}^3_1 : \langle a, a\rangle = -1\right\},$$

respectively. Also, the light cone is given by

$$\Lambda = \left\{a = (a_1, a_2, a_3) \in \mathbb{R}^3_1 : \langle a, a\rangle = 0\right\} - \{(0, 0, 0)\}.$$

It is easy to show that the hyperbolic unit sphere is a spacelike surface, Lorentzian unit sphere is a timelike surface and light cone is a lightlike (null) surface [25, 28].

Let S be a timelike surface and $\alpha = \alpha(s)$ be any timelike curve lying on the surface S. Darboux frame (T, g, n) of α is a solid perpendicular trihedron in $\mathbb{R}^3_1$ associated with each point $M \in \alpha$, where T is the timelike unit tangent vector to the curve α, n is the unit spacelike normal vector to the surface S and $g = -n \times T$ (that is, g is tangential to S which is also a spacelike vector) at the point M. We should note that

$$T \times g = -n, g \times n = T, n \times T = -g,$$

and

$$\langle T, T\rangle = -1, \langle g, g\rangle = 1, \langle n, n\rangle = 1.$$

Then the derivative formulae (the equations of motion) of the Darboux frame (trihedron) is given by

$$\frac{dm}{ds} = T, \quad \frac{d}{ds}\begin{bmatrix} T \\ g \\ n \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ k_g & 0 & -\tau_g \\ k_n & \tau_g & 0 \end{bmatrix}\begin{bmatrix} T \\ g \\ n \end{bmatrix}$$

where $\boldsymbol{m}$ is the position vector of the point M that depends on the choice of the coordinate system. Note that the vector $\boldsymbol{m}$ has three causal characters. The components of the vector $\boldsymbol{m}$ are obtained from the measurement along the axes of the coordinate system. In these formulae, k_g, k_n and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively. It is easy to see that the geodesic curvature k_g, the normal curvature k_n and the geodesic torsion τ_g of the timelike curve α can be given by

$$k_g = \left\langle \frac{dT}{ds}, g \right\rangle, k_n = \left\langle \frac{dT}{ds}, n \right\rangle, \tau_g = -\left\langle \frac{dg}{ds}, n \right\rangle$$

The Darboux instantaneous rotation vector of the Darboux trihedron is defined by [28]

$$\boldsymbol{\omega} = \tau_g T + k_n g - k_g n$$

Then, for a timelike curve $\alpha(s)$ lying on a timelike surface S, we have the following characterizations [28]: $\alpha(s)$ is

(i) geodesic $\Leftrightarrow k_g = 0$,
(ii) asymptotic $\Leftrightarrow k_n = 0$,
(iii) principal $\Leftrightarrow \tau_g = 0$.

Let $x = x(u, v)$ be a parametrization of a timelike surface S. Let $\alpha = \alpha(s)$ be a timelike curve passing through a point M on the surface $x(u, v)$. Let us denote $\alpha_1(s)$ and $\alpha_2(s)$ as u =const. (timelike) and v =const. (spacelike) parametric curves passing through the same point M, respectively. We also assume that the curves $\alpha_1(s)$ and $\alpha_2(s)$ are perpendicular to each other. We denote the unit tangent vectors of α, $\alpha_1(s)$ and $\alpha_2(s)$ at the point M as T, T_1 and T_2, respectively. For the spacelike unit normal vector $\boldsymbol{n}$ of the timelike surface at the point M, the followings are satisfied:

$$T \times g = -n,\, T_1 \times g_1 = -n,\, T_2 \times g_2 = -n.$$

Let φ be a hyperbolic angle between T and T_1. Assume that (T, g, n) is a Darboux frame which can be obtained by a rotation angle φ from (T_1, T_2, n) about the direction of n. Then, the following relations exist:

$$\begin{aligned} &T = \cosh\varphi T_1 + \sinh\varphi T_2, \\ &g = -\sinh\varphi T_1 - \cosh\varphi T_2, \\ &n = n. \end{aligned}$$

Furthermore, if k_{g1} and k_{g2} are geodesic curvatures, k_{n1} and k_{n2} are normal curvatures and τ_{g1} and τ_{g2} are geodesic torsions of the parametric curves α_1 (timelike) and α_2 (spacelike), respectively, then we have the curvatures of the timelike curve α as follows:

$$\begin{aligned} k_n &= k_{n1}\cosh^2\varphi + (\tau_{g1} + \tau_{g2})\cosh\varphi\sinh\varphi - k_{n2}\sinh^2\varphi, \\ \tau_g &= \tau_{g1}\cosh^2\varphi + (k_{n1} - k_{n2})\cosh\varphi\sinh\varphi + \tau_{g2}\sinh^2\varphi, \\ k_g &= k_{g1}\cosh\varphi - k_{g2}\sinh\varphi - \tfrac{d\varphi}{ds} \end{aligned}$$

which are known as generalized Euler's formula, O. Bonnet formula and Liouville's formula, respectively [28].

3. The Inverse Kinematics of Rolling Contact of Timelike Surfaces with Their Spacelike Trajectory Curves

In previous work [2], we studied the forward kinematics of rolling contact motion without sliding and we presented the relative motion of a timelike moving surface rolling on a timelike fixed surface with their timelike trajectory curves by applying the moving-frame method and curvature theory in Lorentzian 3-space. During the relative motion, the moving surface can rotate about any axis through the contact point and maintains its timelike character entirely.

In this section, we study the inverse kinematics of rolling contact of a timelike moving surface rolling on a timelike fixed surface with their timelike trajectory curves. First, we formulate the solution as contact equations in the form of a system of three nonlinear algebraic equations. These contact equations, which are given as the inputs to the inverse kinematics, are the desired angular velocity components ω_1, ω_2 and ω_3. Second, we simplify these equations into a univariate polynomial of degree six. Then, we get the six roots of the polynomial by using the numerical computation. Lastly, we obtain rolling direction φ, rolling rate σ and compensatory spin rate $\hat{\omega}_3$ as the outputs.

The main contribution of this section is that a univariate polynomial formulation of inverse kinematics is formed and six roots of the polynomial is obtained by means of proposed approach rather than traditional approach in Lorentzian 3-space. This brings with a major advantage of the proposed approach which is differed from traditional approach. In traditional approach a system of five nonlinear ordinary differential equations has to be solved and it is difficult to solve the equations in many cases. But in proposed approach, a system of three nonlinear ordinary differential equations has to be solved. Hence, for hard real-time systems, the proposed approach is more suitable than the traditional one.

3.1. **The angular velocity in terms of the contact loci.** Let $x(u,v)$ and $y(\bar{u},\bar{v})$ be parametrizations of a timelike fixed surface S_1 and a timelike moving surface S_2, respectively. T_1 can be thought as the timelike unit vector along the timelike coordinate u-curve of S_1. Similarly, $\bar{T}_1$ can be thought as the timelike unit vector along the timelike coordinate $\bar{u}$-curve of S_2. Since the timelike unit vectors T_1 and $\bar{T}_1$ are on the common timelike tangent plane at the contact point of the timelike surfaces S_1 and S_2, i.e., they span the common timelike tangent plane, then the angle θ between T_1 and $\bar{T}_1$ has to be a hyperbolic angle. Let (T,g,n) be a Darboux frame obtained by a rotation angle φ from (T_1,T_2,n) about the direction of n. Suppose the moving surface S_2 rolls in the direction of the timelike vector T. The unit vectors T_1 and $\bar{T}_1$ span the common timelike tangent plane, and the angle φ between T_1 and $\bar{T}_1$ is also a hyperbolic angle, see Fig. 1.

Assume a timelike curve β lying on the timelike moving surface S_2 passing through the point M and is parameterized by its arc length $\bar{s}$. Then the normal curvature $\bar{k}_n$, geodesic torsion $\bar{\tau}_g$ and geodesic curvature $\bar{k}_g$ of the locus β on the

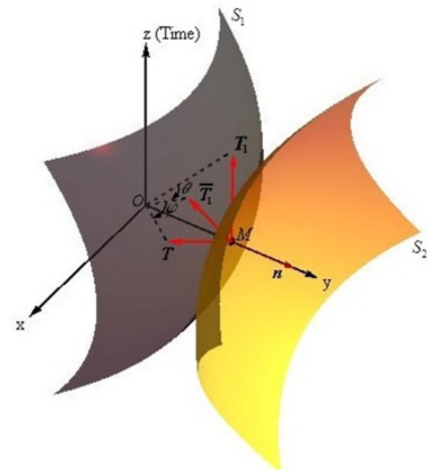

FIGURE 1. The timelike surface S_2 rolling on the timelike surface S_1 in the direction of timelike unit tangent vector T

surface S_2 in the direction of the timelike vector T are given by

$$\left\{\begin{array}{l} \bar{k}_n = \bar{k}_{n1}\cosh^2\varphi + (\bar{\tau}_{g1} + \bar{\tau}_{g2})\cosh\varphi\sinh\varphi - \bar{k}_{n2}\sinh^2\varphi, \\ \bar{\tau}_g = \bar{\tau}_{g1}\cosh^2\varphi + \left(\bar{k}_{n1} - \bar{k}_{n2}\right)\cosh\varphi\sinh\varphi + \bar{\tau}_{g2}\sinh^2\varphi, \\ \bar{k}_g = \bar{k}_{g1}\cosh\varphi - \bar{k}_{g2}\sinh\varphi - \frac{d\varphi}{d\bar{s}} \end{array}\right. \tag{1}$$

where $\bar{k}_{n1}$, $\bar{k}_{n2}$, $\bar{\tau}_{g1}$, $\bar{\tau}_{g2}$, $\bar{k}_{g1}$ and $\bar{k}_{g2}$ are the normal curvatures, geodesic torsions, and geodesic curvatures of the timelike $\bar{u}$- and spacelike $\bar{v}$- parametric curves of the timelike surface S_2 at the contact point M. Let s denote the arc length of the contact locus α on the timelike surface S_1. Then, the curvatures of the timelike curve α can be given by

$$\left\{\begin{array}{l} k_n = k_{n1}\cosh^2(\varphi+\theta) + (\tau_{g1} + \tau_{g2})\cosh(\varphi+\theta)\sinh(\varphi+\theta) - k_{n2}\sinh^2(\varphi+\theta), \\ \tau_g = \tau_{g1}\cosh^2(\varphi+\theta) + (k_{n1} - k_{n2})\cosh(\varphi+\theta)\sinh(\varphi+\theta) + \tau_{g2}\sinh^2(\varphi+\theta), \\ k_g = k_{g1}\cosh(\varphi+\theta) - k_{g2}\sinh(\varphi+\theta) - \frac{d(\varphi+\theta)}{ds} \end{array}\right. \tag{2}$$

where k_{n1}, k_{n2}, τ_{g1}, τ_{g2}, k_{g1} and k_{g2} are the normal curvatures, geodesic torsions, and geodesic curvatures of the timelike u- and spacelike v-parametric curves of the timelike surface S_1 at the contact point M. If the contact loci are given, the angular velocity of the timelike moving surface can be obtained in the authors's previous work (Eq. (9) in Ref. [2]) as

$$\boldsymbol{\omega} = \sigma(-\tau_g^* T - k_n^* g + k_g^* n). \tag{3}$$

Note that $\sigma = \frac{ds}{dt}$ gives the rolling rate and $k_g^* = \bar{k}_g - k_g$, $k_n^* = \bar{k}_n - k_n$, $\tau_g^* = \bar{\tau}_g - \tau_g$. The spacelike unit vectors g and n, and the timelike unit tangent vector T can be given in terms of the frame (T_1, T_2, n) in Fig. 1 as

$$\begin{cases} T = \cosh(\varphi+\theta)\,T_1 + \sinh(\varphi+\theta)\,T_2, \\ g = -\sinh(\varphi+\theta)\,T_1 - \cosh(\varphi+\theta)\,T_2, \\ n = n. \end{cases} \tag{4}$$

Substituting Eq.(4) into Eq.(3) gives the angular velocity $\boldsymbol{\omega}$ in the frame (T_1, T_2, n) as

$$\begin{aligned} \boldsymbol{\omega} &= \sigma\left(-\cosh(\varphi+\theta)\tau_g^* + \sinh(\varphi+\theta)k_n^*\right)\,T_1 \\ &\quad + \sigma\left(-\sinh(\varphi+\theta)\tau_g^* + \cosh(\varphi+\theta)k_n^*\right)\,T_2 + \sigma k_g^* n. \end{aligned} \tag{5}$$

3.2. **The contact equations in terms of curvatures.** Assume the angular velocity of the timelike moving surface S_2 is given in the frame (T_1, T_2, n) as

$$\boldsymbol{\omega} = \omega_1 T + \omega_2 g + \omega_3 n. \tag{6}$$

The angular velocity in the direction of T defined in the frame (T_1, T_2, n) can be obtained by substituting Eqs.(1) and (2) into Eq.(5). If the components in Eqs.(5) and (6) are equated, the three contact equations are obtained as follows:

The first contact equation can be obtained in the direction of T_1 as

$$\omega_1 = \sigma\begin{pmatrix} A_1\cosh^3\varphi + A_2\sinh^3\varphi + A_3\cosh^2\varphi\sinh\varphi \\ + A_4\cosh\varphi\sinh^2\varphi + A_5\cosh\varphi + A_6\sinh\varphi \end{pmatrix} \tag{7}$$

where

$$A_1 = -A_4 = -\bar{\tau}_{g1}\cosh\theta + \bar{k}_{n1}\sinh\theta,\ \ A_2 = -A_3 = -\bar{\tau}_{g2}\sinh\theta - \bar{k}_{n2}\cosh\theta,$$
$$A_5 = \tau_{g1}\cosh\theta - k_{n2}\sinh\theta,\ \ A_6 = \tau_{g1}\sinh\theta - k_{n2}\cosh\theta.$$

The second contact equation can be obtained in the direction of T_2 as

$$\omega_2 = \sigma\begin{pmatrix} B_1\cosh^3\varphi + B_2\sinh^3\varphi + B_3\cosh^2\varphi\sinh\varphi \\ + B_4\cosh\varphi\sinh^2\varphi + B_5\cosh\varphi + B_6\sinh\varphi \end{pmatrix} \tag{8}$$

where

$$B_1 = -B_4 = -\bar{\tau}_{g1}\sinh\theta + \bar{k}_{n1}\cosh\theta,\ \ B_2 = -B_3 = -\bar{\tau}_{g2}\cosh\theta - \bar{k}_{n2}\sinh\theta,$$
$$B_5 = -\tau_{g2}\sinh\theta - k_{n1}\cosh\theta,\ \ B_6 = -\tau_{g2}\cosh\theta - k_{n1}\sinh\theta.$$

The third contact equation can be obtained in the direction of n as

$$\omega_3 = \sigma\left(C_1\cosh\varphi + C_2\sinh\varphi\right) + \hat{\omega}_3 \tag{9}$$

where

$$C_1 = \bar{k}_{g1} - k_{g1}\cosh\theta + k_{g2}\sinh\theta,$$

$$C_2 = -\bar{k}_{g2} - k_{g1} \sinh\theta + k_{g2}\cosh\theta,$$
$$\hat{\omega}_3 = d\theta/dt.$$

In Eq.(9), the first item $\sigma\left(C_1 \cosh\varphi + C_2 \sinh\varphi\right)$ is the spin rate induced by the difference between the geodesic curvatures of the two timelike surfaces. Therefore, the last item $\hat{\omega}_3$ is the spin rate to compensate for the difference between the desired spin rate and the induced spin rate. Since Cui and Dai [11] coined $\hat{\omega}_3$ "compensatory spin rate", we also called $\hat{\omega}_3$ as "compensatory spin rate" in Lorentzian 3-space. Thus, the three contact equations are completed.

3.3. **Inverse rolling contact in Lorentzian 3-space: a polynomial solution.** In Eqs.(7)-(9), the coefficients A_i, B_i and C_i are given with regards to the coordinate curves and the hyperbolic angle θ. It has to be expressed that θ is the intersecting angle between the two coordinate frames. Therefore, the three coefficients and the angle θ are described at a given contact point. When the angular velocity components ω_1, ω_2 and ω_3 are given as inputs, the three unknown terms, which are the rolling direction φ, the rolling rate σ, and the compensatory spin rate $\hat{\omega}_3$, can be obtained as outputs.

The rolling direction φ is the first output that can be obtained by dividing Eq.(7) by Eq.(8). This division eliminates the rolling rate σ and gives a hyperbolic equation with regards to φ as

$$\begin{cases} (A_1\omega_2 - B_1\omega_1)\cosh^3\varphi + (A_2\omega_2 - B_2\omega_1)\sinh^3\varphi \\ + (A_3\omega_2 - B_3\omega_1)\cosh^2\varphi\sinh\varphi + (A_4\omega_2 - B_4\omega_1)\cosh\varphi\sinh^2\varphi \\ + (A_5\omega_2 - B_5\omega_1)\cosh\varphi + (A_6\omega_2 - B_6\omega_1)\sinh\varphi = 0. \end{cases} \tag{10}$$

Let take Eq.(10) as

$$P_1\cosh^3\varphi + P_2\sinh^3\varphi + P_3\cosh^2\varphi\sinh\varphi$$
$$+ P_4\cosh\varphi\sinh^2\varphi + P_5\cosh\varphi + P_6\sinh\varphi = 0.$$

Substituting the half-tangent-Lorentzian timelike angle $x = \tanh(\varphi/2)$ into Eq.(10) gives a polynomial of degree six as

$$(P_1 + P_5)\,x^6 + (2P_3 + 2P_6)\,x^5 + (3P_1 + 4P_4 - P_5)\,x^4$$
$$+ (8P_2 + 4P_3 - 4P_6)\,x^3 + (3P_1 + 4P_4 - P_5)\,x^2 + (2P_3 + 2P_6)\,x + P_1 + P_5 = 0.$$

The solution of the polynomial provides all directions where the timelike moving surface can roll. If all the roots of solution are complex numbers, the desired motion cannot be occurred in no sense.

The rolling rate σ is the second output that can be obtained by substituting the rolling direction φ back to either Eq.(7) or Eq.(8) as

$$\sigma = \sqrt{\frac{(\omega_1)^2 - (\omega_2)^2}{\left(\tau_g^*\right)^2 - (k_n^*)^2}}.$$

These two inputs φ and σ determine the rotational motion of pure-rolling. The compensatory spin rate $\hat{\omega}_3$ is the third output that can be obtained by substituting the values of φ and σ into Eq.(9). These three inputs determine the spin-rolling motion of the timelike moving surface.

3.4. **Example.** (*A* S_1^2 Lorentzian Unit Sphere Rolling on a Timelike Cylinder with radius $\frac{3}{4}$)

Let a timelike moving surface S_2 be a Lorentzian unit sphere parameterized as $y(\bar{u}, \bar{v}) = (\cos \bar{v} \cosh \bar{u}, \sin \bar{v} \cosh \bar{u}, \sinh \bar{u})$. Consider that a timelike cylinder is parameterized as $x_1(u, v) = \left(\frac{1+3\cos v}{4}, \frac{3\sin v}{4}, -u\right)$. Suppose the timelike cylinder is a fixed surface S_1 that is formed by rotating around x-axis in negative direction with the hyperbolic angle $\operatorname{arccos h}\left(\frac{\sqrt{5}}{2}\right)$. Then, a new parametrization of the cylinder with radius 3/4 is given by

$$x(u, v) = \left(\frac{1+3\cos v}{4}, \frac{3\sqrt{5}\sin v}{8} - \frac{u}{2}, \frac{3\sin v}{8} - \frac{\sqrt{5}u}{2}\right) (See\, Fig.\, 2).$$

Suppose that the timelike moving surface S_2 can rotate about any axis through the contact point with respect to the timelike fixed surface S_1. During the rolling contact motion, the moving surface with the spacelike unit normal vector maintains its causal character entirely. The parametric curves α_1 ($v = const.$) and α_2 ($u = const.$) on the cylinder are timelike and spacelike coordinate curves, respectively. Let the arc lengths of the curves α_1 and α_2 be s_1 and s_2, respectively. The unit vectors and the curvatures of these coordinate curves on the cylinder are obtained as

$$\left\{\begin{array}{l} T_1 = \frac{d\boldsymbol{\alpha}_1}{ds_1} = \frac{d\boldsymbol{\alpha}_1}{du} \Big/ \frac{ds_1}{du} = \left(0, -\frac{1}{2}, -\frac{\sqrt{5}}{2}\right), \\ g_1 = \left(\sin v_0, -\frac{\sqrt{5}}{2}\cos v_0, -\frac{1}{2}\cos v_0\right), \\ T_2 = \frac{d\boldsymbol{\alpha}_2}{ds_2} = \frac{d\boldsymbol{\alpha}_2}{dv} \Big/ \frac{ds_2}{dv} = \left(-\sin v, \frac{\sqrt{5}\cos v}{2}, \frac{\cos v}{2}\right), \\ g_2 = \left(0, -\frac{1}{2}, -\frac{\sqrt{5}}{2}\right), \\ n = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|} = \left(-\cos v,\ -\frac{\sqrt{5}}{2}\sin v,\ -\frac{1}{2}\sin v\right), \end{array}\right. \tag{11}$$

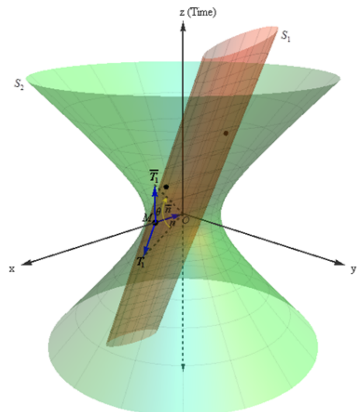

FIGURE 2. A Lorentzian Unit Sphere S_1^2 Rolling on a Timelike Cylinder with radius $\frac{3}{4}$

and

$$
\begin{array}{ll}
k_{g1} = \langle d\boldsymbol{T}_1/du\ ,\boldsymbol{g}_1\rangle/\frac{ds_1}{du} \ = 0, & k_{g2} = -\langle d\boldsymbol{T}_2/dv\ ,\boldsymbol{g}_2\rangle/\frac{ds_2}{dv} \ = 0, \\
k_{n1} = \langle d\boldsymbol{T}_1/du\ ,\boldsymbol{n}\rangle/\frac{ds_1}{du} \ = 0, & k_{n2} = -\langle d\boldsymbol{T}_2/dv\ ,\boldsymbol{n}\rangle/\frac{ds_2}{dv} \ = -\frac{4}{3}, \\
\tau_{g1} = -\langle d\boldsymbol{g}_1/du\ ,\boldsymbol{n}\rangle/\frac{ds_1}{du} \ = 0, & \tau_{g2} = \langle d\boldsymbol{g}_2/dv\ ,\boldsymbol{n}\rangle/\frac{ds_2}{dv} \ = 0
\end{array}
\tag{12}
$$

where $\frac{ds_1}{du} = \sqrt{\left|\left\langle \frac{d\boldsymbol{\alpha}_1}{du}, \frac{d\boldsymbol{\alpha}_1}{du}\right\rangle\right|} = 1$ and $\frac{ds_2}{dv} = \sqrt{\left\langle \frac{d\boldsymbol{\alpha}_2}{dv}, \frac{d\boldsymbol{\alpha}_2}{dv}\right\rangle} = \frac{3}{4}$. Note that we accept the unit normal vector $\boldsymbol{n}$ of the cylinder is outward.

Similarly, the parametric curves β_1 ($\bar{v} = const.$) and β_2 ($\bar{u} = const.$) on the Lorentzian unit sphere are timelike and spacelike coordinate curves, respectively. Let the arc lengths of the curves β_1 and β_2 be $\bar{s}_1$ and $\bar{s}_2$, respectively. The unit vectors and the curvatures of these coordinate curves on Lorentzian unit sphere are obtained as

$$
\left\{
\begin{array}{l}
\bar{T}_1 = \frac{d\boldsymbol{\beta}_1}{d\bar{s}_1} = \frac{d\boldsymbol{\beta}_1}{d\bar{u}} \Big/ \frac{d\bar{s}_1}{d\bar{u}} = (\cos\bar{v}_0 \sinh\bar{u}, \sin\bar{v}_0 \sinh\bar{u}, \cosh\bar{u}), \\
\bar{g}_1 = (-\sin\bar{v}_0, \cos\bar{v}_0, 0), \\
\bar{T}_2 = \frac{d\boldsymbol{\beta}_2}{d\bar{s}_2} = \frac{d\boldsymbol{\beta}_2}{d\bar{v}} \Big/ \frac{d\bar{s}_2}{d\bar{v}} = (-\sin\bar{v}, \cos\bar{v}, 0), \\
\bar{g}_2 = (-\cos\bar{v}\sinh\bar{u}_0, -\sin\bar{v}\sinh\bar{u}_0, -\cosh\bar{u}_0), \\
\bar{\boldsymbol{n}} = -\frac{\boldsymbol{y}_{\bar{u}} \times \boldsymbol{y}_{\bar{v}}}{\|\boldsymbol{y}_{\bar{u}} \times \boldsymbol{y}_{\bar{v}}\|} = (-\cos\bar{v}\cosh\bar{u}, -\sin\bar{v}\cosh\bar{u}, -\sinh\bar{u}),
\end{array}
\right.
\tag{13}
$$

and

$$\begin{array}{ll} \bar{k}_{g1} = \left\langle d\bar{T}_1/d\bar{u}\ ,\bar{\boldsymbol{g}}_1\right\rangle/\frac{d\bar{s}_1}{d\bar{u}} \ = 0, & \bar{k}_{g2} = -\left\langle d\bar{T}_2/d\bar{v}\ ,\bar{\boldsymbol{g}}_2\right\rangle/\frac{d\bar{s}_2}{d\bar{v}} \ = -\tanh \bar{u}_0, \\ \bar{k}_{n1} = \left\langle d\bar{T}_1/d\bar{u}\ ,\bar{\boldsymbol{n}}\right\rangle/\frac{d\bar{s}_1}{d\bar{u}} \ = -1, & \bar{k}_{n2} = -\left\langle d\bar{T}_2/d\bar{v}\ ,\bar{\boldsymbol{n}}\right\rangle/\frac{d\bar{s}_2}{d\bar{v}} \ = -1, \\ \bar{\tau}_{g1} = -\left\langle d\bar{\boldsymbol{g}}_1/d\bar{u}\ ,\bar{\boldsymbol{n}}\right\rangle/\frac{d\bar{s}_1}{d\bar{u}} \ = 0, & \bar{\tau}_{g2} = \left\langle d\bar{\boldsymbol{g}}_2/d\bar{v}\ ,\bar{\boldsymbol{n}}\right\rangle/\frac{d\bar{s}_2}{d\bar{v}} \ = 0 \end{array} \tag{14}$$

where $\frac{d\bar{s}_1}{d\bar{u}} = \sqrt{\left|\left\langle \frac{d\boldsymbol{\beta}_1}{d\bar{u}}, \frac{d\boldsymbol{\beta}_1}{d\bar{u}}\right\rangle\right|} = 1$ and $\frac{d\bar{s}_2}{d\bar{v}} = \sqrt{\left\langle \frac{d\boldsymbol{\beta}_2}{d\bar{v}}, \frac{d\boldsymbol{\beta}_2}{d\bar{v}}\right\rangle} = \cosh \bar{u}_0$. Note that we accept the unit normal vector $\bar{\boldsymbol{n}}$ of the Lorentzian unit sphere is inward.

Substituting the curvatures in Eqs.(12) and (14) into Eqs.(2) and (1), respectively, yields

$$k_n = \frac{4}{3}\sinh^2(\varphi+\theta)\,, \quad \tau_g = \frac{4}{3}\cosh(\varphi+\theta)\sinh(\varphi+\theta)\,, \quad k_g = -\frac{d(\varphi+\theta)}{ds}$$

and

$$\bar{k}_n = -1\,, \quad \bar{\tau}_g = 0\,, \quad \bar{k}_g = \tanh\bar{u}\sinh\varphi - \frac{d\varphi}{d\bar{s}}.$$

Since $k_g^* = \bar{k}_g - k_g,\ k_n^* = \bar{k}_n - k_n,\ \tau_g^* = \bar{\tau}_g - \tau_g$, then the induced curvatures are obtained as

$$\begin{array}{l} k_n^* = -1 - \frac{4}{3}\sinh^2(\varphi+\theta), \\ \tau_g^* = -\frac{4}{3}\cosh(\varphi+\theta)\sinh(\varphi+\theta), \\ k_g^* = \tanh\bar{u}\sinh\varphi + \frac{d\theta}{ds}. \end{array} \tag{15}$$

By substituting the induced curvatures into (5) gives three contact equations as

$$\begin{array}{l} \omega_1 = \sigma\sinh(\varphi+\theta)/3, \\ \omega_2 = -\sigma\cosh(\varphi+\theta), \\ \omega_3 = \sigma\left(\tanh\bar{u}\sinh\varphi\right) + \hat{\omega}_3. \end{array} \tag{16}$$

Eliminating the rolling rate σ from the first two parts of (16) gives the following equation:

$$P_1\cosh(\varphi+\theta) + P_2\sinh(\varphi+\theta) = 0 \tag{17}$$

where $P_1 = -3\omega_1,\ P_2 = -\omega_2$. Since the closed-form solution cannot be obtained, Eq.(17) can be solved by using a numerical method. Putting the half-tangent hyperbolic angle $x = \tanh\left((\varphi+\theta)/2\right)$ into Eq.(17) gives the polynomial of degree six as

$$(P_1)\,x^6 + (2P_2)\,x^5 + (\ -P_1)\,x^4 + (\ -4P_2)\,x^3 + (\ -P_1)\,x^2 + (2P_2)\,x + P_1 = 0. \tag{18}$$

Suppose that the coordinates of the contact point M on the cylinder are ($u = -2$ and $v = -\pi$), and the coordinates of the contact point M on the Lorentzian unit sphere are ($\bar{u} = 1$ and $\bar{v} = \pi/6$). Also, let the desired angular velocity be

$$\boldsymbol{\omega}_{\text{desired}} = 1\boldsymbol{T}_1 - 6\boldsymbol{T}_2 + 1\boldsymbol{n}.$$

By putting these parameters into Eq.(17) yields $P_1 = -3$ and $P_2 = 6$. From (18), we get

$$x^6 - 4x^5 - x^4 + 8x^3 - x^2 - 4x + 1 = 0.$$

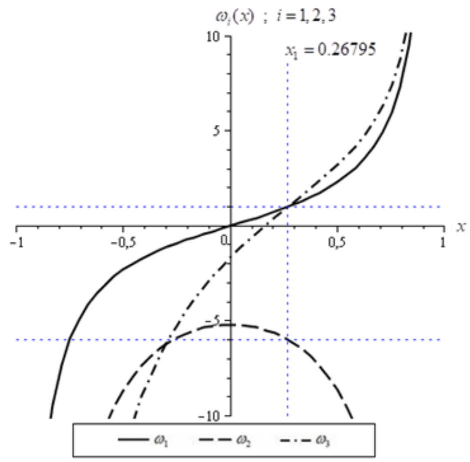

FIGURE 3. The angular velocity components with respect to $x = \tanh\left(\left(\varphi+\theta\right)/2\right)$

Then, the numerical approximations of the roots of the polynomial gives

$$x_1 = 0.26795, \quad x_2 = 3.73205, \quad x_3 = x_4 = 1, \quad x_5 = x_6 = -1.$$

Since all the roots are real roots, they give the potential rolling directions. Substituting the first real root x_1 and second real root x_2 into the first part of Eq.(16) gives the values of the rolling rate σ as 5.19615 and -5.19615, respectively. When substituting the last four roots into (16) one by one, the rolling rates become zero. Since the rolling rate can only be positive physically, the non-positive solutions are neglected. When the coordinates ($u = -2$ and $v = -\pi$) and ($\bar{u} = 1$ and $\bar{v} = \pi/6$) are taken into account, the hyperbolic angle θ between the timelike vectors $\boldsymbol{T}_1$ and $\bar{\boldsymbol{T}}_1$ in (11) and (13) is computed as

$$\theta = \operatorname{arccosh}\left(\left\langle \boldsymbol{T}_1, \bar{\boldsymbol{T}}_1 \right\rangle\right) = 0.89837.$$

In this case, the rolling direction φ is obtained as

$$\varphi = 2\operatorname{arctanh}(x_1) - \theta = -0.34906.$$

When substituting $\omega_3 = 1$, the rolling direction $\varphi = -0.34906$ and rolling rate $\sigma = 5.19615$ into the third term of Eq.(16) gives the compensatory spin rate as $\hat{\omega}_3 = 2.40958$. Then, the first part of the third term in Eq.(16) gives the induced spin rate and the value of this part is obtained as $\sigma\left(\tanh \bar{u} \sinh \varphi\right) = -1.40958$. Thus, the inverse kinematics of a Lorentzian unit sphere rolling on a timelike cylinder with radius $3/4$ is completed. Since the compensatory spin rate is 2.40958, then the value of the desired angular velocity components can be easily seen when the value of $x = x_1$ is given as 0.26795 (See Fig. 3).

4. Conclusion

In this paper, we aim to investigate the inverse kinematics of rolling contact motion without sliding of one timelike surface on another timelike surface along their timelike trajectory curves by applying curvature theory and the moving-frame method from Lorentzian geometry. When the desired angular velocity and the coordinates of the contact point of the timelike surfaces are given, we present a polynomial formulation by using three nonlinear algebraic equations via the proposed approach. Then, the solution of the polynomial gives six roots enabling us to analyze the rolling rate. Moreover, we engender two fundamental parts of the spin velocity in Lorentzian 3-space: the induced spin velocity and the compensatory spin velocity.

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