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AUTOMATIC STRUCTURE FOR GENERALIZED BRUCK-REILLY $*$-EXTENSION OF A MONOID

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Abstract. In the present paper, we study the automaticity of generalized Bruck-Reilly $*$-extension of a monoid. Under some certain situations, we prove that the automaticity of the monoid implies the automaticity of the generalized Bruck-Reilly $*$-extension of this monoid.

1. Introduction and Preliminaries

One of the most popular areas of computational algebra has recently been the theory of *automatic groups*. The description of a group by an automatic structure allows one efficiently to perform various computations involving the group, which may be hard or impossible given only a presentation. Groups which admit automatic structure also share a number of interesting structural and geometric properties [8]. Recently, many authors have followed a suggestion of Hudson [12] by considering a natural generalization to the broader class of monoids or, even more generally, of semigroups, and a coherent theory has begun to develop from the point of geometric aspects [21], computational and decidebility aspects [17, 18, 19], other notions of automaticity for semigroups [9, 10].

Many results about automatic semigroups concern automaticity of semigroup constructions. For instance, in [5] free product of semigroups, in [4] direct product of semigroups, in [7] Rees matrix semigroups, in [1, 3] Bruck-Reilly extension of monoids and wreath product of semigroups were studied. In [6], the author showed that a Bruck-Reilly extension $BR(S,\theta)$ of an automatic monoid S is itself automatic

- if S is finite (Theorem 5.1),
- if the mapping $\theta : S \to S$ sends every element of S to 1_S (Theorem 5.2),
- if $\theta : S \to S$ is the identity mapping (Theorem 5.3),
- if S is a finite geometric type automatic monoid and $S\theta$ is finite (Theorem 5.4).

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These results and their proofs are reproduced in a survey article by Andrade et al. [1]. In the present paper, by considering the results given in [6], we study on generalized Bruck-Reilly $*$-extension of a monoid of which presentation was firstly defined in [14]. A generalized Bruck-Reilly $*$-extension was first introduced in [2]. Since then many research papers have been published see for example [13, 15, 16, 20]. We prove the following results:

Theorem 4 *If T is a finite monoid then generalized Bruck-Reilly $*$-extension of T is automatic.*

Theorem 6 *If T is an automatic monoid and $\gamma, \beta : T \to H_1^*$; $t \mapsto 1_T$ then generalized Bruck-Reilly $*$-extension of T is automatic.*

Theorem 7 *If T is an automatic monoid and γ, β are identity homomorphisms of T then generalized Bruck-Reilly $*$-extension of T is automatic.*

Theorem 10 *Let T be a finite geometric type automatic monoid and let $\gamma, \beta : T \to H_1^*$ be homomorphisms. If $T\gamma$, $T\beta$ are finite then generalized Bruck-Reilly $*$-extension of T is automatic.*

Let A be an alphabet. We denote by A^+ the free semigroup generated by A consisting of finite sequences of elements of A, which we call words, under the concatenation; and by A^* the free monoid generated by A consisting of A^+ with the empty word ϵ, the identity in A^*. For a word $w \in A^*$, we denote the length of w by $|w|$. Let S be a semigroup and $\phi : A \to S$ a mapping. We say that A is a *finite generating set for S with respect to ϕ* if the unique extension of ϕ to a semigroup homomorphism $\psi : A^+ \to S$ is surjective. For $u, v \in A^+$ we write $u \equiv v$ to mean that u and v are equal as words and $u = v$ to mean that u and v represent the same element in the semigroup. In other words $u\psi = v\psi$. We say that a subset L of A^*, usually called a *language*, is *regular* if there is a finite state automaton accepting L ([5]). To be able to deal with automata that accept pairs of words and to define automatic semigroups we need to define the set $A(2, \$) = ((A \cup \{\$\}) \times (A \cup \{\$\})) - \{(\$, \$)\}$ where \$ is a symbol not in A (called the padding symbol) and the function $\delta_A : A^* \times A^* \to A(2, \$)^*$ defined by

$$(a_1 \cdots a_m, b_1 \cdots b_n)\delta_A = \begin{cases} \epsilon & \text{if } 0 = m = n \\ (a_1, b_1) \cdots (a_m, b_m) & \text{if } 0 < m = n \\ (a_1, b_1) \cdots (a_m, b_m)(\$, b_{m+1}) \cdots (\$, b_n) & \text{if } 0 \leq m < n \\ (a_1, b_1) \cdots (a_n, b_n)(a_{n+1}, \$) \cdots (a_m, \$) & \text{if } m > n \geq 0. \end{cases}$$

Let S be a semigroup and A a finite generating set for S with respect to $\psi : A^+ \to S$. The pair (A, L) is an *automatic structure for S* (*with respect to ψ*) if

- L is a regular subset of A^+ and $L\psi = S$,
- $L_= = \{(\alpha, \beta) : \alpha, \beta \in L, \alpha = \beta\}\delta_A$ is a regular in $A(2, \$)^+$, and
- $L_a = \{(\alpha, \beta) : \alpha, \beta \in L, \alpha a = \beta\}\delta_A$ is a regular in $A(2, \$)^+$ for each $a \in A$.

We say that a semigroup is *automatic* if it has an automatic structure.

We say that the pair (A, L) is an *automatic structure with uniqueness* (with respect to ψ) for a semigroup S, if it is an automatic structure and each element in S is represented by an unique word in L (the restriction of ψ to L is a bijection).

2. Generalized Bruck-Reilly ∗-Extension

Let T be a monoid with H_1^* and H_1 as the $\mathcal{H}^*$- and $\mathcal{H}$- class which contains the identity 1_T of T, respectively. Assume that β and γ are morphisms from T into H_1^*. Let u be an element in H_1 and let λ_u be the inner automorphism of H_1^* defined by $x \mapsto uxu^{-1}$ such that $\gamma\lambda_u = \beta\gamma$. Now we can consider $\mathbb{N}^0 \times \mathbb{N}^0 \times T \times \mathbb{N}^0 \times \mathbb{N}^0$ into a semigroup by defining multiplication

$$(m, n, v, p, q)(m', n', v', p', q') =$$
$$\begin{cases} (m, n-p+t, (v\beta^{t-p})(v'^{t-n'}), p'-n'+t), q') & \text{if } q = m' \\ (m, n, v(((u^{-n'}(v'^{p'})\gamma^{q-m'-1})\beta^p), p, q'-m'+q) & \text{if } q > m' \\ (m-q+m', n'^{-n}(v\gamma)u^p)\gamma^{m'-q-1})\beta^{n'})v', p', q') & \text{if } q < m', \end{cases}$$

where $t = max(p, n')$ and β^0, γ^0 are interpreted as the identity map of T and u^0 is interpreted as the identity 1_T of T. The monoid $\mathbb{N}^0 \times \mathbb{N}^0 \times T \times \mathbb{N}^0 \times \mathbb{N}^0$ constructed above is called *generalized Bruck-Reilly ∗-extension* of T determined by the morphisms β, γ and the element u. This monoid is denoted by $GBR^*(T; \beta, \gamma; u)$ and the identity of it is the element $(0, 0, 1_T, 0, 0)$ ([20]). For some information concerning semigroup theory such as $\mathcal{H}^*$- and $\mathcal{H}$-Green relations, see [11].

In [14], the authors have obtained the following results.

Lemma 1. *Suppose that X is a generating set for the monoid T. Then*

$$\{(0,0,x,0,0) : x \in X\} \cup \{(1,0,1_T,0,0) \cup (0,1,1_T,0,0) \cup (0,0,1_T,1,0)$$
$$\cup (0,0,1_T,0,1)\}$$

is a generating set for the monoid $GBR^*(T; \beta, \gamma; u)$.

Theorem 2. *Let T be a monoid defined by the presentation $< X; R >$, and let β, γ be morphisms from T into H_1^*. Therefore the monoid $GBR^*(T; \beta, \gamma; u)$ is defined by the presentation*

$$\begin{aligned} < X, y, z, a, b \quad ; \quad & R, \quad yz = 1, \quad ba = 1, \\ & yx = (x\gamma)y, \quad xz = z(x\gamma), \quad bx = (x\beta)b, \quad xa = a(x\beta) \ (x \in X), \\ & yb = uy, \quad ya = u^{-1}y, \quad bz = zu, \quad az = zu^{-1} > . \end{aligned}$$

As a consequence of Theorem 2, we have the following result.

Corollary 3. *Let v be an arbitrary word in X^*. The relations*

$$\begin{aligned} y^m v = (v\gamma^m)y^m, &\quad vz^m = z^m(v\gamma^m), \\ b^n v = (v\beta^n)b^n, &\quad va^n = a^n(v\beta^n), \\ y^m b^n = (u\gamma^{m-1})^n y^m, &\quad y^m a^n = (u^{-1}\gamma^{m-1})^n y^m, \end{aligned}$$

$$b^n z^m = z^m(u\gamma^{m-1})^n, \quad a^n z^m = z^m(u^{-1}\gamma^{m-1})^n$$

hold in $GBR^*(T;\beta,\gamma;u)$ *for all* $m,n \in \mathbb{N}^0$. *As a consequence, every word* $w \in (X \cup \{y,z,a,b\})^*$ *is equal in* $GBR^*(T;\beta,\gamma;u)$ *to a word of the form* $z^m a^n v b^p y^q$ *for some* $v \in X^*$ *and* $m,n,p,q \in \mathbb{N}^0$.

3. Main Results

We give the first result of this paper.

Theorem 4. *If* T *is a finite monoid then any generalized Bruck-Reilly ∗-extension of* T *is automatic.*

Proof. Let $T = \{t_1, t_2, \cdots, t_l\}$ and let $\overline{T} = \{\overline{t_1}, \overline{t_2}, \cdots, \overline{t_l}\}$ be an alphabet in bijection with T. We define the alphabet $A = \{y,z,a,b\} \cup \overline{T}$ and the regular language

$$L = \{z^m a^n \bar{t} b^p y^q : m,n,p,q \geq 0,\ \bar{t} \in \overline{T}\}$$

on A. Defining the homomorphism

$$\begin{array}{rcl} \psi : A^+ & \to & GBR^*(T;\beta,\gamma;u); \\ \bar{t} & \mapsto & (0,0,t,0,0), \\ y & \mapsto & (0,0,1_T,0,1), \\ z & \mapsto & (1,0,1_T,0,0), \\ a & \mapsto & (0,1,1_T,0,0), \\ b & \mapsto & (0,0,1_T,1,0), \end{array}$$

it is clear that A is a generating set for $GBR^*(T;\beta,\gamma;u)$ with respect to ψ and, in fact, given an element $(m,n,t,p,q) \in \mathbb{N}^0 \times \mathbb{N}^0 \times T \times \mathbb{N}^0 \times \mathbb{N}^0$ the unique word in L representing it is $z^m a^n \bar{t} b^p y^q$.

In order to prove that (A,L) is an automatic structure with uniqueness for $GBR^*(T;\beta,\gamma;u)$ we have to prove that, for each generator $k \in A$ the language L_k is regular. To prove that L_y, L_z, L_a and L_b are regular we observe that

$$(z^m a^n \overline{t_i} b^p y^q)y = z^m a^n \overline{t_i} b^p y^{q+1},$$

$$(z^m a^n \overline{t_i} b^p y^q)z = \begin{cases} z^m a^n \overline{t_i} b^p y^{q-1} & \text{if } q \geq 1, \\ z^{m+1}(\overline{t_i \gamma}) & \text{if } q = 0, \end{cases}$$

$$(z^m a^n \overline{t_i} b^p y^q)a = \begin{cases} z^m a^n (\overline{t_i((u^{-1}\gamma^{q-1})\beta^p)}) b^p y^q & \text{if } q \geq 1, \\ z^m a^n \overline{t_i} b^{p-1} & \text{if } q = 0,\ p \geq 1, \\ z^m a^{n+1}(\overline{t_i \beta}) & \text{if } q = p = 0, \end{cases}$$

$$(z^m a^n \overline{t_i} b^p y^q)b = \begin{cases} z^m a^n (\overline{t_i((u\gamma^{q-1})\beta^p)}) b^p y^q & \text{if } q \geq 1, \\ z^m a^n \overline{t_i} b^{p+1} & \text{if } q = 0, \end{cases}$$

and so we can write

$$L_y = \bigcup_{i=1}^{l} \{(z^m a^n \overline{t_i} b^p y^q, z^m a^n \overline{t_i} b^p y^{q+1})\delta_A : m,n,p,q \in \mathbb{N}^0\}$$

$$= \bigcup_{i=1}^{l}(\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(\overline{t_i},\overline{t_i})\}^* \cdot \{(b,b)\}^* \cdot \{(y,y)\}^* \cdot \{(\$,y)\})$$

which is a regular language. We have

$$\begin{aligned} L_z &= \bigcup_{i=1}^{l}\{(z^m a^n \overline{t_i} b^p y^q, z^m a^n \overline{t_i} b^p y^{q-1})\delta_A : m,n,p \in \mathbb{N}^0, q \geq 1\} \\ &\quad \cup \bigcup_{i=1}^{l}\{(z^m \overline{t_i}, z^{m+1}(\overline{t_i}\gamma))\delta_A : m \in \mathbb{N}^0\} \\ &= \bigcup_{i=1}^{l}(\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(\overline{t_i},\overline{t_i})\}^* \cdot \{(b,b)\}^* \cdot \{(y,y)\}^* \cdot \{(y,\$)\}) \\ &\quad \cup \bigcup_{i=1}^{l}(\{(z,z)\}^* \cdot \{(\overline{t_i},z)(\$,\overline{t_i\gamma})\}), \end{aligned}$$

and we conclude that L_z is a regular language. Now we consider the language L_a

$$\begin{aligned} L_a &= \bigcup_{i=1}^{l}\{(z^m a^n \overline{t_i} b^p y^q, z^m a^n(\overline{t_i}((u^{-1}\gamma^{q-1})\beta^p))b^p y^q)\delta_A : m,n,p \in \mathbb{N}^0, q \geq 1\} \\ &\quad \cup \bigcup_{i=1}^{l}\{(z^m a^n \overline{t_i} b^p, z^m a^n \overline{t_i} b^{p-1})\delta_A : m,n \in \mathbb{N}^0, p \geq 1\} \\ &\quad \cup \bigcup_{i=1}^{l}\{(z^m a^n \overline{t_i}, z^m a^{n+1}(\overline{t_i\beta}))\delta_A : m,n \in \mathbb{N}^0\}. \end{aligned}$$

Since T is finite the set H_1^* is finite as well. So $\{(u^{-1}\gamma^{q-1})\beta^p, (u\gamma^{q-1})\beta^p : p,q \in \mathbb{N}^0, q \geq 1\}$ is finite. Then we get

$$\begin{aligned} L_a &= \bigcup_{i=1}^{l}(\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(\overline{t_i},\overline{t_i})(\$,((u^{-1}\gamma^{q-1})\beta^p))\} \cdot \{(b,b)\}^* \cdot \{(y,y)\}^+) \\ &\quad \cup \bigcup_{i=1}^{l}(\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(\overline{t_i},\overline{t_i})\} \cdot \{(b,b)\}^* \cdot \{(b,\$)\}) \\ &\quad \cup \bigcup_{i=1}^{l}(\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(\overline{t_i},a)(\$,\overline{t_i\beta})\}) \end{aligned}$$

which is a finite union of regular languages and so is regular.

$$L_b = \bigcup_{i=1}^{l}\{(z^m a^n \overline{t_i} b^p y^q, z^m a^n(\overline{t_i}((u\gamma^{q-1})\beta^p))b^p y^q)\delta_A : m,n,p \in \mathbb{N}^0, q \geq 1\}$$

$$\begin{aligned}
& \cup \bigcup_{i=1}^{l}\{(z^m a^n \overline{t_i} b^p, z^m a^n \overline{t_i} b^{p+1})\delta_A : m,n,p \in \mathbb{N}^0\} \\
= & \bigcup_{i=1}^{l}(\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(\overline{t_i},\overline{t_i})(\$,((u\gamma^{q-1})\beta^p))\} \cdot \{(b,b)\}^* \cdot \{(y,y)\}^+) \\
& \cup \bigcup_{i=1}^{l}(\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(\overline{t_i},\overline{t_i})\} \cdot \{(b,b)\}^* \cdot \{(\$,b)\}),
\end{aligned}$$

and we conclude that L_b is a regular language as well.

Now for $\overline{t} \in \overline{T}$ we have

$$(z^m a^n \overline{t_i} b^p y^q)\overline{t} = \begin{cases} z^m a^n \overline{t_i}((t\gamma^q)\beta^p) b^p y^q & \text{if } q \geq 1, \\ z^m a^n \overline{t_i}(t\beta^p) b^p & \text{if } q = 0,\ p \geq 1. \end{cases}$$

Since T is finite the sets $\{(t\gamma^q)\beta^p : p,q \in \mathbb{N}^0, q \geq 1\}$ and $\{t\beta^p : p \in \mathbb{N}^0\}$ are finite as well. Thus we have

$$\begin{aligned}
L_{\overline{t}} = & \bigcup_{i=1}^{l}\{(z^m a^n \overline{t_i} b^p y^q, z^m a^n \overline{t_i}((t\gamma^q)\beta^p) b^p y^q)\delta_A : m,n,p \in \mathbb{N}^0, q \geq 1\} \\
& \cup \bigcup_{i=1}^{l}\{(z^m a^n \overline{t_i} b^p, z^m a^n \overline{t_i}(t\beta^p) b^p)\delta_A : m,n \in \mathbb{N}^0, p \geq 1\} \\
= & \bigcup_{i=1}^{l}(\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(\overline{t_i},\overline{t_i})(\$,((t\gamma^q)\beta^p))\} \cdot \{(b,b)\}^* \cdot \{(y,y)\}^+) \\
& \cup \bigcup_{i=1}^{l}(\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(\overline{t_i},\overline{t_i})(\$,t\beta^p)\} \cdot \{(b,b)\}^+)
\end{aligned}$$

which is a finite union of regular languages and so is regular.

Hence the result. □

Now on we assume that T is an automatic monoid and we fix an automatic structure (X,K) with uniqueness for T, where $X = \{x_1, \cdots, x_n\}$ is a set of semigroup generators for T with respect to the homomorphism

$$\phi : X^+ \to T.$$

We define the alphabet

$$A = \{y,z,a,b\} \cup X \tag{1}$$

to be a set of semigroup generators for $GBR^*(T;\gamma,\beta;u)$ with respect to the homomorphism

$$\begin{aligned}
\psi : A^+ & \to GBR^*(T;\gamma,\beta;u); \\
x_i & \mapsto (0,0,x_i\phi,0,0), \\
y & \mapsto (0,0,1_T,0,1),
\end{aligned}$$

$$\begin{array}{rcl} z & \mapsto & (1,0,1_T,0,0), \\ a & \mapsto & (0,1,1_T,0,0), \\ b & \mapsto & (0,0,1_T,1,0), \end{array}$$

and the regular language

$$L = \{z^m a^n w b^p y^q : w \in K; m,n,p,q \in \mathbb{N}^0\} \tag{2}$$

on A^+, which is a set of unique normal forms for $GBR^*(T;\gamma,\beta;u)$, since we have $(z^m a^n w b^p y^q)\psi = (m,n,w\phi,p,q)$ for $w \in K, m,n,p,q \in \mathbb{N}^0$. As usual, to simplify notation, we will avoid explicit use of the homomorphisms ψ and ϕ, associated with the generating sets, and it will be clear from the context whenever a word $w \in X^+$ is being identified with an element of T, with an element of $GBR^*(T;\gamma,\beta;u)$ or considered as a word. In particular, for a word $w \in X^+$ we write $w\theta$ instead of $(w\phi)\theta$, seeing θ also as a homomorphism $\theta : X^+ \to T$, and we will often write (m,n,w,p,q) instead of $(m,n,w\phi,p,q)$ for $m,n,p,q \in \mathbb{N}^0$.

To show that $GBR^*(T;\gamma,\beta;u)$ has automatic structure (A,L), the languages

$$\begin{array}{rcl} L_y & = & \{(z^m a^n w b^p y^q, z^m a^n w b^p y^{q+1})\delta_A : w \in K; m,n,p,q \in \mathbb{N}^0\}, \\ L_z & = & \{(z^m a^n w b^p y^q, z^m a^n w b^p y^{q-1})\delta_A : w \in K; m,n,p \in \mathbb{N}^0, q \geq 1\} \\ & & \cup\{(z^m w_1, z^{m+1} w_2)\delta_A : w_1,w_2 \in K; m \in \mathbb{N}^0; w_2 = w_1\gamma\}, \\ L_a & = & \{(z^m a^n w_1 b^p y^q, z^m a^n w_2 b^p y^q)\delta_A : w_1,w_2 \in K; m,n,p \in \mathbb{N}^0, q \geq 1; \\ & & w_2 = w_1((u^{-1}\gamma^{q-1})\beta^p)\} \\ & & \cup\{(z^m a^n w b^p, z^m a^n w b^{p-1})\delta_A : w \in K; m,n \in \mathbb{N}^0, p \geq 1\} \\ & & \cup\{(z^m a^n w_1, z^m a^{n+1} w_2)\delta_A : w_1,w_2 \in K; m,n \in \mathbb{N}^0; w_2 = w_1\beta\}, \\ L_b & = & \{(z^m a^n w_1 b^p y^q, z^m a^n w_2 b^p y^q)\delta_A : w_1,w_2 \in K; m,n,p \in \mathbb{N}^0, q \geq 1; \\ & & w_2 = w_1((u\gamma^{q-1})\beta^p)\} \\ & & \cup\{(z^m a^n w b^p, z^m a^n w b^{p+1})\delta_A : w \in K; m,n,p \in \mathbb{N}^0\}, \\ L_{x_r} & = & \{(z^m a^n w_1 b^p y^q, z^m a^n w_2 b^p y^q)\delta_A : (w_1,w_2)\delta_X \in K_{(x_r\gamma^q)\beta^p}; \\ & & m,n,p,q \in \mathbb{N}^0, (x_r \in X)\}, \end{array}$$

must be regular. We note that the language L_y is regular, since we have

$$L_y = \{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(y,y)\}^* \cdot \{(\$,y)\},$$

but there is no obvious reason why the languages L_z, L_a, L_b and L_{x_r} should also be regular. Hence we will consider particular situations where (A,L) is an automatic structure for $GBR^*(T;\gamma,\beta;u)$. We will use the notion of padded product of languages and the following result. The proof of the following result can be found in [6]. Now we fix an alphabet A, and take two regular languages M,N in $(A^* \times A^*)\delta$. Then the *padded product of languages* M and N is

$$M \odot N = \{(w_1 w_1^{'}, w_2 w_2^{'})\delta : (w_1,w_2)\delta \in M, (w_1^{'}, w_2^{'})\delta \in N\}.$$

The result is as follows.

Lemma 5. *Let A be an alphabet and let M, N be regular languages on $(A^* \times A^*)\delta$. If there exists a constant C such that for any two words $w_1, w_2 \in A^*$ we have*

$$(w_1, w_2)\delta \in M \Rightarrow ||w_1| - |w_2|| \leq C,$$

then the language $M \odot N$ is regular.

Now we give our result.

Theorem 6. *If T is an automatic monoid and $\gamma, \beta : T \to H_1^*$; $t \mapsto 1_T$ then $GBR^*(T; \gamma, \beta; u)$ is automatic.*

Proof. To show that the pair (A, L) defined by (1) and (2) is an automatic structure for $GBR^*(T; \gamma, \beta; u)$, we have to prove that the languages L_z, L_a, L_b and L_x $(x \in X)$ are regular. But now we denote by w_{1_T} the unique word in K representing 1_T. Then we have

$$\begin{aligned} L_z &= \{(z^m a^n w b^p y^q, z^m a^n w b^p y^{q-1})\delta_A : w \in K; m, n, p \in \mathbb{N}^0, q \geq 1\} \\ &\quad \cup \{(z^m w, z^{m+1} w_{1_T})\delta_A : w \in K; m \in \mathbb{N}^0\} \\ &= (\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(y,y)\}^* \cdot \{(y,\$)\}) \\ &\quad \cup (\{(z,z)\}^* \odot (K \times \{w_{1_T}\})\delta_X) \end{aligned}$$

and

$$\begin{aligned} L_a &= \{(z^m a^n w b^p y^q, z^m a^n w b^p y^q)\delta_A : w \in K; m, n, p \in \mathbb{N}^0, q \geq 1\} \\ &\quad \cup \{(z^m a^n w b^p, z^m a^n w b^{p-1})\delta_A : w \in K; m, n \in \mathbb{N}^0, p \geq 1\} \\ &\quad \cup \{(z^m a^n w, z^m a^n w_{1_T})\delta_A : w \in K; m, n \in \mathbb{N}^0\} \\ &= (\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(y,y)\}^+) \\ &\quad \cup \{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(b,\$)\} \\ &\quad \cup ((\{(z,z)\}^* \cdot \{(a,a)\}^*) \odot (K \times \{w_{1_T}\})\delta_X), \end{aligned}$$

which are regular languages by Lemma 5. Now we consider the language L_b and then we have

$$\begin{aligned} L_b &= \{(z^m a^n w b^p y^q, z^m a^n w b^p y^q)\delta_A : w \in K; m, n, p \in \mathbb{N}^0, q \geq 1\} \\ &\quad \cup \{(z^m a^n w b^p, z^m a^n w b^{p+1})\delta_A : w \in K; m, n, p \in \mathbb{N}^0\} \\ &= (\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(y,y)\}^+) \\ &\quad \cup (\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(\$,b)\}), \end{aligned}$$

which is a regular language. Since, for any $z^m a^n w b^p y^q \in L$ with $q \geq 1$, we have

$$(z^m a^n w b^p y^q)x = z^m a^n w b^p y^q,$$

and for $z^m a^n w b^p \in L$ with $p \geq 1$, we have

$$(z^m a^n w b^p)x = z^m a^n w b^p,$$

and for $z^m a^n w \in L$ we have

$$(z^m a^n w)x = z^m a^n w x,$$

we get

$$\begin{aligned} L_x &= \{(z^m a^n w b^p y^q, z^m a^n w b^p y^q)\delta_A : w \in K; m,n,p \in \mathbb{N}^0, q \geq 1\} \\ &\quad \cup \{(z^m a^n w b^p, z^m a^n w b^p)\delta_A : w \in K; m,n \in \mathbb{N}^0, p \geq 1\} \\ &\quad \cup \{(z^m a^n w_1, z^m a^n w_2)\delta_A : (w_1,w_2)\delta_X \in K_x; m,n \in \mathbb{N}^0\} \\ &= (\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(y,y)\}^+) \\ &\quad \cup (\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^+) \\ &\quad \cup (\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot K_x). \end{aligned}$$

Hence L_x is a regular language and so $GBR^*(T;\gamma,\beta;u)$ is automatic. □

Theorem 7. *If T is an automatic monoid and γ,β are identity homomorphisms of T then $GBR^*(T;\gamma,\beta;u)$ is automatic.*

Proof. To show that the pair (A,L) defined by (1) and (2) is an automatic structure for $GBR^*(T;\gamma,\beta;u)$ we have to prove that the languages L_z, L_a, L_b and L_x $(x \in X)$ are regular. To do that we have

$$\begin{aligned} L_z &= \{(z^m a^n w b^p y^q, z^m a^n w b^p y^{q-1})\delta_A : w \in K; m,n,p \in \mathbb{N}^0, q \geq 1\} \\ &\quad \cup \{(z^m w, z^{m+1} w)\delta_A : w \in K; m \in \mathbb{N}^0\} \\ &= (\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(y,y)\}^* \cdot \{(y,\$)\}) \\ &\quad \cup ((\{(z,z)\}^* \cdot \{(\$,y)\}) \odot \{(w,w)\delta_X : w \in K\}), \end{aligned}$$

and

$$\begin{aligned} L_a &= \{(z^m a^n w b^p y^q, z^m a^n w u^{-1} b^p y^q)\delta_A : w, u^{-1} \in K; m,n,p \in \mathbb{N}^0, q \geq 1\} \\ &\quad \cup \{(z^m a^n w b^p, z^m a^n w b^{p-1})\delta_A : w \in K; m,n \in \mathbb{N}^0, p \geq 1\} \\ &\quad \cup \{(z^m a^n w, z^m a^{n+1} w)\delta_A : w \in K; m,n \in \mathbb{N}^0\} \\ &= (\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(\$,u^{-1})\delta_X : u^{-1} \in K\} \cdot \\ &\quad \{(b,b)\}^* \cdot \{(y,y)\}^+) \\ &\quad \cup (\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(b,\$)\}) \\ &\quad \cup ((\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(\$,a)\}) \odot \{(w,w)\delta_X : w \in K\}), \end{aligned}$$

which are regular languages by Lemma 5. We have

$$\begin{aligned} L_b &= \{(z^m a^n w b^p y^q, z^m a^n w u b^p y^q)\delta_A : w, u \in K; m,n,p \in \mathbb{N}^0, q \geq 1\} \\ &\quad \cup \{(z^m a^n w b^p, z^m a^n w b^{p+1})\delta_A : w \in K; m,n,p \in \mathbb{N}^0\} \\ &= (\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(\$,u)\delta_X : u \in K\} \cdot \\ &\quad \{(b,b)\}^* \cdot \{(y,y)\}^+) \\ &\quad \cup (\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(\$,b)\}), \end{aligned}$$

which is a regular language. Also we have

$$L_x = \{(z^m a^n w_1 b^p y^q, z^m a^n w_2 b^p y^q)\delta_A : (w_1,w_2)\delta_X \in K_x;$$

$$\begin{aligned}&\quad m,n,p,q\in\mathbb{N}^0,(x\in X)\}\\ &= \{(z,z)\}^*\cdot\{(a,a)\}^*\cdot K_x\cdot\{(b,b)\}^*\cdot\{(y,y)\}^*\end{aligned}$$

which is a regular language. So (A,L) is an automatic structure for $GBR^*(T;\gamma,\beta;u)$. □

A semigroup T is called of *finite geometric type* (fgt) (see [21]) if for every $t_1\in T$, there exists $k\in\mathbb{N}$ such that the equation $xt_1=t_2$ has at most k solutions for every $t_2\in T$.

To prove the next theorem we need the following two lemmas which were proved in [6].

Lemma 8. *Let T be a finite geometric type monoid with an automatic structure with uniqueness (X,K). Then for every $w\in X^+$ there is a constant C such that $(w_1,w_2)\delta_X\in K_w$ implies $||w_1|-|w_2||<C$.*

Lemma 9. *Let S be a finite semigroup, X be a finite set and $\psi:X^+\to S$ be a surjective homomorphism. For any $s\in S$ the set $s\psi^{-1}$ is a regular language.*

Theorem 10. *Let T be a finite geometric type automatic monoid and let $\gamma,\beta:T\to H_1^*$ be homomorphisms. If $T\gamma$, $T\beta$ are finite then $GBR^*(T;\gamma,\beta;u)$ is automatic.*

Proof. We will prove that the pair (A,L) defined by (1) and (2) is an automatic structure for $GBR^*(T;\gamma,\beta;u)$. To do that we have

$$\begin{aligned}L_z &= \{(z^ma^nwb^py^q,z^ma^nwb^py^{q-1})\delta_A:w\in K;m,n,p\in\mathbb{N}^0,q\geq 1\}\\ &\quad\cup\{(z^mw_1,z^{m+1}w_2)\delta_A:w_1,w_2\in K;m\in\mathbb{N}^0;w_2=w_1\gamma\}.\end{aligned}$$

It is seen that the language

$$\{(z^ma^nwb^py^q,z^ma^nwb^py^{q-1})\delta_A:w\in K;m,n,p\in\mathbb{N}^0,q\geq 1\}=$$
$$\{(z,z)\}^*\cdot\{(a,a)\}^*\cdot\{(w,w)\delta_X:w\in K\}\cdot\{(b,b)\}^*\cdot\{(y,y)\}^*\cdot\{(y,\$)\}$$

is regular. Thus we just have to prove that the language

$$M=\{(z^mw_1,z^{m+1}w_2)\delta_A:w_1,w_2\in K;m\in\mathbb{N}^0;w_2=w_1\gamma\}$$

is also regular. For any $t\in T\gamma$, let w_t be the unique word in K representing t. Let

$$\begin{aligned}N &= \{(w_1,w_2)\delta_X:w_1,w_2\in K;w_2=w_1\gamma\}\\ &= \bigcup_{t\in T\gamma}\{(w_1,w_2)\delta_X:w_1,w_2\in K;w_2=w_1\gamma=t\}\\ &= \bigcup_{t\in T\gamma}\{(w_1,w_t)\delta_X:w_1\in K;w_1\in(t\gamma^{-1})\phi^{-1}\}\\ &= \bigcup_{t\in T\gamma}(((t\gamma^{-1})\phi^{-1}\cap K)\times\{w_t\})\delta_X.\end{aligned}$$

We can define $\psi : X^+ \to T\gamma$; $w \mapsto w\phi\gamma$ and, since $T\gamma$ is finite, for any $t \in T\gamma$, we can apply Lemma 9 and conclude that $(t\gamma^{-1})\phi^{-1} = t\psi^{-1}$ is regular. Therefore, N is a regular language. By Lemma 5, the language

$$\begin{aligned} M &= \{(z^m w_1, z^{m+1} w_2)\delta_A : (w_1, w_2)\delta_X \in N; m \in \mathbb{N}^0\} \\ &= (\{(z, z)\}^* \cdot \{(\$, z)\}) \odot N \end{aligned}$$

is regular. Now we will show that the language

$$\begin{aligned} L_a = \; & \{(z^m a^n w_1 b^p y^q, z^m a^n w_2 b^p y^q)\delta_A : w_1, w_2 \in K; m, n, p \in \mathbb{N}^0, q \geq 1; \\ & w_2 = w_1((u^{-1}\gamma^{q-1})\beta^p)\} \\ & \cup\{(z^m a^n w b^p, z^m a^n w b^{p-1})\delta_A : w \in K; m, n \in \mathbb{N}^0, p \geq 1\} \\ & \cup\{(z^m a^n w_1, z^m a^{n+1} w_2)\delta_A : w_1, w_2 \in K; m, n \in \mathbb{N}^0; w_2 = w_1\beta\} \end{aligned}$$

is regular. Since the language

$$\{(z^m a^n w b^p, z^m a^n w b^{p-1})\delta_A : w \in K; m, n \in \mathbb{N}^0, p \geq 1\} =$$
$$\{(z, z)\}^* \cdot \{(a, a)\}^* \cdot \{(w, w)\delta_X : w \in K\} \cdot \{(b, b)\}^* \cdot \{(b, \$)\}$$

is regular, we have to prove that

$$\begin{aligned} M_1 = \; & \{(z^m a^n w_1 b^p y^q, z^m a^n w_2 b^p y^q)\delta_A : w_1, w_2 \in K; m, n, p \in \mathbb{N}^0, q \geq 1; \\ & w_2 = w_1((u^{-1}\gamma^{q-1})\beta^p)\}, \end{aligned}$$

and

$$M_2 = \{(z^m a^n w_1, z^m a^{n+1} w_2)\delta_A : w_1, w_2 \in K; m, n \in \mathbb{N}^0; w_2 = w_1\beta\}$$

are regular. It is seen that the language

$$\begin{aligned} M_1 &= \{(z^m a^n w_1 b^p y^q, z^m a^n w_2 b^p y^q)\delta_A : w_1, w_2 \in K; m, n, p \in \mathbb{N}^0, q \geq 1; \\ & \quad w_2 = w_1((u^{-1}\gamma^{q-1})\beta^p)\} \\ &= \{(z, z)\}^* \cdot \{(a, a)\}^* \cdot \{(w_1, w_1)\delta_X : w_1 \in K\} \cdot \{(\$, (u^{-1}\gamma^{q-1})\beta^p)\} \cdot \\ & \quad \{(b, b)\}^* \cdot \{(y, y)\}^* \end{aligned}$$

is regular. Now for any $t \in T\beta$, let w_t be the unique word in K representing t. Let

$$\begin{aligned} N_2 &= \{(w_1, w_2)\delta_X : w_1, w_2 \in K; w_2 = w_1\beta\} \\ &= \bigcup_{t \in T\beta} \{(w_1, w_2)\delta_X : w_1, w_2 \in K; w_2 = w_1\beta = t\} \\ &= \bigcup_{t \in T\beta} \{(w_1, w_t)\delta_X : w_1 \in K; w_1 \in (t\beta^{-1})\phi^{-1}\} \\ &= \bigcup_{t \in T\beta} (((t\beta^{-1})\phi^{-1} \cap K) \times \{w_t\})\delta_X. \end{aligned}$$

We can define $\psi_2 : X^+ \to T\beta$; $w \mapsto w\phi\beta$ and, since $T\beta$ is finite, for any $t \in T\beta$, we can apply Lemma 9 and conclude that $(t\beta^{-1})\phi^{-1} = t\psi^{-1}$ is regular. Therefore,

N_2 is a regular language. By Lemma 5, we have that the language

$$\begin{aligned} M_2 &= \{(z^m a^n w_1, z^m a^{n+1} w_2)\delta_A : (w_1, w_2)\delta_X \in N_2; m, n \in \mathbb{N}^0\} \\ &= (\{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(\$, a)\}) \odot N_2, \end{aligned}$$

is regular.

Now we will prove that the language

$$\begin{aligned} L_b = & \{(z^m a^n w_1 b^p y^q, z^m a^n w_2 b^p y^q)\delta_A : w_1, w_2 \in K; m, n, p \in \mathbb{N}^0, q \geq 1; \\ & w_2 = w_1((u\gamma^{q-1})\beta^p)\} \\ & \cup \{(z^m a^n w b^p, z^m a^n w b^{p+1})\delta_A : w \in K; m, n, p \in \mathbb{N}^0\} \end{aligned}$$

is regular. Since the languages

$$\{(z^m a^n w b^p, z^m a^n w b^{p+1})\delta_A : w \in K; m, n, p \in \mathbb{N}^0\} = \\ \{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w,w)\delta_X : w \in K\} \cdot \{(b,b)\}^* \cdot \{(\$, b)\}$$

and

$$\{(z^m a^n w_1 b^p y^q, z^m a^n w_2 b^p y^q)\delta_A : w_1, w_2 \in K, m, n, p \in \mathbb{N}^0, q \geq 1; \\ w_2 = w_1((u\gamma^{q-1})\beta^p)\} = \{(z,z)\}^* \cdot \{(a,a)\}^* \cdot \{(w_1, w_1)\delta_X : w_1 \in K\} \cdot \\ \{(\$, (u\gamma^{q-1})\beta^p)\} \cdot \{(b,b)\}^* \cdot \{(y,y)\}^*$$

are regular, L_b is regular as well.

Now it remains to prove that the language

$$\begin{aligned} L_x = & \{(z^m a^n w_1 b^p y^q, z^m a^n w_2 b^p y^q)\delta_A : (w_1, w_2)\delta_X \in K_{(x\gamma^q)\beta^p}; \\ & m, n, p, q \in \mathbb{N}^0 \ (x \in X)\} \end{aligned}$$

is regular. We have

$$L_x = \{(z,z)\}^* \cdot \{(a,a)\}^* \cdot (K_{(x\gamma^q)\beta^p} \odot \{(b,b)\}^*) \cdot \{(y,y)\}^*.$$

Since T is finite geometric type, by Lemma 8 there is a constant C such that $(w_1, w_2)\delta_X \in K_{(x\gamma^q)\beta^p}$ implies $||w_1| - |w_2|| < C$, for any $p, q \in \mathbb{N}^0$, and thus we can apply Lemma 5 and we conclude that L_x is a regular language. □

As known, for a given construction, natural questions are:

(1) Is the class of automatic semigroups closed under this construction?
(2) If a semigroup resulting from such a construction is automatic, is the original semigroup (or are the original semigroups) automatic?

In this paper, we answered the first question "yes" under some certain situations for generalized Bruck-Reilly $*$-extension. But the second question is still open.

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