Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (2) (2020), 498 – 504
DOI : 10.15672/hujms.471101

Research Article

A note on the paper "Best constants for the Hardy–Littlewood maximal operator on finite graphs"

Zaryab Hussain*[1], Sadia Talib[2]

[1] Department of Mathematics, University of Central Punjab, Faisalabad-38000, Pakistan
[2] Department of Mathematics, Government College University, Faisalabad-38000, Pakistan

Abstract

Let G_n^m be a simple, connected and finite graph. Suppose $\phi : \mathbb{N} \to \mathbb{R}^+$ is a positive and increasing function. We consider the action of generalized maximal operator $M^\phi_{G_n^m}$ on ℓ^p spaces and find optimal bound for the quasi norm $\|M^\phi_{G_n^m}\|_p$ for the case $0 < p \leq 1$. In addition we find bounds for the norm $\|M^\phi_{G_n^m}\|_p$ for the case $1 < p < \infty$. We also prove some general results for $0 < p \leq 1$.

Mathematics Subject Classification (2010). 05C12, 42B25

Keywords. generalized maximal operator, ℓ^p-estimates, G_n^m graph

1. Introduction

Let $G(V, E)$ be a connected, finite and simple graph where $V(G)$ is the set of vertices and $E(G)$ is the set of edges between the vertices of graph G. Let $d_G : V(G) \times V(G) \to \mathbb{R}$ be the geodesic metric space defined for $u, v \in V(G)$ as the number of edges in shortest path between u and v written as $d_G(u, v)$. The set $N_G(u) = \{x \in V(G) \mid d_G(u, x) = 1\}$ is the neighborhood of u in graph G, cardinality of neighborhood set is called degree of u and is denoted as $d_G(u)$. For any function $f : V(G) \to \mathbb{R}$ we can consider the generalized maximal operator [1] $M^\phi_G : \ell^p \to \ell^p$, such as

$$M^\phi_G f(j) = \sup_{r\geq 0} \frac{1}{\phi(|B(j, r)|)} \sum_{s\in B(j,r)} |f(s)| \tag{1.1}$$

where $\phi : \mathbb{N} \to \mathbb{R}^+$ is a positive, increasing function and $B(j, r) = \{x \in V(G) \mid d_G(j, x) \leq r\}$ is the ball of radius r with center at j. Note that M^t_G is the classical Hardy-Littlewood maximal operator and $M^{t^{1-\frac{s}{r}}}_G$, where $0 < s < r$, is the fractional maximal operator. As distance takes only natural numbers as values, the radius $r \geq 0$ considered in the definition

*Corresponding Author.
Email addresses: zaryabhussain2139@gmail.com (Z. Hussain), sadiatalib2015@gmail.com (S. Talib)
Received: 16.10.2018; Accepted: 27.12.2018

of generalized maximal operator can be taken to be a natural number also the diameter of the graph of n vertices is at most $n-1$, so we can write the equation (1.1) such as

$$M_G^{\phi} f(j) = \max_{r=0,1,\ldots,n-1} \frac{1}{\phi\left(|B(j,r)|\right)} \sum_{s\in B(j,r)} |f(s)|. \tag{1.2}$$

For $0 < p < \infty$, the norm of M_G^{ϕ} is define as

$$\|M_G^{\phi}\|_p := \sup_{f\neq 0} \frac{\|M_G^{\phi} f\|_p}{\|f\|_p}$$

where $\|f\|_p = \left(\sum_{s\in V(G)} |f(s)|^p\right)^{\frac{1}{p}}$.

In the paper [2] authors proved that if $0 < p \le 1$, then

$$\|M_{K_n}\|_p = \left(1 + \frac{n-1}{n^p}\right)^{\frac{1}{p}},$$

if $1 < p < \infty$, then

$$\left(1 + \frac{n-1}{n^p}\right)^{\frac{1}{p}} \le \|M_{K_n}\|_p \le \left(1 + \frac{n-1}{n}\right)^{\frac{1}{p}}, \tag{1.3}$$

where K_n is a complete graph. In this paper we generalize the results given in [2].

2. Preliminaries

Definition 2.1. A family of graphs G_n^m as those simple graphs having n vertices with one vertex say k (central vertex) of degree $n-1$ and all other vertices of degree m, where $1 \le m \le n-1$.

G_n^m is a very large family of graphs as it contains both star graph $S_n \sim G_n^1$ as well as complete graph $K_n = G_n^{n-1}$ as end-points, it has also many more important graphs in it.

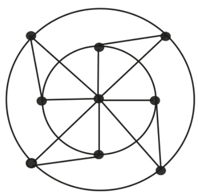

Figure 1. G_9^4 graph

For example if we take $m = 3$, then $G_n^3 \sim W_n$ (wheel graph). For $k \in V$(central vertex) the $B(k,r)$ for G_n^m is

$$B(k,r) = \begin{cases} \{k\}, & \text{for} \quad r = 0, \\ V, & \text{for} \quad r \ge 1. \end{cases}$$

For $j \in V$ other than k, the $B(j,r)$ for G_n^m is

$$B(j,r) = \begin{cases} \{j\}, & \text{for} \quad r = 0, \\ \{j\} \bigcup N_{G_n^m}(j), & \text{for} \quad r = 1, \\ V, & \text{for} \quad r \ge 2. \end{cases}$$

Suppose $j \in V$, the generalized maximal operator for G_n^m is

$$M^{\phi}_{G_n^m} f(j) = \begin{cases} \max\left\{\frac{1}{\phi(1)}\left|f(j)\right|, \frac{1}{\phi(n)}\sum\limits_{x\in V}\left|f(x)\right|\right\}, & \text{if} \quad j = k, \\ \max\left\{\frac{1}{\phi(1)}\left|f(j)\right|, \frac{1}{\phi(m+1)}\sum\limits_{v\in B(j,1)}\left|f(v)\right|, \frac{1}{\phi(n)}\sum\limits_{x\in V}\left|f(x)\right|\right\}, & \text{if} \quad j \neq k. \end{cases} \tag{2.1}$$

Let see a particular example for the norm of generalized maximal operator on G_7^2.

Example 2.2. Let $G_7^2 \sim F_7$ (friendship graph of 7 vertices) with $V = \{1, 2, 3, 4, 5, 6, 7\}$ be the vertex set, 1 is the central vertex. There are 9 edges in this graph 1-2, 1-3, 1-4, 1-5, 1-6, 1-7, 2-3, 4-5 and 6-7, now it is easy to draw this graph. Take Dirac delta as function, $\phi(t) = t^2$ and $p = \frac{1}{2}$, then we have

$$M^{t^2}_{G_7^2}\delta_1(j) = \begin{cases} 1, & \text{for} \quad j = 1, \\ \frac{1}{9}, & \text{for} \quad j = 2, 3, 4, 5, 6, 7, \end{cases}$$

and

$$M^{t^2}_{G_7^2}\delta_2(j) = \begin{cases} 1, & \text{for} \quad j = 2, \\ \frac{1}{9}, & \text{for} \quad j = 3, \\ \frac{1}{49}, & \text{for} \quad j = 1, 4, 5, 6, 7. \end{cases}$$

Hence $\|M^{t^2}_{G_7^2}\delta_1\|_{\frac{1}{2}} = 9$ and $\|M^{t^2}_{G_7^2}\delta_2\|_{\frac{1}{2}} = 4.1927$. By symmetry, we also have the estimates for the remaining vertices: $\|M^{t^2}_{G_7^2}\delta_3\|_{\frac{1}{2}} = \|M^{t^2}_{G_7^2}\delta_4\|_{\frac{1}{2}} = \|M^{t^2}_{G_7^2}\delta_5\|_{\frac{1}{2}} = \|M^{t^2}_{G_7^2}\delta_6\|_{\frac{1}{2}} = \|M^{t^2}_{G_7^2}\delta_7\|_{\frac{1}{2}} = 4.1927$, so $\|M^{t^2}_{G_7^2}\|_{\frac{1}{2}} = 9$. This calculation can be obtained directly from Proposition 3.1.

The operator $M^{\phi}_{G_n^{n-1}}$ ($G_n^{n-1} = K_n$) is the smallest, in the pointwise ordering, among all M^{ϕ}_G, with G a graph of n vertices. That is for each $f : V \to \mathbb{R}$ and every $j \in V$, we have that

$$M^{\phi}_{G_n^{n-1}} f(j) \leq M^{\phi}_G f(j). \tag{2.2}$$

Consequently for every $0 < p < \infty$,

$$\|M^{\phi}_{G_n^{n-1}}\|_p^p \leq \|M^{\phi}_G\|_p^p. \tag{2.3}$$

Lemma 2.3 ([2])**.** *Let G be the graph, and $\Omega : \ell^p(G) \to \ell^p(G)$ be a sublinear operator with $0 < p \leq 1$. Then,*

$$\|\Omega\|_p = \max_{j\in V} \|\Omega\delta_j\|_p.$$

3. Main results

Proposition 3.1. *If $0 < p \leq 1$, then*

$$\|M^{\phi}_{G_n^m}\|_p = \left(\frac{1}{\phi^p(1)} + \frac{n-1}{\phi^p(m+1)}\right)^{\frac{1}{p}}$$

and if $1 < p < \infty$, then

$$\left(\frac{1}{\phi^p(1)} + \frac{n-1}{\phi^p(m+1)}\right)^{\frac{1}{p}} \leq \|M^{\phi}_{G_n^m}\|_p \leq \left(\frac{1}{\phi^p(1)} + (n-1)\max\left\{\frac{(m+1)^{p-1}}{\phi^p(m+1)}, \frac{n^{p-1}}{\phi^p(n)}\right\}\right)^{\frac{1}{p}}.$$

Proof. Let $f : V \to \mathbb{R}$ be a function such that $\|f\|_p = 1$. Suppose that $k \in V(G_n^m)$ is the central vertex of the graph define δ_k, then for $0 < p < \infty$ we have

$$\begin{aligned}\|M^{\phi}_{G_n^m}\delta_k\|_p &= \left(\left(M^{\phi}_{G_n^m}\delta_k(k)\right)^p + \sum_{i\in V\setminus\{k\}}\left(M^{\phi}_{G_n^m}\delta_k(i)\right)^p\right)^{\frac{1}{p}}\\ &= \left(\frac{1}{\phi^p(1)} + \frac{n-1}{\phi^p(m+1)}\right)^{\frac{1}{p}}.\end{aligned}$$

Now suppose $r \in V(G_n^m)$ such that $r \neq k$, we define δ_r, then we have

$$\begin{aligned}\|M^{\phi}_{G_n^m}\delta_r\|_p &= \Bigg(\left(M^{\phi}_{G_n^m}\delta_r(r)\right)^p + \sum_{i\in N_{G_n^m}(r)\setminus\{k\}}\left(M^{\phi}_{G_n^m}\delta_r(i)\right)^p\\ &\quad + \sum_{b\in\{k\}\bigcup\{x:\ x\notin N_{G_n^m}(r)\}}\left(M^{\phi}_{G_n^m}\delta_r(b)\right)^p\Bigg)^{\frac{1}{p}}\\ &= \left(\frac{1}{\phi^p(1)} + \frac{m-1}{\phi^p(m+1)} + \frac{n-m}{\phi^p(n)}\right)^{\frac{1}{p}}.\end{aligned}$$

As $\|\delta_k\|_p = 1$ so we have for $0 < p < \infty$

$$\|M^{\phi}_{G_n^m}\|_p \geq \max\left\{\left(\frac{1}{\phi^p(1)} + \frac{n-1}{\phi^p(m+1)}\right)^{\frac{1}{p}}, \left(\frac{1}{\phi^p(1)} + \frac{m-1}{\phi^p(m+1)} + \frac{n-m}{\phi^p(n)}\right)^{\frac{1}{p}}\right\}.$$

Due to the monotonicity of ϕ, the maximum is always attained at the first term, so

$$\|M^{\phi}_{G_n^m}\|_p \geq \left(\frac{1}{\phi^p(1)} + \frac{n-1}{\phi^p(m+1)}\right)^{\frac{1}{p}}.$$

For $0 < p \leq 1$ using Lemma 2.3 we get

$$\|M^{\phi}_{G_n^m}\|_p = \left(\frac{1}{\phi^p(1)} + \frac{n-1}{\phi^p(m+1)}\right)^{\frac{1}{p}}.$$

Now we will prove the upper bound for $1 < p < \infty$

$$\begin{aligned}\|M^{\phi}_{G_n^m}f\|_p &= \left(\left(M^{\phi}_{G_n^m}f(k)\right)^p + \sum_{i\in V\setminus\{k\}}\left(M^{\phi}_{G_n^m}f(i)\right)^p\right)^{\frac{1}{p}}\\ &= \Bigg(\max\left\{\frac{1}{\phi^p(1)}|f(k)|^p, \frac{1}{\phi^p(n)}\left(\sum_{w\in V}|f(w)|\right)^p\right\} + \sum_{i\in V\setminus\{k\}}\max\Bigg\{\frac{1}{\phi^p(1)}|f(i)|^p,\\ &\quad \frac{1}{\phi^p(m+1)}\left(\sum_{x\in B(j,1)}|f(x)|\right)^p, \frac{1}{\phi^p(n)}\left(\sum_{w\in V}|f(w)|\right)^p\Bigg\}\Bigg)^{\frac{1}{p}}\end{aligned}$$

after applying Hölder's inequality we get

$$\|M^{\phi}_{G_n^m}\|_p \leq \sup\left(\max\left\{\frac{1}{\phi^p(1)}|f(k)|^p, \frac{1}{\phi^p(n)}n^{p-1}\right\} + \sum_{i\in V\setminus\{k\}}\max\left\{\frac{1}{\phi^p(1)}|f(i)|^p, \frac{1}{\phi^p(j)}j^{p-1}\right\}\right)^{\frac{1}{p}}$$

where $\frac{1}{\phi^p(j)}j^{p-1} = \max\left\{\frac{1}{\phi^p(m+1)}(m+1)^{p-1}, \frac{1}{\phi^p(n)}n^{p-1}\right\}$. If $\frac{1}{\phi^p(1)}|f(k)|^p \le \frac{1}{\phi^p(n)}n^{p-1}$ and $\frac{1}{\phi^p(1)}|f(i)|^p \le \frac{1}{\phi^p(j)}j^{p-1}$ for all vertices then we have

$$\begin{aligned}\|M^{\phi}_{G^m_n}\|_p &\le \left(\frac{1}{\phi^p(n)}n^{p-1} + \sum_{i\in V\setminus\{k\}} \frac{1}{\phi^p(j)}j^{p-1}\right)^{\frac{1}{p}} \\ &= \left(\frac{n^{p-1}}{\phi^p(n)} + \frac{(n-1)\,j^{p-1}}{\phi^p(j)}\right)^{\frac{1}{p}}.\end{aligned}$$

If $\frac{1}{\phi^p(1)}|f(k)|^p \le \frac{1}{\phi^p(n)}n^{p-1}$ and $\frac{1}{\phi^p(1)}|f(i_\circ)|^p > \frac{1}{\phi^p(j)}j^{p-1}$ for some $i_\circ$ then

$$\begin{aligned}\|M^{\phi}_{G^m_n}\|_p &\le \sup\left(\frac{1}{\phi^p(n)}n^{p-1} + \sum_{i_\circ\in\frac{1}{\phi^p(1)}|f(i_\circ)|^p>\frac{1}{\phi^p(j)}j^{p-1}} \frac{1}{\phi^p(1)}|f(i_\circ)|^p\right. \\ &\quad \left. + \sum_{i\in\frac{1}{\phi^p(1)}|f(i)|^p\le\frac{1}{\phi^p(j)}j^{p-1}} \frac{1}{\phi^p(j)}j^{p-1}\right)^{\frac{1}{p}} \\ &\le \left(\frac{n^{p-1}}{\phi^p(n)} + \frac{1}{\phi^p(1)} + \frac{(n-2)\,j^{p-1}}{\phi^p(j)}\right)^{\frac{1}{p}}.\end{aligned}$$

If $\frac{1}{\phi^p(1)}|f(k)|^p \ge \frac{1}{\phi^p(n)}n^{p-1}$ and $\frac{1}{\phi^p(1)}|f(i_\circ)|^p > \frac{1}{\phi^p(j)}j^{p-1}$ for some $i_\circ$ then

$$\begin{aligned}\|M^{\phi}_{G^m_n}\|_p &\le \sup\left(\frac{1}{\phi^p(1)}|f(k)|^p + \sum_{i_\circ\in\frac{1}{\phi^p(1)}|f(i_\circ)|^p>\frac{1}{\phi^p(j)}j^{p-1}} \frac{1}{\phi^p(1)}|f(i_\circ)|^p\right. \\ &\quad \left. + \sum_{i\in\frac{1}{\phi^p(1)}|f(i)|^p\le\frac{1}{\phi^p(j)}j^{p-1}} \frac{1}{\phi^p(j)}j^{p-1}\right)^{\frac{1}{p}} \\ &= \sup\left(\sum_{y\in\{k\}\bigcup\{i_\circ\}} \frac{1}{\phi^p(1)}|f(y)|^p + \sum_{i\in\frac{1}{\phi^p(1)}|f(i)|^p\le\frac{1}{\phi^p(j)}j^{p-1}} \frac{1}{\phi^p(j)}j^{p-1}\right)^{\frac{1}{p}} \\ &\le \left(\frac{1}{\phi^p(1)} + \frac{(n-2)\,j^{p-1}}{\phi^p(j)}\right)^{\frac{1}{p}}.\end{aligned}$$

If $\frac{1}{\phi^p(1)}|f(k)|^p \ge \frac{1}{\phi^p(n)}n^{p-1}$ and $\frac{1}{\phi^p(1)}|f(i)|^p \le \frac{1}{\phi^p(j)}j^{p-1}$ then we have

$$\begin{aligned}\|M^{\phi}_{G^m_n}\|_p &\le \sup\left(\frac{1}{\phi^p(1)}|f(k)|^p + \sum_{i\in\frac{1}{\phi^p(1)}|f(i)|^p\le\frac{1}{\phi^p(j)}j^{p-1}} \frac{1}{\phi^p(j)}j^{p-1}\right)^{\frac{1}{p}} \\ &\le \left(\frac{1}{\phi^p(1)} + \frac{(n-1)\,j^{p-1}}{\phi^p(j)}\right)^{\frac{1}{p}}.\end{aligned}$$

□

Now we will prove some general results. For rest of the paper we assume $0 < p \le 1$.

Theorem 3.2. *For the general graph G with n vertices we have*

$$\|M^{\phi}_{G^{n-1}_n}\|^p_p \le \|M^{\phi}_{G}\|^p_p \le \|M^{\phi}_{G^1_n}\|^p_p.$$

Proof. Lower bound of this theorem is trivial. We have to prove only the upper bound. Let $j \in V$ and define δ_j, then we have

$$\begin{aligned}\|M_G^{\phi}\delta_j\|_p^p &= \left(M_G^{\phi}\delta_j(j)\right)^p + \sum_{x\in V\setminus\{j\}}\left(M_G^{\phi}\delta_j(x)\right)^p \\ &= \frac{1}{\phi^p(1)} + \sum_{x\in V\setminus\{j\}}\left\{\frac{1}{\phi\left(|B(j,r)|\right)}\sum_{w\in B(j,r)}\delta_j(w)\right\}^p\end{aligned}$$

clearly $2 \le |B(j,r)|$ for the radius $r \ge 1$, so we get

$$\|M_G^{\phi}\delta_j\|_p^p \le \frac{1}{\phi^p(1)} + \frac{n-1}{\phi^p(2)}$$

by using Lemma 2.3, we get

$$\|M_G^{\phi}\|_p^p \le \|M_{G_n^1}^{\phi}\|_p^p.$$

□

Theorem 3.3. $G = G_n^{n-1}$ *if and only if* $\|M_G^{\phi}\|_p^p = \|M_{G_n^{n-1}}^{\phi}\|_p^p$.

Proof. If $G = G_n^{n-1}$ then $\|M_G^{\phi}\|_p = \|M_{G_n^{n-1}}^{\phi}\|_p$ is a trivial case. We have only to prove the converse part, for that let $G \neq G_n^{n-1}$ then there exist two different vertices x and y in V such that $d_G(x,y) > 1$. Let consider two sets $X = B(x,1) = \{j \in V \;:\; d_G(x,j) \le 1\}$ and $Y = B(y,1) = \{j \in V \;:\; d_G(y,j) \le 1\}$. It is clear that $|X|, |Y| \ge 2$. Thus, we consider two cases.

Case 1. $\min\{|X|,|Y|\} \le \frac{n}{2}$.
We assume that $|X| \le \frac{n}{2}$. Let $k \in X$ such that it is different from x and we define δ_k, then

$$\begin{aligned}\|M_G^{\phi}\delta_k\|_p^p &= \sum_{v\in V}\left(M_G^{\phi}\delta_k(v)\right)^p \\ &= \left(M_G^{\phi}\delta_k(k)\right)^p + \left(M_G^{\phi}\delta_k(x)\right)^p + \sum_{v\in V\setminus\{x,k\}}\left(M_G^{\phi}\delta_k(v)\right)^p\end{aligned}$$

since $M_G^{\phi}\delta_k(v) \ge \frac{1}{\phi(n)}$ for each $v \in V$, so we get

$$\begin{aligned}\|M_G^{\phi}\|_p^p &\ge \|M_G^{\phi}\delta_k\|_p^p \ge \frac{1}{\phi^p(1)} + \frac{1}{\phi^p\left(|X|\right)} + \frac{n-2}{\phi^p(n)} \\ &\ge \frac{1}{\phi^p(1)} + \frac{1}{\phi^p\left(\frac{n}{2}\right)} + \frac{n-2}{\phi^p(n)} \\ &> \|M_{G_n^{n-1}}^{\phi}\|_p^p,\end{aligned}$$

which completes the proof of case 1.

Case 2. $\min\{|X|,|Y|\} > \frac{n}{2}$.
It is easy to see that $X \cap Y \neq \varnothing$. Let $k \in X \cap Y$ and define δ_k, then

$$\begin{aligned}\|M_G^{\phi}\delta_k\|_p^p &= \left(M_G^{\phi}\delta_k(k)\right)^p + \left(M_G^{\phi}\delta_k(x)\right)^p + \left(M_G^{\phi}\delta_k(y)\right)^p + \sum_{v\in V\setminus\{x,y,k\}}\left(M_G^{\phi}\delta_k(v)\right)^p \\ &\ge \frac{1}{\phi^p(1)} + \frac{1}{\phi^p\left(|X|\right)} + \frac{1}{\phi^p\left(|Y|\right)} + \frac{n-3}{\phi^p(n)}.\end{aligned}$$

Clearly $|X|, |Y| \leq n-1$, so

$$\begin{aligned}\|M_G^{\phi}\|_p^p &\geq \|M_G^{\phi}\delta_k\|_p^p \geq \frac{1}{\phi^p(1)} + \frac{2}{\phi^p(n-1)} + \frac{n-3}{\phi^p(n)} \\ &> \|M_{G_n^{n-1}}^{\phi}\|_p^p.\end{aligned}$$

□

Theorem 3.4. *$G \sim G_n^1$ if and only if $\|M_G^{\phi}\|_p^p = \|M_{G_n^1}^{\phi}\|_p^p$.*

Proof. $G \sim G_n^1 \Rightarrow \|M_G^{\phi}\|_p = \|M_{G_n^1}^{\phi}\|_p$ is trivial. Now suppose that $G \nsim G_n^1$ and hence $n \geq 3$ then $\exists\, x \neq y$ in V such that $d_G(x), d_G(y) > 1$. Suppose that $\|f\|_1 \leq 1$, for every function $f : V \to \mathbb{R}$ then, either $M_G^{\phi}f(j) = \frac{|f(j)|}{\phi(1)}$ or $M_G^{\phi}f(j) \leq \frac{1}{\phi(d_G(j)+1)}$. Take the set $X = \left\{j \in V : M_G^{\phi}f(j) = \frac{f(j)}{\phi(1)}\right\}$, then we have

$$\begin{aligned}\|M_G^{\phi}f\|_p^p &= \sum_{j\in X}\left(M_G^{\phi}f(j)\right)^p + \sum_{j\notin X}\left(M_G^{\phi}f(j)\right)^p \\ &\leq \sum_{j\in X}\left(M_G^{\phi}f(j)\right)^p + \sum_{j\notin X}\frac{1}{\phi^p\left(d_G(j)+1\right)}.\end{aligned}$$

If $x, y \in X$, then

$$\|M_G^{\phi}f\|_p^p \leq \frac{1}{\phi^p(1)} + \frac{n-2}{\phi^p(2)}.$$

If $x \notin X$, then since $X \neq \emptyset$, we have

$$\begin{aligned}\|M_G^{\phi}f\|_p^p &\leq \frac{1}{\phi^p(1)} + \frac{1}{\phi^p(d_G(j)+1)} + \frac{n-2}{\phi^p(2)} \\ &\leq \frac{1}{\phi^p(1)} + \frac{1}{\phi^p(3)} + \frac{n-2}{\phi^p(2)}.\end{aligned}$$

Similarly case when $y \notin X$. So

$$\|M_G^{\phi}\|_p^p \leq \sup\left\{\frac{1}{\phi^p(1)} + \frac{n-2}{\phi^p(2)}, \frac{1}{\phi^p(1)} + \frac{1}{\phi^p(3)} + \frac{n-2}{\phi^p(2)}\right\} < \|M_{G_n^1}^{\phi}\|_p^p,$$

which completes our arguments. □

If we put $\phi(t) = t$ and $m = n-1$ in the result of Proposition 3.1, then we get the expression (1.3). Moreover, if we put $\phi(t) = t$ in the results of Theorems 3.2, 3.3 and 3.4 then we get the same results as proved in [2], which shows that this work is a generalization of [2].

Acknowledgment. The authors would like to thank the referee for the useful corrections and invaluable comments which improved the first version of this paper. Moreover they said special thanks to their respected teacher Dr. Irshad Ahmad for his support and guidance.

References

[1] I. Ahmad and W. Nazeer, *Optimal Couples of Rearrangement Invariant Spaces for Generalized Maximal Operators*, J. Funct. Spaces **2014**, Article ID 647123, 5 pages, 2014.

[2] J. Soria and P. Tradacete, *Best constants for the Hardy-Littlewood maximal operator on finite graphs*, J. Math. Anal. Appl. **436** (2), 661–682, 2016.