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On second-order linear recurrent homogeneous differential equations with period k

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Abstract

We say that $w(x)\colon \mathbb{R} \to \mathbb{C}$ is a solution to a second-order linear recurrent homogeneous differential equation with period k ($k \in \mathbb{N}$), if it satisfies a homogeneous differential equation of the form

$$w^{(2k)}(x) = pw^{(k)}(x) + qw(x), \quad \forall x \in \mathbb{R},$$

where $p, q \in \mathbb{R}^+$ and $w^{(k)}(x)$ is the k^{th} derivative of $w(x)$ with respect to x. On the other hand, $w(x)$ is a solution to an odd second-order linear recurrent homogeneous differential equation with period k if it satisfies

$$w^{(2k)}(x) = -pw^{(k)}(x) + qw(x), \quad \forall x \in \mathbb{R}.$$

In the present paper, we give some properties of the solutions of differential equations of these types. We also show that if $w(x)$ is the general solution to a second-order linear recurrent homogeneous differential equation with period k (resp. odd second-order linear recurrent homogeneous differential equation with period k), then the limit of the quotient $w^{((n+1)k)}(x)/w^{(n)}(x)$ as n tends to infinity exists and is equal to the positive (resp. negative) dominant root of the quadratic equation $x^2 - px - q = 0$ as x increases (resp. decreases) without bound.

Keywords: Homogenous differential equations, second-order linear recurrence sequences, solutions.

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1. Introduction

Problems involving Fibonacci numbers and its various generalizations have been extensively studied by many authors. Its beauty and applications have been greatly appreciated since its introduction. In 1965, a certain generalization of the sequence of Fibonacci numbers was introduced by A. F. Horadam in [1], which is called as a second-order linear recurrence sequence and is now known as *Horadam sequence*. Properties of these type of sequences have also been studied by Horadam in [1]. In [2], J. S. Han, H. S. Kim, and J. Neggers studied a Fibonacci norm of positive integers. These authors [3] have also studied Fibonacci sequences in groupoids and introduced the concept of Fibonacci functions in [4]. They developed the notion of this type of functions using the concept of f-even and f-odd functions. Later on, a certain generalization of Fibonacci function has been investigated by B. Sroysang in [5]. In particular, Sroysang defined a function $f(x)\colon \mathbb{R} \to \mathbb{R}$ as a Fibonacci function of period $k, (k \in \mathbb{N})$ if it satisfies the equation $f(x + 2k) = f(x + k) + f(x)$ for all $x \in \mathbb{R}$. Recently, the notion of Fibonacci function has been further generalized by the author in [6]. The concept of second-order linear recurrent functions with period k which has been introduced by the author in [6] gave rise to the concept of Pell and Jacobsthal functions with period k, which are analogues of Fibonacci functions. Some elementary properties of these newly defined functions were also presented by the author in [6]. Now, inspired by these results, we present in this work the concept of second-order (resp. odd second-order) linear recurrent homogeneous differential equations with period k, or simply SOLRHDE-k (resp. oSOLRHDE-k), and study some of its properties.

The next section, which discusses our main results, is organized as follows. First, we present some elementary results on second-order (and odd second-order) linear recurrent homogeneous differential equation with period k, and then provide the form of its general solution. Afterwards, we investigate the quotient $w^{((n+1)k)}(x)/w^{(n)}(x)$, where $w(x)$ is the general solution to a SOLRHDE-k (or an oSOLRHDE-k), and find its limit as n tends to infinity. Each of our results is accompanied by an example for validation and illustration.

2. Main Results

We start-off this section with the following definition.

2.1. Definition. Let $k \in \mathbb{N}$, $p, q \in \mathbb{R}^+$ and $w\colon \mathbb{R} \to \mathbb{C}$ be differentiable on $\mathbb{R}$ infinitely many times. We say that $w(x)$ is a solution to a SOLRHDE-k if it satisfies a differential equation of the form given by

$$w^{(2k)}(x) = pw^{(k)}(x) + qw(x), \tag{2.1}$$

for all $x \in \mathbb{R}$, where $w^{(k)}(x)$ is the k^{th} derivative of $w(x)$ with respect to x. If $(p, q) = (1, 1), (1, 2), (2, 1)$, then w is a solution to a Fibonacci-like, Jacobsthal-like, and Pell-like homogeneous differential equation with period k, respectively.

2.2. Example. Let $p, q \in \mathbb{R}^+$ and $0 \neq t \in \mathbb{R}$. Define $w(x) = a^{tx}$, where $a > 0$. Suppose that $w(x)$ is a solution to a SOLRHDE-k then $(t \ln a)^{2k} a^{tx} = p(t \ln a)^k a^{tx} + q a^{tx}$. Hence, $r^2 - pr - q = 0$ where $r = (t \ln a)^k$. Solving for r, we have $r = (p \pm \sqrt{p^2 + 4q})/2$. So, $a = \exp\left(t^{-1}\Phi_\pm^{1/k}\right)$, where $\Phi_\pm = (p \pm \sqrt{p^2 + 4q})/2$. Thus, $w(x) = A\exp\left(\alpha^{1/k}x\right) + B\exp\left(\beta^{1/k}x\right)$, where $\alpha = \Phi_+$ and $\beta = \Phi_-$ and, A, B are any arbitrary real numbers. If we set $k = 1$, and $w(0) = 0$ and $w'(0) = 1$, then we get $A + B = 0$ and $\alpha A + \beta B = 1$. Here we obtain,

$$w(x) = \frac{1}{\alpha - \beta}\left(e^{\alpha x} - e^{\beta x}\right). \tag{2.2}$$

Thus, (2.2) is a solution to a SOLRHDE-k, with $k = 1$ and initial boundary conditions $w(0) = 0$ and $w'(0) = 1$. Using the identity $e^X = \sum_{n=0}^{\infty}(X^n/n!)$, we can express (2.2) in terms of power series, i.e. we have

$$w(x) = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} = \sum_{n=0}^{\infty} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{W_n}{n!} x^n,$$

where W_n is the number sequence obtained from the recurrence relation given by

$$W_0 = 0, \quad W_1 = 1, \quad W_{n+1} = pW_n + qW_{n-1}, \quad \forall n \in \mathbb{N}. \tag{2.3}$$

We note that $\alpha + \beta = p, \alpha - \beta = \sqrt{p^2 + 4q}$, and $\alpha\beta = -q$. Hence, for some particular values of p and q, we have the following examples.

(1) For $(p, q) = (1, 1)$, the function defined by

$$f(x) = \frac{1}{\sqrt{5}} \left(e^{\phi x} - e^{(1-\phi)x} \right) = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n,$$

where ϕ is the golden ratio and F_n is the n^{th} Fibonacci number, is a solution to a Fibonacci-like homogeneous differential equation. By letting $x = 1$, we obtain the identity

$$\sum_{n=0}^{\infty} \frac{F_n}{n!} = \frac{e^{\phi} - e^{1-\phi}}{\sqrt{5}}.$$

(2) For $(p, q) = (1, 2)$, the function defined by

$$j(x) = \frac{1}{3} \left(e^{2x} - e^{-x} \right) = \sum_{n=0}^{\infty} \frac{J_n}{n!} x^n,$$

where J_n is the n^{th} Jacobsthal number, is a solution to a Jacobsthal-like homogeneous differential equation. By letting $x = 1$, we obtain the identity

$$\sum_{n=0}^{\infty} \frac{J_n}{n!} = \frac{e^2 - e^{-1}}{3}.$$

(3) For $(p, q) = (2, 1)$, the function defined by

$$p(x) = \frac{1}{2\sqrt{2}} \left(e^{\sigma x} - e^{(2-\sigma)x} \right) = \sum_{n=0}^{\infty} \frac{P_n}{n!} x^n,$$

where σ is the silver ratio and P_n is the n^{th} Pell number, is a solution to a Pell-like homogeneous differential equation. By letting $x = 1$, we obtain the identity

$$\sum_{n=0}^{\infty} \frac{P_n}{n!} = \frac{e^{\sigma} - e^{2-\sigma}}{2\sqrt{2}}.$$

2.3. Proposition. *Let $k \in \mathbb{N}$, $p, q, \in \mathbb{R}^+$ and $w(x)$ be a solution to the differential equation* (2.1)*. If $g_m(x) := w^{(m)}(x)$, then $g(x)$ is also a solution to* (2.1)*.*

Proof. Let $k \in \mathbb{N}$ and $p, q, \in \mathbb{R}^+$. Suppose $g_m(x) = w^{(m)}(x)$ where $w(x)$ is a solution to (2.1). Then,

$$g_m^{(2k)}(x) = \frac{d^{2k}\left[w^{(m)}(x)\right]}{dx^{2k}} = p\frac{d^m\left[w^{(k)}(x)\right]}{dx^m} + q\frac{d^m\left[w(x)\right]}{dx^m} = pg_m^{(k)}(x) + qg_m(x),$$

proving the proposition. □

2.4. Example. Let $j(x) = e^{(-1)^{1/k}x}$ where $k \in \mathbb{N}$. It can be verified easily that $j(x) = e^{(-1)^{1/2}x} = e^{\pm ix}$ is a solution to a Jacobsthal-like homogeneous differential equation with period 2, *i.e.*

$$j^{(4)}(x) = e^{\pm ix} = -e^{\pm ix} + 2e^{\pm ix} = j''(x) + 2j(x), \quad \forall x \in \mathbb{R}.$$

Now, define $g(x) = \pm ie^{\pm ix}$. We show that $g(x)$ is also a solution to a Jacobsthal-like homogeneous differential equation with period 2, *i.e.*

$$g^{(4)}(x) = g''(x) + 2g(x), \quad \forall x \in \mathbb{R}.$$

We note that,

$$g'(x) = -e^{\pm ix}, \quad g''(x) = \mp ie^{\pm ix}, \quad g'''(x) = e^{\pm ix}, \quad g^{(4)}(x) = \pm ie^{\pm ix}.$$

Hence,

$$g^{(4)}(x) = \pm ie^{\pm ix} = \mp ie^{\pm ix} + 2 \pm ie^{\pm ix} = g''(x) + 2g(x).$$

We can also show this via Proposition (2.3). Since $g(x) = j'(x)$, and $j(x)$ is a solution to a Jacosthal-like homogeneous differential equation with period 2, then so is $g(x)$ by Proposition (2.3).

2.5. Proposition. *Let $k \in \mathbb{N}$, $p, q, \in \mathbb{R}^+$ and, $g(x)$ and $h(x)$ be any two solutions of the differential equation* (2.1)*. Then, any linear combination of $g(x)$ and $h(x)$, say $w(x) = Ag(x) + Bh(x)$ where $A, B \in \mathbb{R}$, is again a solution to* (2.1).

Proof. The proof is straightforward. Let $k \in \mathbb{N}, p, q, \in \mathbb{R}^+$, and $g(x)$ and $h(x)$ be any two solutions to the differential equation (2.1). Consider the function $w(x) = Ag(x) + Bh(x)$ where $A, B \in \mathbb{R}$. Then,

$$\begin{aligned} w^{(2k)}(x) &= Ag^{(2k)}(x) + Bh^{(2k)}(x) \\ &= p\left[Ag^{(k)}(x) + Bh^{(k)}(x)\right] + q\left[Ag(x) + Bh(x)\right] \\ &= pw^{(k)}(x) + qw(x). \end{aligned}$$

This proves the proposition. □

2.6. Example. Let $j(x) = e^{(-1)^{1/k}x}$ where $k \in \mathbb{N}$. It can be verified diretly that the function $j(x) = e^{(-1)^{1/3}x} = e^{tx}$, where $t \in \{-1, (1 \pm \sqrt{3}i)/2\}$, is a solution to a Jacobsthal-like homogeneous differential equation with period 3, *i.e.*

$$j^{(6)}(x) = j'''(x) + 2j(x), \quad \forall x \in \mathbb{R}, \tag{2.4}$$

Define $w(x) = Ae^{-x} + Be^{\frac{1}{2}(1\pm\sqrt{3})ix}$, where $A, B \in \mathbb{R}$. Then,

$$\begin{aligned} w^{(6)}(x) &= Ae^{-x} + Be^{\frac{1}{2}(1\pm\sqrt{3})ix} \\ &= -\left[Ae^{-x} + Be^{\frac{1}{2}(1\pm\sqrt{3}i)x}\right] + 2\left[Ae^{-x} + Be^{\frac{1}{2}(1\pm\sqrt{3}i)x}\right] \\ &= w'''(x) + 2w(x). \end{aligned}$$

In fact, this can also be shown using Proposition (2.5). Since $g(x) = e^{-x}$ and $h(x) = \exp(\frac{1}{2}(1 \pm \sqrt{3})ix)$ are solutions of (2.4), then the function defined by $w(x) = Ag(x) + Bh(x)$, where $A, B \in \mathbb{R}$, is also a solution to (2.4) by Proposition (2.5).

2.7. Theorem. *Let $k \in \mathbb{N}$, $p, q, \in \mathbb{R}^+$ and $w(x)$ be a solution to the differential equation* (2.1)*. Furthermore, let $\{W_n\}_{n=0}^{\infty}$ be a number sequence obtained from a second-order linear recurrence relation defined by* (2.3)*. Then,*

$$w^{(nk)}(x) = W_n w^{(k)}(x) + qW_{n-1}w(x), \quad \forall x \in \mathbb{R},\ n \in \mathbb{N}. \tag{2.5}$$

Proof. We prove this using induction on n. Let $k \in \mathbb{N},\ p, q \in \mathbb{R}^+$, and $w(x)$ be a solution to the differential equation (2.1). Then,

$$\begin{aligned} w^{(k)}(x) &= (1)w^{(k)}(x) + q(0)w(x) = W_1 w^{(k)}(x) + qW_0 w(x), \\ w^{(2k)}(x) &= pw^{(k)}(x) + q(1)w(x) = W_2 w^{(k)}(x) + qW_1 w(x), \\ w^{(3k)}(x) &= \frac{d^k}{dx^k}\left(w^{(2k)}(x)\right) = pw^{(2k)}(x) + qw^{(k)}(x) \\ &= p\left[pw^{(k)}(x) + qw(x)\right] + qw^{(k)}(x) \\ &= (p^2+q)w^{(k)}(x) + qpw(x) \\ &= W_3 w^{(k)}(x) + qW_2 w(x). \end{aligned}$$

Now we assume that the following equation is true for some natural number n,

$$w^{(nk)}(x) = W_n w^{(k)}(x) + qW_{n-1} w(x).$$

Hence,

$$\begin{aligned} w^{((n+1)k)}(x) &= \frac{d^k}{dx^k}\left[w^{(nk)}\right] = \frac{d^k}{dx^k}\left[W_n w^{(k)}(x) + qW_{n-1}w(x)\right] \\ &= W_n w^{(2k)}(x) + qW_{n-1} w^{(k)}(x) \\ &= W_n\left[pw^{(k)}(x) + qw(x)\right] + qW_{n-1}w^{(k)}(x) \\ &= \left(pW_n + qW_{n-1}\right) w^{(k)}(x) + qW_n w(x) \\ &= W_{n+1} w^{(k)}(x) + qW_n w(x). \end{aligned}$$

This proves the theorem. □

2.8. Corollary. *Let $k \in \mathbb{N}$ and $f(x)$ be a solution to a Fibonacci-like differential equation with period k. If $\{F_n\}_{n=0}^{\infty}$ is the sequence of Fibonacci numbers, then*

$$f^{(nk)}(x) = F_n f^{(k)}(x) + F_{n-1} f(x), \quad \forall x \in \mathbb{R},\ n \in \mathbb{N}.$$

2.9. Example. Consider the solution $f(x) = e^{\sqrt[4]{\phi}x}$ to a Fibonacci-like differential equation with period 4 given by the equation

$$f^{(8)}(x) = f^{(4)}(x) + f(x), \quad \forall x \in \mathbb{R}.$$

Furthermore, let $\{F_n\}$ be the sequence of Fibonacci numbers. By Corollary (2.8), we see that

$$\begin{aligned} f^{(12)}(x) &= (2+\sqrt{5})e^{\sqrt[4]{\phi}x} = 2\phi e^{\sqrt[4]{\phi}x} + e^{\sqrt[4]{\phi}x} = F_3 f^{(4)}(x) + F_2 f(x), \\ f^{(16)}(x) &= \frac{1}{2}(7+3\sqrt{5})e^{\sqrt[4]{\phi}x} = 3\phi e^{\sqrt[4]{\phi}x} + 2e^{\sqrt[4]{\phi}x} = F_4 f^{(4)}(x) + F_3 f(x). \end{aligned}$$

Similarly, for Jacobsthal-like and Pell-like differential equations with period k we have the following corollaries.

2.10. Corollary. *Let $k \in \mathbb{N}$ and $j(x)$ be a solution to a Jacobsthal-like differential equation with period k. If $\{J_n\}_{n=0}^{\infty}$ is the sequence of Jacobsthal numbers, then*

$$j^{(nk)}(x) = J_n j^{(k)}(x) + 2J_{n-1} j(x), \quad \forall x \in \mathbb{R},\ n \in \mathbb{N}.$$

2.11. Example. Consider the solution $j(x) = e^{-x}$ to a Jacobsthal-like differential equation given by

$$j''(x) = j'(x) + 2j(x), \quad \forall x \in \mathbb{R}.$$

Furthermore, let $\{J_n\}_{n=0}^{\infty}$ be the sequence of Jacobsthal numbers, *i.e.* $\{J_n\} = \{0, 1, 1, 3, 5, 11, 21, 43, 85, 171, \dots\}$. By Corollary (2.10), we see that

$$\begin{aligned} j^{(7)}(x) &= -e^{-x} = 43(-e^{-x}) + 2(21)e^{-x} = J_7 j'(x) + 2J_6 j(x), \\ j^{(8)}(x) &= e^{-x} = 85(-e^{-x}) + 2(43)e^{-x} = J_8 j'(x) + 2J_7 j(x), \\ j^{(9)}(x) &= -e^{-x} = 171(-e^{-x}) + 2(85)e^{-x} = J_9 j'(x) + 2J_8 j(x). \end{aligned}$$

2.12. Corollary. *Let $k \in \mathbb{N}$ and $p(x)$ be a solution to a Pell-like differential equation with period k. If $\{P_n\}_{n=0}^{\infty}$ is the sequence of Pell numbers, then*

$$p^{(nk)}(x) = P_n p^{(k)}(x) + P_{n-1} p(x), \quad \forall x \in \mathbb{R},\ n \in \mathbb{N}.$$

2.13. Example. Consider the solution $p(x) = e^{\sqrt[3]{\sigma}x}$ to a Pell-like differential equation with period 3 given by the equation

$$p^{(6)}(x) = 2p'''(x) + p(x), \quad \forall x \in \mathbb{R}. \tag{2.6}$$

Furthermore, let $\{P_n\}_{n=0}^{\infty}$ be the sequence of Pell numbers, *i.e.* $\{P_n\} = \{0, 1, 2, 5, 12, 29, \dots\}$. By Corollary (2.12), we see that

$$\begin{aligned} p^{(9)}(x) &= (7 + 5\sqrt{2})e^{\sqrt[3]{\sigma}x} = 5\sigma e^{\sqrt[3]{\sigma}x} + 2e^{\sqrt[3]{\sigma}x} = P_3 p'''(x) + P_2 p(x), \\ p^{(12)}(x) &= (17 + 12\sqrt{2})e^{\sqrt[3]{\sigma}x} = 12\sigma e^{\sqrt[3]{\sigma}x} + 5e^{\sqrt[3]{\sigma}x} = P_4 p'''(x) + P_3 p(x), \\ p^{(15)}(x) &= (41 + 29\sqrt{2})e^{\sqrt[3]{\sigma}x} = 29\sigma e^{\sqrt[3]{\sigma}x} + 12e^{\sqrt[3]{\sigma}x} = P_5 p'''(x) + P_4 p(x). \end{aligned}$$

In solving for the solution of equation (2.6), we obtain an approximation of the golden ratio involving the silver ratio σ. In particular, we obtain

$$\phi \approx 10\left(\sqrt[3]{\sigma}\sin(2\pi/3) - 1\right).$$

This gives us a motivation to obtain a better approximation which is given by

$$\phi \approx 10\left(\sqrt[3]{\sigma}\sin\left(\frac{2^{20} \cdot 5^6 - 315611}{2^{19} \cdot 3 \cdot 5^6}\pi\right) - 1\right).$$

Looking at this approximation, it might be interesting to get a better approximation of ϕ in terms of σ by altering the coefficient of π inside the sine function.

2.14. Corollary. *Let $k = 1$, $p, q, \in \mathbb{R}^+$ and $w(x) = e^{\alpha x}$ be a solution to* (2.1)*. Furthermore, let $\{W_n\}_{n=0}^{\infty}$ be a number sequence obtained from* (2.3)*. Then,*

$$\alpha^n = \alpha W_n + qW_{n-1}, \quad \forall n \in \mathbb{N}. \tag{2.7}$$

Furthermore, if $\{F_n\}, \{J_n\}$, and $\{P_n\}$ are the sequence of Fibonacci, Jacobsthal and Pell numbers, respectively, then

$$\phi^n = \phi F_n + F_{n-1}, \quad \forall n \in \mathbb{N}, \tag{2.8}$$

$$2^{n-1} = J_n + J_{n-1}, \quad \forall n \in \mathbb{N}, \tag{2.9}$$

$$\sigma^n = 2\sigma P_n + P_{n-1}, \quad \forall n \in \mathbb{N}, \tag{2.10}$$

where ϕ and σ are the golden and silver ratio, respectively.

Proof. We note that $w(x) = e^{\alpha x}$ is a solution to equation (2.1) with period $k = 1$. So, by Theorem (2.7), we have

$$\alpha^n e^{\alpha x} = \alpha W_n e^{\alpha x} + qW_{n-1} e^{\alpha x},$$

proving equation (2.7). By letting $(p, q) = (1, 1), (1, 2), (2, 1)$, we obtain equations (2.8), (2.9), and (2.10), respectively. □

In the following discussion, we study differential equations of the form

$$w^{(2k)}(x) = -pw^{(k)} + qw(x), \quad \forall x \in \mathbb{R}, \tag{2.11}$$

where $k \in \mathbb{N}$ and $p, q \in \mathbb{R}^+$. We call such equation as an *odd second-order linear recurrent homogeneous differential equation with period* k, or simply, oSOLRHDE-k.

Solving equation (2.11) we obtain the solution

$$w(x) = ae^{\alpha^{1/k}\zeta_n x} + be^{\beta^{1/k}\zeta_n x},$$

where $\zeta_n = \cos\left(\frac{\pi+2n\pi}{k}\right) + i\sin\left(\frac{\pi+2n\pi}{k}\right)$, $n = 0, 1, \ldots, k-1$, and $a, b \in \mathbb{R}$. If $(p, q, k) = (1, 1, 1)$, then we see that $f(x) = e^{-\phi x}$ is a solution to the following differential equation

$$w''(x) = -w'(x) + w(x), \quad \forall x \in \mathbb{R}.$$

Similarly, for $(p, q, k) = (1, 2, 1), (2, 1, 1)$, we see that the functions $j(x) = e^{-2x}$ and $p(x) = e^{-\sigma x}$ are solutions to the differential equations

$$j''(x) = -j'(x) + 2j(x), \quad \forall x \in \mathbb{R},$$
$$p''(x) = -2p'(x) + p(x), \quad \forall x \in \mathbb{R},$$

respectively. Also, if $(p, q, k) = (1, 1, 3)$, then the function defined by $f(x) = e^{tx}$, where $t \in \{-\sqrt[3]{\phi}, \sqrt[3]{\phi}(1 \pm \sqrt{3}i)/2\}$, is a solution to an odd Fibonacci-like homogeneous differential equation with period 3. *i.e.*, $f(x) = e^{tx}$ is a solution to

$$f^{(6)}(x) = -f^{(3)}(x) + f(x), \quad \forall x \in \mathbb{R}. \tag{2.12}$$

2.15. Theorem. *Let $k \in \mathbb{N}$, $p, q, \in \mathbb{R}^+$ and $w(x)$ be a solution to the differential equation (2.11). Furthermore, let $\{W_{-n}\}_{n=0}^{\infty}$, where $W_{-n} = (-1)^{n+1}W_n$ be a number sequence obtained from a second-order linear recurrence relation defined by*

$$W_0 = 0, \quad W_{-1} = 1, \quad W_{-(n+1)} = -pW_{-n} + qW_{-n+1}, \quad \forall n \in \mathbb{N}. \tag{2.13}$$

Then,

$$w^{(nk)}(x) = W_{-n}w^{(k)}(x) + qW_{-n+1}w(x), \quad \forall x \in \mathbb{R},\ n \in \mathbb{N}. \tag{2.14}$$

Proof. We follow the proof of Theorem (2.7). Let $k \in \mathbb{N}$, $p, q, \in \mathbb{R}^+$, and $w(x)$ be a solution to the differential equation (2.11). Then,

$$\begin{aligned}
w^{(k)}(x) &= (1)w^{(k)}(x) + q(0)w(x) = W_{-1}w^{(k)}(x) + qW_0w(x),\\
w^{(2k)}(x) &= -pw^{(k)}(x) + q(1)w(x) = W_{-2}w^{(k)}(x) + qW_{-1}w(x),\\
w^{(3k)}(x) &= \frac{d^k}{dx^k}\left(w^{(2k)}(x)\right) = -pw^{(2k)}(x) + qw^{(k)}(x)\\
&= -p\left[-pw^{(k)}(x) + qw(x)\right] + qw^{(k)}(x)\\
&= (p^2 + q)w^{(k)}(x) + qpw(x)\\
&= W_{-3}w^{(k)}(x) + qW_{-2}w(x).
\end{aligned}$$

Now we assume that the following equation is true for some natural number n,

$$w^{(nk)}(x) = W_{-n}w^{(k)}(x) + qW_{-n+1}w(x).$$

Hence,

$$\begin{aligned}
w^{((n+1)k)}(x) &= \frac{d^k}{dx^k}\left[w^{(nk)}\right] = \frac{d^k}{dx^k}\left[W_{-n}w^{(k)}(x) + qW_{-n+1}w(x)\right] \\
&= W_{-n}w^{(2k)}(x) + qW_{-n+1}w^{(k)}(x) \\
&= W_{-n}\left[-pw^{(k)}(x) + qw(x)\right] + qW_{-n+1}w^{(k)}(x) \\
&= (-pW_{-n} + qW_{-n+1})\, w^{(k)}(x) + qW_{-n}w(x) \\
&= W_{-(n+1)}w^{(k)}(x) + qW_{-n}w(x),
\end{aligned}$$

proving the theorem. □

2.16. Corollary. *Let $k \in \mathbb{N}$ and $f(x)$ be a solution to an odd Fibonacci-like differential equation with period k. If $\{F_n\}_{n=0}^{\infty}$ is the sequence of Fibonacci numbers then,*

$$f^{(nk)}(x) = F_{-n}f^{(k)}(x) + F_{-n+1}f(x), \quad \forall x \in \mathbb{R},\ n \in \mathbb{N}.$$

2.17. Example. Consider the solution $f(x) = e^{(\sqrt[3]{\phi}/2)(1+\sqrt{3}i)x}$ to the differential equation (2.12). By Corollary (2.16), we see that

$$\begin{aligned}
f^{(15)}(x) &= -\frac{1}{2}(11 + 5\sqrt{5})e^{(\sqrt[3]{\phi}/2)(1+\sqrt{3}i)x} \\
&= -5\phi e^{(\sqrt[3]{\phi}/2)(1+\sqrt{3}i)x} + -3e^{(\sqrt[3]{\phi}/2)(1+\sqrt{3}i)x} \\
&= F_{-5}f^{(3)}(x) + F_{-4}f(x).
\end{aligned}$$

2.18. Corollary. *Let $k \in \mathbb{N}$ and $j(x)$ be a solution to an odd Jacobsthal-like differential equation with period k. If $\{J_n\}_{n=0}^{\infty}$ is the sequence of Jacobsthal numbers then,*

$$j^{(nk)}(x) = J_{-n}j^{(k)}(x) + 2J_{-n+1}j(x), \quad \forall x \in \mathbb{R},\ n \in \mathbb{N}.$$

2.19. Example. Consider the solution $j(x) = e^{-\sqrt[5]{2}x}$ to the odd Jacobsthal-like differential equation with period 5 given by

$$j^{(10)}(x) = -j^{(5)}(x) + 2j(x), \quad \forall x \in \mathbb{R}.$$

By Corollary (2.18), we see that

$$j^{(25)}(x) = -32e^{-\sqrt[5]{2}x} = 11(-2e^{-\sqrt[5]{2}x}) + 2(-5)e^{-\sqrt[5]{2}x} = J_{-5}j^{(3)}(x) + 2J_{-4}f(x).$$

2.20. Corollary. *Let $k \in \mathbb{N}$ and $p(x)$ be a solution to an odd Pell-like differential equation with period k. If $\{P_n\}_{n=0}^{\infty}$ is the sequence of Pell numbers then,*

$$p^{(nk)}(x) = P_{-n}p^{(k)}(x) + P_{-n+1}p(x), \quad \forall x \in \mathbb{R},\ n \in \mathbb{N}.$$

2.21. Theorem. *Let $k \in \mathbb{N}$, $p, q \in \mathbb{R}^+$, and consider the SOLRHDE-k defined by (2.1). Then,*

$$\Omega_{W,k}(x) = \sum_{j=1}^{k}\left(c_j e^{r_j x} + \bar{c}_j e^{t_j x}\right), \quad \forall x \in \mathbb{R}, \tag{2.15}$$

where $c_j, \bar{c}_j \in \mathbb{R}$ and, r_j and t_j, for all $j = 1, 2, \ldots, k$ are roots of α and β, respectively, is the general solution of the given homogeneous differential equation.

Proof. Let $\{r_j\}_{j=1}^{k}$ and $\{t_j\}_{j=1}^{k}$ be the set of k^{th} roots of α and β, *i.e.*

$$r_j = |\alpha|^{1/k}\left[\cos\left(\frac{\theta_r + 2\pi j}{k}\right) + i\sin\left(\frac{\theta_r + 2\pi j}{k}\right)\right],$$

and

$$t_j = |\beta|^{1/k}\left[\cos\left(\frac{\theta_t + 2\pi j}{k}\right) + i\sin\left(\frac{\theta_t + 2\pi j}{k}\right)\right],$$

where $j = 1, 2, \ldots, k$, $\theta_r = \arg(\alpha)$ and $\theta_t = \arg(\beta)$. Note that $r_{j's}$ and $t_{j's}$ are all distinct then, $\{e^{r_1x}, e^{r_2x}, \ldots, e^{r_kx}\}$ and $\{e^{t_1x}, e^{t_2x}, \ldots, e^{t_kx}\}$ are linearly independent sets of solutions of the homogeneous linear equation defined in (2.1). Hence, by Proposition (2.5), conclusion follows. □

2.22. Example. Consider the Jacobsthal-like homogeneous differential equation (2.4) with period 3. By Theorem (2.21), we have the general solution

$$\Omega_{J,3}(x) = c_1 e^{\sqrt[3]{2}x} + c_2 e^{-\frac{1}{2}\sqrt[3]{2}(1+\sqrt{3}i)x} + c_3 e^{-\frac{1}{2}\sqrt[3]{2}(1-\sqrt{3}i)x}$$
$$+\ \bar{c}_1 e^{-x} + \bar{c}_2 e^{\frac{1}{2}(1+\sqrt{3}i)x} + \bar{c}_3 e^{\frac{1}{2}(1-\sqrt{3}i)x}.$$

Also, if ϕ and σ are the golden ratio and silver ratio, respectively, then the general solution to a Fibonacci-like and Pell-like homogeneous differential equation are given by

$$\Omega_{F,k}(x) = \sum_{j=1}^{k} c_j \exp\left(\phi^{1/k}\Theta_{2j}x\right) + \sum_{j=1}^{k} \bar{c}_j \exp\left((\phi-1)^{1/k}\Theta_{2j+1}x\right)$$

and

$$\Omega_{P,k}(x) = \sum_{j=1}^{k} c_j \exp\left(\sigma^{1/k}\Theta_{2j}x\right) + \sum_{j=1}^{k} \bar{c}_j \exp\left((2-\sigma)^{1/k}\Theta_{2j+1}x\right),$$

where $\Theta_m = \cos(m\pi/k) + i\sin(m\pi/k)$ and $c_{j's}, \bar{c}_{j's} \in \mathbb{R}$, for all $x \in \mathbb{R}$, respectively.

In the rest of our discussion, we investigate the quotient of solutions of a second-order linear recurrent homogeneous differential equation with period k.

2.23. Theorem. *Let $p, q \in \mathbb{R}^+$ and $k \in \mathbb{N}$ be the period of a SOLRHDE-k defined in* (2.1) *and let $w(x)$ be its general solution. Then, the limit $\lim_{n\to\infty} \frac{w^{((n+1)k)}(x)}{w^{(n)}(x)}$ exists and is given by*

$$\lim_{n\to\infty} \frac{w^{((n+1)k)}(x)}{w^{(n)}(x)} = \alpha \ (\textit{resp. } \beta), \quad \textit{as} \quad x \to \infty \ (\textit{resp. } x \to -\infty), \tag{2.16}$$

where α and β are the roots of the quadratic equation $x^2 - px - q = 0$. Particularly, if $f(x), j(x)$, and $p(x)$ are solutions to a Fibonacci-like, Jacobsthal-like, and Pell-like homogeneous differential equation with period k, respectively, then

$$\lim_{n\to\infty} \frac{f^{((n+1)k)}(x)}{f^{(n)}(x)} = \phi \ (\textit{resp. } 1-\phi), \quad \textit{as} \quad x \to \infty \ (\textit{resp. } x \to -\infty) \tag{2.17}$$

$$\lim_{n\to\infty} \frac{j^{((n+1)k)}(x)}{j^{(n)}(x)} = 2 \ (\textit{resp. } -1), \quad \textit{as} \quad x \to \infty \ (\textit{resp. } x \to -\infty) \tag{2.18}$$

$$\lim_{n\to\infty} \frac{p^{((n+1)k)}(x)}{p^{(n)}(x)} = \sigma \ (\textit{resp. } 1-\sigma), \quad \textit{as} \quad x \to \infty \ (\textit{resp. } x \to -\infty). \tag{2.19}$$

Proof. Let $k, n \in \mathbb{N}$, $p, q \in \mathbb{R}^+$, and consider the quotient $Q(x) := \frac{\omega^{(k)}(x)}{\omega(x)}$, where $\omega(x) = w^{(nk)}(x)$ satisfies a SOLRHDE-k. We suppose $x \to \infty$. The case when $x \to -\infty$ can be proven in a similar fashion.

We consider two cases: (i) $Q(x) < 0$, and (ii) $Q(x) > 0$.

CASE 1. Suppose that $Q(x) < 0$. Hence, we can assume without loss of generality (WLOG) that $\omega(x) > 0$ and $\omega^{(k)}(x) < 0$. By assumption, $w(x)$ satisifes (2.1), so we have

$$\begin{aligned}
w^{(2k)}(x) &= -pw^{(k)}(x) + qw(x),\\
w^{(3k)}(x) &= pw^{(2k)}(x) - qw^{(k)}(x) = p(-pw^{(k)}(x) + qw(x)) - qw^{(k)}(x)\\
&= -(p^2+q)w^{(k)}(x) + pqw(x),\\
w^{(4k)}(x) &= pw^{(3k)}(x) + qw^{(2k)}(x)\\
&= p(-(p^2+q)w^{(k)}(x) + pqw(x)) + q(-pw^{(k)}(x) + qw(x))\\
&= -(p^3+2pq)w^{(2k)}(x) + q(p^2+q)w^{(k)}(x),\\
&\vdots\\
w^{(nk)}(x) &= -W_n w^{(k)}(x) + qW_{n-1}w(x), \quad \forall n \in \mathbb{N},
\end{aligned}$$

where W_n is the number sequence satisfying equation (2.3). We let $\omega(x) = w^{(nk)}(x)$. Hence, by Proposition (2.3), $\omega(x)$ is also a solution to (2.1). It follows that

$$\begin{aligned}
\frac{\omega^{(k)}(x)}{\omega(x)} &= \frac{1}{w^{(nk)}(x)}\frac{d^k}{dx^k}\left(w^{(nk)}(x)\right) = \frac{-W_{n+1}w^{(k)}(x) + qW_n w(x)}{-W_n w^{(k)}(x) + qW_{n-1}w(x)}\\
&= \frac{-w^{(k)}(x)\frac{W_{n+1}}{W_n} + qw(x)}{-w^{(k)}(x) + qw(x)\frac{W_{n-1}}{W_n}}.
\end{aligned}$$

So we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{\omega^{(k)}(x)}{\omega(x)} &= \lim_{n\to\infty}\frac{-w^{(k)}(x)\frac{W_{n+1}}{W_n} + qw(x)}{-w^{(k)}(x) + qw(x)\frac{W_{n-1}}{W_n}}\\
&= \frac{-w^{(k)}(x)\left(\lim_{n\to\infty}\frac{W_{n+1}}{W_n}\right) + qw(x)}{-w^{(k)}(x) + qw(x)\left(\lim_{n\to\infty}\frac{W_{n-1}}{W_n}\right)}.
\end{aligned}$$

Since $\beta = (p - \sqrt{p^2+4q})/2 \in (-1, 0)$, then $\lim_{n\to\infty}\beta^n = 0$. Thus,

$$\lim_{n\to\infty}\frac{\omega^{(k)}(x)}{\omega(x)} = \frac{-\alpha w^{(k)}(x) + qw(x)}{-w^{(k)}(x) + \alpha^{-1}qw(x)} = \alpha < \infty,$$

because $\lim_{n\to\infty}\frac{W_{n+1}}{W_n} = \lim_{n\to\infty}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha^n-\beta^n} = \alpha$ and $\alpha > \beta$.

CASE 2. Suppose (WLOG) that $\omega(x)$ and $\omega^{(k)}(x)$ are both positive. By Proposition (2.3), $\omega(x) = w^{(nk)}(x)$ is also a solution to (2.1). Hence,

$$\begin{aligned}
\lim_{n\to\infty}\frac{\omega^{(k)}(x)}{\omega(x)} &= \lim_{n\to\infty}\frac{w^{((n+1)k)}(x)}{w^{(nk)}(x)} = \lim_{n\to\infty}\frac{W_{n+1}w^{(k)}(x) + qW_n w(x)}{W_n w^{(k)}(x) + qW_{n-1}w(x)}\\
&= \lim_{n\to\infty}\frac{w^{(k)}(x)\frac{W_{n+1}}{W_n} + qw(x)}{w^{(k)}(x) + qw(x)\frac{W_{n-1}}{W_n}}\\
&= \frac{w^{(k)}(x)\left(\lim_{n\to\infty}\frac{W_{n+1}}{W_n}\right) + qw(x)}{w^{(k)}(x) + qw(x)\left(\lim_{n\to\infty}\frac{W_{n-1}}{W_n}\right)}\\
&= \alpha.
\end{aligned}$$

By letting $(p, q) = (1, 1), (1, 2), (2, 1)$, we obtain equations (2.17), (2.18), and (2.19), respectively. This completes the proof of the theorem. $\square$

We also have the following theorem for oSOLRHDE-k.

2.24. Theorem. *Let $p, q \in \mathbb{R}^+$ and $k \in \mathbb{N}$ be the period of an oSOLRHDE-k defined by (2.11) and let $w(x)$ be its solutions. Then, the limit $\lim_{n\to\infty} \frac{w^{((n+1)k)}(x)}{w^{(n)}(x)}$ exists and is given by*

$$\lim_{n\to\infty} \frac{w^{((n+1)k)}(x)}{w^{(n)}(x)} = -\beta \ (\textit{resp.}\ -\alpha), \quad \textit{as} \quad x \to \infty \ (\textit{resp.}\ x \to -\infty), \tag{2.20}$$

where α and β are the roots of the quadratic equation $x^2 - px - q = 0$. Particularly, if $f(x), j(x)$, and $p(x)$ are solutions to an odd Fibonacci-like, odd Jacobsthal-like, and odd Pell-like homogeneous differential equation with period k, respectively, then

$$\lim_{n\to\infty} \frac{f^{((n+1)k)}(x)}{f^{(n)}(x)} = -(1-\phi) \ (\textit{resp.}\ -\phi), \quad \textit{as} \quad x \to \infty \ (\textit{resp.}\ x \to -\infty)$$

$$\lim_{n\to\infty} \frac{j^{((n+1)k)}(x)}{j^{(n)}(x)} = 1 \ (\textit{resp.}\ -2), \quad \textit{as} \quad x \to \infty \ (\textit{resp.}\ x \to -\infty)$$

$$\lim_{n\to\infty} \frac{p^{((n+1)k)}(x)}{p^{(n)}(x)} = -(1-\sigma) \ (\textit{resp.}\ -\sigma), \quad \textit{as} \quad x \to \infty \ (\textit{resp.}\ x \to -\infty).$$

The proof of the above theorem follows the same argument as in the proof of Theorem (2.23), so we omit it.

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