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On Lagrangian submersions

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Abstract

In this paper, we study Riemannian, anti-invariant Riemannian and Lagrangian submersions. We prove that the horizontal distribution of a Lagrangian submersion from a Kählerian manifold is integrable. We also give some applications of this result. Moreover, we investigate the effect of the submersion to the geometry of its total manifold and its fibers.

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1. Introduction

The theory of Riemannian submersions was initiated by O'Neill [11]. In [18], the Riemannian submersions were considered between almost Hermitian manifolds by Watson under the name of almost Hermitian submersions. In this case, the Riemannian submersion is also an almost complex mapping and consequently the vertical and horizontal distribution are invariant with respect to the almost complex structure of the total manifold of the submersion. Afterwards, almost Hermitian submersions have been actively studied between different kind of subclasses of almost Hermitian manifolds, for example, see [5]. We note that almost Hermitian submersions have been extended to different kind of subclasses of almost contact manifolds, for example, see [14]. Most of the studies related to Riemannian or almost Hermitian submersions can be found in the book [4]. The study of anti-invariant Riemannian submersions from almost Hermitian manifolds were initiated by Şahin [15]. In this case, the fibres are anti-invariant with respect to the almost complex structure of the total manifold. A Lagrangian submersion is a special case of an anti-invariant Riemannian submersion such that the almost complex structure of the total manifold reverses the vertical and horizontal distributions. In this paper, we consider Riemannian, anti-invariant Riemannian and Lagrangian submersions. We will focus Lagrangian submersions from a Kählerian manifold onto a Riemannian manifold

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and prove that the horizontal distribution of such a submersion is integrable and totally geodesic. Using this result we obtain that such a submersion is a totally geodesic map if and only if it has totally geodesic fibers. We also obtained other applications of the result. In the last section, we show that non-existence of a Lagrangian submersion with totally geodesic fibers from a non-flat Kählerian manifold. We also proved that if the fibers of a Lagrangian submersion are totally umbilical, then the fibers are minimal.

2. Riemannian submersions

In this section, we give necessary background for Riemannian submersions.

Let (M, g) and (N, g_N) be Riemannian manifolds, where $dim(M) > dim(N)$. A surjective mapping $\pi : (M, g) \to (N, g_N)$ is called a *Riemannian submersion* [11] if:

(S1) π has maximal rank, and

(S2) π_*, restricted to $(ker\pi_*)^\perp$, is a linear isometry.

In this case, for each $q \in N$, $\pi^{-1}(q)$ is a k-dimensional submanifold of M and called *fiber*, where $k = dim(M) - dim(N)$. A vector field on M is called *vertical* (resp. *horizontal*) if it is always tangent (resp. orthogonal) to fibers. A vector field X on M is called *basic* if X is horizontal and π-related to a vector field X_* on N, i.e., $\pi_* X_p = X_{*\pi(p)}$ for all $p \in M$. As usual, we denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $ker\pi_*$ and the horizontal distribution $(ker\pi_*)^\perp$, respectively. The geometry of Riemannian submersions is characterized by O'Neill's tensors T and A, defined as follows:

$$\mathrm{T}_E F = \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F, \tag{2.1}$$

$$\mathrm{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F \tag{2.2}$$

for any vector fields E and F on M, where ∇ is the Levi-Civita connection of g_M. It is easy to see that T_E and A_E are skew-symmetric operators on the tangent bundle of M reversing the vertical and the horizontal distributions. We summarize the properties of the tensor fields T and A. Let V, W be vertical and X, Y be horizontal vector fields on M, then we have

$$\mathrm{T}_V W = \mathrm{T}_W V, \tag{2.3}$$

$$\mathrm{A}_X Y = -\mathrm{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y]. \tag{2.4}$$

On the other hand, from (2.1) and (2.2), we obtain

$$\nabla_V W = \mathrm{T}_V W + \mathcal{V}\nabla_V W, \tag{2.5}$$

$$\nabla_V X = \mathrm{T}_V X + \mathcal{H}\nabla_V X, \tag{2.6}$$

$$\nabla_X V = \mathrm{A}_X V + \mathcal{V}\nabla_X V, \tag{2.7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathrm{A}_X Y, \tag{2.8}$$

and if X is basic, then $\mathcal{H}\nabla_V X = \mathrm{A}_X V$. It is not difficult to observe that T acts on the fibers as the second fundamental form while A acts on the horizontal distribution and measures of the obstruction to the integrability of this distribution. For details on the Riemannian submersions, we refer to O'Neill's paper [11] and to the book [4].

3. Anti-invariant Riemannian submersions

A smooth manifold M is called almost Hermitian [19] if its tangent bundle has an almost complex structure J and a Riemannian metric g such that

$$(3.1)\qquad g(E,F)=g(JE,JF)$$

for any vector fields E and F on M. Let M be a $2m$-dimensional almost Hermitian manifold with Hermitian metric g and almost complex structure J, and N be a Riemannian manifold with Riemannian metric g_N. Suppose that there exists a Riemannian submersion $\pi : M \to N$ such that $ker\pi_*$ is anti-invariant with respect to J, i.e., $J(ker\pi_*) \subseteq (ker\pi_*)^{\perp}$. Then the Riemannian submersion π is called an *anti-invariant Riemannian submersion*. For the details, see [15].

There are some other recent paper which involve other structures such as almost product [6], almost contact [9], Sasakian [7] and cosymplectic [8]. In any cases, the definition of anti-invariant Riemannian submersion is the same as the above definition. Besides there are many other notions related with that of anti-invariant Riemannian submersion, such as slant submersion [16] and semi-invariant submersion [17]. The key of this definitions consists on considering the fibres as submanifolds of the almost Hermitian manifold M having the corresponding property. Because of that, we may consider that the following names are more convenient: totally real, instead of anti-invariant, but semi-invariant (cfr. [17]) of CR-submersion (cfr. e.g. [10]) because definition of a CR-submersion depends on certain CR-submanifold of the total manifold, instead of the fact the fibres are CR-submanifolds. As one can see, names are quite complex in this field.

An almost Hermitian manifold M is called a *Kählerian manifold* if

$$(3.2)\qquad (\nabla_E J)F=0$$

for any vector fields E and F on M, where ∇ is the Levi-Civita connection on M. Let (M,g,J) be a Kählerian manifold. The Riemannian curvature tensor [19] of (M,g,J) is defined by $\mathrm{R}(E,F)G=\nabla_{[E,F]}G-[\nabla_E,\nabla_F]G$ for vector fields E,F and G on M. We put $\mathrm{R}(E,F,G,K)=g(\mathrm{R}(E,F)G,K)$ where K is a vector field on M. The *holomorphic sectional curvature* [19] of M is defined for any unit vector field E tangent to M via

$$(3.3)\qquad \mathrm{H}(E)=\mathrm{R}(E,JE,E,JE).$$

We note that a Kählerian manifold with vanishing holomorphic sectional curvature is flat [19]. The manifold M is called a *complex space form* if it is of constant holomorphic sectional curvature. We denote by $M(c)$ a complex space form of constant holomorphic sectional curvature c. Then the Riemannian curvature tensor R of $M(c)$ is given by

$$(3.4)\qquad \begin{aligned}\mathrm{R}(E,F)G=\tfrac{c}{4}\{&g(F,G)E-g(E,G)F+g(JF,G)JE\\&-g(JE,G)JF+2g(E,JF)JG\}\end{aligned}$$

for any vector fields E,F and G on $M(c)$ [19]. In this point, we give the following proposition.

3.1. Proposition. *Let $\pi : M(c) \to N$ be a Riemannian submersion from a complex space form $M(c)$ with $c \neq 0$ onto a Riemannian manifold N. Then the fibers of $M(c)$ are invariant or anti-invariant with respect to the almost complex structure J of $M(c)$ if and only if*

$$(3.5)\qquad g((\nabla_U \mathrm{T})_V W, X)=g((\nabla_V \mathrm{T})_U W, X),$$

where U,V and W are vertical vector fields and X is a horizontal vector field on $M(c)$.

Proof. Let U, V and W be vertical vector fields and X be a horizontal vector field on $M(c)$. Then from (3.4), we have

$$\begin{aligned}\mathrm{R}(U,V)W = \tfrac{c}{4}\{&g(V,W)U - g(U,W)V + g(JV,W)JU \\ &-g(JU,W)JV + 2g(U,JV)JW\}\end{aligned} \quad (3.6)$$

From (14), we see that $\mathrm{R}(U,V)W$ is vertical, if the fibers are invariant or anti-invariant with respect to the almost complex structure J of $M(c)$. So, we get easily, $\mathrm{R}(U,V,W,X) = 0$. Therefore, (3.5) follows from the following O'Neill curvature formula {1} [11]:

$$\mathrm{R}(U,V,W,X) = g((\nabla_V \mathrm{T})_U W, X) - g((\nabla_U \mathrm{T})_V W, X).$$

Conversely, assume that (3.5) holds. Then for U, V and W, it is not difficult to see that $\mathrm{R}(U,V)W$ is vertical from the above O'Neill curvature formula. If we put $W = U$ in (3.6), then we have

$$\mathrm{R}(U,V)U = \frac{c}{4}\{g(V,U)U - g(U,U)V + g(U,JV)JU\}. \quad (3.7)$$

Thus, we see that $g(U,JV)JU$ is vertical from (15), since $\mathrm{R}(U,V)U$ is vertical. So, we conclude that either JU is vertical or $g(U,JV) = 0$. It means that either $J(ker\pi_*) \subseteq ker\pi_*$ or $J(ker\pi_*) \subseteq (ker\pi_*)^{\perp}$, i.e., either the fibers are invariant or anti-invariant with respect to the almost complex structure J of $M(c)$. □

3.2. Corollary. *Let $\pi : M(c) \to N$ be an anti-invariant Riemannian submersion from a complex space form $M(c)$ with $c \neq 0$ onto a Riemannian manifold N. Then the equality (3.5) holds.*

4. Lagrangian submersions

Let M be a $2m$-dimensional almost Hermitian manifold with Hermitian metric g and almost complex structure J, and N be a Riemannian manifold with Riemannian metric g_N and let $\pi : M \to N$ be an anti-invariant Riemannian submersion. Then we call π a *Lagrangian Riemannian submersion* or briefly, a *Lagrangian submersion*, if $dim(ker\pi_*) = dim((ker\pi_*)^{\perp})$. In this case, the almost complex structure J of M reverses the vertical and the horizontal distributions, i.e., $J(ker\pi_*) = (ker\pi_*)^{\perp}$ and $J((ker\pi_*)^{\perp}) = ker\pi_*$.

In Symplectic Geometry, a Lagrangian submersion $\pi : (M,\omega) \to N$ from a symplectic manifold onto a manifold is a submersion having the fibres Lagrangian submanifolds (see, e.g. [1]), i.e., $\omega|_{\pi^{-1}(q)} = 0$.

An almost Hermitian structure (J,g) defines an almost symplectic structure $\omega(X,Y) = g(JX,Y)$, and then we can consider compare both definitions. It is easily shown that they coincide:

4.1. Lemma. *Let $\pi : (M,J,g) \to N$ be a submersion from an almost Hermitian manifold onto a manifold. Then the following conditions are equivalent:*
(1) The fibres of π are Lagrangian submanifolds.
(2) $J(ker\pi_*) = (ker\pi_*)^{\perp}$.
Moreover, the horizontal distribution $(ker\pi_*)^{\perp}$ is also Lagrangian.

Proof. (1) ⇒ (2). Let X and Y be vertical, that is; $X, Y \in ker\pi_*$. Then $g(JX,Y) = \omega(X,Y) = 0$, thus proving (2). Reversing the reasoning, one has the other implication. □

In order to have a Lagrangian submersion $\pi : (M, J, g) \to N$ dimensions must be related in the following way: $dim(M) = 2dim(N)$. The most natural examples of manifolds having this relation are given by the tangent (resp. cotangent) bundle of $M = TN \to N$ (resp. $M = T^*N \to N$). In the seminal paper [2], Dombrowski introduces the almost complex structure J on the tangent bundle TN of a manifold N having a linear connection, which is given by the conditions $J(X^H) = X^V; \quad J(X^V) = -X^H$, H and V being the horizontal and vertical lifts. On the other hand, Sasaki [13] introduced the diagonal lift g^D, or Sasaki metric, over the tangent bundle of a Riemannian manifold (N, g), given by $g(X^H, Y^H) = g(X^V, Y^V) = g(X, Y); \quad g(X^H, Y^V) = 0$. Thus, the tangent bundle (TN, J, g^D) of a Riemannian manifold (N, g) is an almost Hermitian manifold. Then one easily obtains:

4.2. Lemma. *With the above notation, $\pi : (TN, J, g^D) \to (N, g)$ is a Lagrangian submersion.*

We want to emphasize that the same considerations can be done about the cotangent bundle.

Let M be a Kählerian manifold with Hermitian metric g and almost complex structure J, and N be a Riemannian manifold with Riemannian metric g_N. Now we examine how the Kählerian structure on M places restrictions on the tensor fields T and A of a Lagrangian submersion $\pi : M \to N$.

4.3. Lemma. *Let $\pi : M \to N$ be a Lagrangian submersion from a Kählerian manifold M onto a Riemannian manifold N. Then we have*

$$\textbf{\textit{a)}} \quad \mathrm{T}_V JE = J\mathrm{T}_V E \qquad\qquad \textbf{\textit{b)}} \quad \mathrm{A}_X JE = J\mathrm{A}_X E$$

where V is a vertical vector field, X is a horizontal vector field, and E is a vector field on M.

Proof. Using (2.5)-(2.8), we obtain easily both assertions from (3.2). □

We remark that Lemma 4.3 was proved partially in [15].

4.4. Corollary. *Let $\pi : M \to N$ be a Lagrangian submersion from a Kählerian manifold M onto a Riemannian manifold N. Then we have*

$$\mathrm{A}_X JY = -\mathrm{A}_Y JX$$

where X and Y are any horizontal vector fields on M.

Proof. Let X and Y be any horizontal vector fields on M, from Lemma 4.3-b), we have $\mathrm{A}_X JY = J\mathrm{A}_X Y$. Since the tensor A has the alternation property, we get $J\mathrm{A}_X Y = -J\mathrm{A}_Y X = -\mathrm{A}_Y JX$. □

4.1. The Horizontal Distribution. We now prove that the horizontal distribution $(ker\pi_*)^\perp$ is integrable and totally geodesic. It is well-known that the vertical distribution $ker\pi_*$ is always integrable.

4.5. Theorem. *Let $\pi : M \to N$ be a Lagrangian submersion from a Kählerian manifold M onto a Riemannian manifold N. Then the horizontal distribution $(ker\pi_*)^\perp$ is integrable and totally geodesic.*

Proof. Let X and Y be any horizontal vector fields on M, since $\mathrm{A}_X Y = \frac{1}{2}\mathcal{V}[X, Y]$, it is sufficient to show that $\mathrm{A}_X = 0$. If Z is a horizontal vector field on M, then using (2.4), (2.8), (3.1), (3.2) and Corollary 4.4, we have

$$
\begin{aligned}
&g(\mathrm{A}_X JY, Z) = -g(\mathrm{A}_Y JX, Z) = -g(\nabla_Y JX, Z) = -g(J\nabla_Y X, Z)\\
&= g(\nabla_Y X, JZ) = -g(\mathrm{A}_X Y, JZ) = g(\mathrm{A}_X JZ, Y) = -g(\mathrm{A}_Z JX, Y)\\
&= g(\mathrm{A}_Z Y, JX) = -g(\mathrm{A}_Y Z, JX) = g(\mathrm{A}_Y JX, Z) = -g(\mathrm{A}_X JY, Z).
\end{aligned}
$$

Therefore $\mathrm{A}_X JY = 0$. By Proposition 2.7-(e) ([18]), we get $\mathrm{A}_X = 0$. □

4.2. Applications. In this subsection, we give some applications of Theorem 4.5.

4.6. Corollary. *Let $\pi : M \to N$ be a Lagrangian submersion from a Kählerian manifold M onto a Riemannian manifold N. Then we have*

$$(\nabla\pi_*)(X, JY) = (\nabla\pi_*)(JX, Y) = 0,$$

where X and Y are any horizontal vector fields on M, and $\nabla\pi_*$ is the second fundamental form [15] *of π.*

Proof. It follows immediately from our main result Theorem 4.5, Corollary 3.1([15]) and Corollary 3.2([15]). □

It is well-known that a differential map π between two Riemannian manifolds is called totally geodesic if $\nabla\pi_* = 0$. Now we give a necessary and sufficient condition for a Lagrangian submersion to be a totally geodesic map.

4.7. Theorem. *Let $\pi : M \to N$ be a Lagrangian submersion from a Kählerian manifold M onto a Riemannian manifold N. Then π is a totally geodesic map if and only if it has totally geodesic fibers.*

Proof. Let V and W be any vertical vector fields on M, if $\mathrm{T}_V JW = 0$, then from Lemma 4.3, we get $\mathrm{T}_V W = 0$. On the other hand, from Proposition 2.7-(d)([18]), it follows that $\mathrm{T}_V = 0$, which means that the Lagrangian submersion π has totally geodesic fibers. Thus the assertion follows from Theorem 4.5 and Theorem 3.4([15]). □

Now, we simply decompose theorems given in [15]. First, we recall the following facts given in [12].

Let $B = M \times N$ be a Riemannian manifold with metric g. Assume that the canonical foliations $\mathcal{D}_M$ and $\mathcal{D}_N$ intersect perpendicularly everywhere. Then g is the metric tensor of

(i) a twisted product $M \times_f N$ if and only if $\mathcal{D}_M$ is a totally geodesic foliation and $\mathcal{D}_N$ is a totally umbilical foliation,

(ii) a usual product of Riemannian manifolds if and only if $\mathcal{D}_M$ and $\mathcal{D}_N$ are totally geodesic foliations.

Thus, from Theorem 4.5, Theorem 4.2([15]) and Theorem 4.3([15]), we have the following result.

4.8. Theorem. *Let $\pi : M \to N$ be a Lagrangian submersion from a Kählerian manifold M onto a Riemannian manifold N. Then*
a) *M is a locally twisted product manifold of the form $M_{(ker\pi_*)^\perp} \times_f N_{ker\pi_*}$ if and only if π has totally umbilical fibers,*
b) *M is a locally product of manifold if and only if π has totally geodesic fibers.*

5. The Geometry of Total Manifold and Fibers

In this section, we prove some characterization results for a Lagrangian submersion from a Kählerian manifold onto a Riemannian manifold.

Let M be a Kählerian manifold with Hermitian metric g and almost complex structure J and let $\pi : M \to N$ be a Lagrangian submersion from the manifold M onto a Riemannian manifold N. Since $\mathrm{A} \equiv 0$, the O'Neill's curvature formula {2} [11] reduces to

$$\mathrm{R}(X, V, Y, W) = g((\nabla_X \mathrm{T})_V W, Y) - g(\mathrm{T}_V X, \mathrm{T}_W Y), \tag{5.1}$$

where V and W are vertical, and X and Y are horizontal vector fields on M.

5.1. Theorem. *Let $\pi : M \to N$ be a Lagrangian submersion from a Kählerian manifold M onto a Riemannian manifold N. Then the holomorphic sectional curvature* H *of M satisfies*

a) $\mathrm{H}(X) = g_M((\nabla_X \mathrm{T})_{JX} JX, X) - \|\mathrm{T}_{JX} X\|^2,$

b) $\mathrm{H}(V) = g_M((\nabla_{JV} \mathrm{T})_V V, JV) - \|\mathrm{T}_V V\|^2,$

where X is a unit horizontal and V is a unit vertical vector field on M.

Proof. Both assertion **a)** and assertion **b)** follow easily from (3.3), (5.1), Lemma 4.3 and (3.1). □

We know from Proposition 1.2([3]) that if T is parallel, i.e., $\nabla_E \mathrm{T} = 0$, for any vector field E on M, then $\mathrm{T} = 0$. Therefore, by Theorem 5.1 we obtain the following result.

5.2. Theorem. *Let $\pi : M \to N$ be a Lagrangian submersion from a Kählerian manifold M onto a Riemannian manifold N. If the tensor field* T *is parallel, then the holomorphic sectional curvature* H *of M vanishes. Namely, M is flat.*

We remark that Theorem 5.2 describes the geometry of the total manifold of the Lagrangian submersion studied above. On the other hand, if the tensor T vanishes, then the fibers are totally geodesic. Thus, from Theorem 5.1 and Theorem 5.2, we have the following result.

5.3. Corollary. *Let M be a non-flat Kählerian manifold. Then there is no Lagrangian submersion π with totally geodesic fibers from M onto a Riemannian manifold N.*

Now, we recall that any fiber of a Riemannian submersion $\pi : (M, g) \to (N, g_N)$ is called *totally umbilical* if

$$\mathrm{T}_U V = g(U, V)\eta \tag{5.2}$$

for any $U, V \in ker\pi_*$, where η is the mean curvature vector field of the fiber in M. The fiber is called *minimal*, if $\eta = 0$, identically [4].

5.4. Proposition. *Let $\pi : M \to N$ be a Lagrangian submersion from a Kählerian manifold M onto a Riemannian manifold N. If the fibers of M are totaly umbilical, then either $ker\pi_* = \{0\}$ or 1-dimensional or the mean curvature vector field η vanishes, i.e., the fibers are minimal.*

Proof. If $ker\pi_* = \{0\}$ or $Dim(ker\pi_*) = 1$, then the conclusion is obvious. If $Dim(ker\pi_*) \geq 2$, then we can choose $U, V \in ker\pi_*$, such that $g(U, V) = 0$ and $\|U\| = 1$. By Lemma 4.3-(a) and (5.2), we have

$g(\eta, JV) = g(\mathrm{T}_U U, JV) = -g(\mathrm{T}_U JV, U) = -g(J\mathrm{T}_U V, U) = 0$. Hence, it follows that $\eta = 0$. □

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