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Laguerre spectral method with different basis functions

Hadi DARVISHI[1], Ali BADERAN[1,*], Morteza MOHAMMAD NEZHAD KIASARY[1]

[1]*Department of Applied Mathematics, Iran University of Science and*

Technology, P.O.Box, 16844-13114, Narmak, Tehran, Iran

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Abstract. In this paper a model has been solved using Laguerre spectral method with two different basis functions. Unlike classical method, all the boundary conditions will be satisfied using linear combination of Laguerre functions as basis functions. Numerical results show that using a new basis has less error and also leads to three diagonal matrix that has a conditions number which is smaller than that of the classical method.

Keywords: Linear differential equations, Basis functions, Combination of Laguerre functions, Laguerre spectral method

1. INTRODUCTION

Nowadays the Laguerre spectral method has an extensive use in solving ODEs and PDEs in unbounded domains. Such method has been used by different mathematicians and authors in var ious math fields to solve linear and nonlinear differential equations associated with Dirichlet bo undary conditions[1-16]. In this paper, an ordinary differential equation model with Neumann boundary conditions has been considered. Wang Zhong in [17] has solved weak formulation of this model using combination of Laguerre functions as basis functions. We approximate strong form instead of weak form. In this paper we solve this model with two differ ent basis functions and consequently compare the different results. In section2 we introduce the Laguerre polynomials and functions. In section3 we present the Galerkin approximation in stron g form of differential equations. In section4 we solve the model using classical approach (CLM). In section5 we solve this model using Laguerre spectral method with a new basis functions (LM). In the final section numerical results indicate that using combination of Laguerre functions has less error.

2. LAGUERRE POLYNOMIALS AND FUNCTIONS

Let $\Omega = (0, \infty)$ and $\psi(x)$ be a certain weight function on Ω in the usual sense. We define the weighted space $L^2_\psi(\Omega)$ as usual with inner product $(u, v)_\psi$ and the norm$\|u\|_\psi$. We omit the subscript ψ in the notation whenever $\psi(x) = 1$. The Laguerre polynomials define by

$$\mathcal{L}_l(x) = \frac{1}{l!} e^x \partial_x^l \left(x^l e^{-x}\right), \quad l = 0, 1, 2, \ldots$$

form a complete $L^2_w(\Omega)$ orthogonal system with respect to the weight function $w(x) = e^{-x}$. We have

$$(\mathcal{L}_l, \mathcal{L}_m)_w = \delta_{lm} \qquad (2.1)$$

*Corresponding author. *Email address: a_baderan@yahoo.com*

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Where δ_{lm} is the Kronecher symbol. Thus, for any $v \in L_w^2(\Omega)$ we can write

$$v(x) = \sum_{l=0}^{\infty} a_l \mathcal{L}_l(x), \quad a_l = \int_0^{\infty} v(x)\mathcal{L}_l(x).$$

Due to (7.6) in [18] we have

$$\mathcal{L}_l(0) = 1. \tag{2.2}$$

To design a proper spectral method for solving Neumann problems, we use the Laguerre functions as follows (cf. [19])

$$\widehat{\mathcal{L}}_l(x) = e^{-\frac{1}{2}x}\mathcal{L}_l(x), \quad l = 0, 1, 2, \dots$$

as pointed out in [19], the set of $\widehat{\mathcal{L}}_l(x)$forms a complete $L^2(\Omega)$ orthogonal system.

3. GALERKIN SPECTRAL METHOD

We first consider an ordinary differential equation;

$$\begin{cases} -\partial_x^2 u(x) + \lambda u(x) = f(x), & x \in \Omega, \lambda \geq 0 \\ \lim\limits_{x\to\infty} u(x) = \lim\limits_{x\to\infty} \partial_x u(x) = \partial_x u(0) = 0. \end{cases} \tag{3.1}$$

In the Galerkin approach we consider a space with finite dimension like

$\emptyset_N = < \emptyset_0(x), \emptyset_1(x), \dots, \emptyset_{N-1}(x) >$ as approximation space. Approximate solution

would be

$$u_N(x) = \sum_{l=0}^{N-1} a_l \emptyset_l(x). \tag{3.2}$$

In this approach in order to obtain coefficients a_l equation (3.1) should be project in approximation space. We have

$$-\partial_x^2 u_N(x) + \lambda u_N(x) = f(x). \tag{3.3}$$

Now by inner product of basic functions into (3.3) coefficients a_l will be yielded. Hence,

$$\left(-\partial_x^2 u_N(x), \emptyset_l(x)\right) + \lambda(u_N(x), \emptyset_l(x)) = \left(f(x), \emptyset_l(x)\right), \quad 0 \leq l \leq N-1. \tag{3.4}$$

4. LAGUERRE SPECTRAL METHOD WITH LAGUERRE FUNCTIONS

In this section we use Laguerre functions as basic functions and we call that classical method. Since$\partial_x \widehat{\mathcal{L}}_l(0) \neq 0$, one of the Neumann boundary conditions is not fully satisfied. Now

by taking $\emptyset_l(x) = \tilde{\mathcal{L}}_l(x)$ for $0 \leq l \leq N-1$, in (3.4) we find that (3.4) is equivalent to following system;

$$(A + \lambda B)\vec{X} = \vec{F} \qquad (4.1)$$

Where the matrices are given by

$$A = (a_l), \qquad B = (I)_{N-1}, \quad 0 \leq l \leq N-1,$$

with the rows

$$a_{l',l} = [-(\partial_x^2 \emptyset_0 . \emptyset_l), \quad -(\partial_x^2 \emptyset_1 . \emptyset_l), \qquad \ldots, -(\partial_x^2 \emptyset_{N-1} . \emptyset_l)]$$

and the vectors are define by

$$\vec{X} = (a_0, a_1, \ldots, a_{N-1})^T, \qquad \vec{F} = (f_0, f_1, \ldots, f_{N-1})^T$$

$$f_{l'} = (f, \emptyset_{l'}), \qquad 0 \leq l' \leq N-1.$$

The numerical solution is

$$u_N(x) = \sum_{l=0}^{N-1} a_l \emptyset_l(x).$$

5. LAGUERRE SPECTRAL METHOD WITH COMBINATION OF LAGUERRE FUNCTIONS

In this section in order satisfying all the boundary conditions we use linear combination of Laguerre functions as basis functions. Therefore the basis functions are assumed as

$$\emptyset_l(x) = \tilde{\mathcal{L}}_l(x) + \alpha_l \bar{\mathcal{L}}_{l+1}(x), \quad 0 \leq l \leq N-1.$$

Now we select α_l in a way that we would have $\partial_x \emptyset_l(0) = 0$, hence

$$\partial_x \tilde{\mathcal{L}}_l(0) + \alpha_l \partial_x \bar{\mathcal{L}}_{l+1}(0) = 0.$$

Using (2.2) we have, $\alpha_l = -\frac{2l+1}{2l+3}$. By taking $\emptyset_l(x) = \hat{\mathcal{L}}_l(x) - \frac{2l+1}{2l+3}\widehat{\mathcal{L}_{l+1}}(x)$ in(3.4), we find that (3.4) is equivalent to the following system:

$$(A + \lambda B)\vec{X} = \vec{F}. \qquad (5.1)$$

Where the matrices are given by

$$A = (a_{l',l}), \qquad B = (b_{l',l}), \quad 0 \leq l', l \leq N-1,$$

with the entries

$$a_{l',l} = \int_0^\infty \partial_x \emptyset_{l'}(x) \partial_x \emptyset_l(x) dx, \; b_{l',l} = \int_0^\infty \emptyset_{l'}(x) \emptyset_l(x) dx.$$

Due to (18) and (21) of [14], we obtain that for $0 \leq l', l \leq N-1$,

$$a_{l',l} = \begin{cases} \dfrac{2l'+1}{4(2l'+3)}, & l' = l-1, \\ \dfrac{4l'^2+8l'+1}{2(2l'+3)^2}, & l' = l, \\ \dfrac{2l+1}{4(2l+3)}, & l' = l+1, \end{cases} \quad , otherwise \quad a_{l',l} = 0,$$

$$b_{l',l} = \begin{cases} -\dfrac{2l'+1}{2l'+3}, & l' = l-1, \\ 1 + \dfrac{(2l')^2}{(2l'+3)^2}, & l' = l, \\ -\dfrac{2l+1}{2l+3}, & l' = l+1, \end{cases} \quad , otherwise \ = 0.$$

Convergence of this system solution has been proved in [17]

6. NUMERICAL RESULTS

We denote by E_N the $L^2(\Omega)$ error. Let $\lambda = 1.5$ in(3. 1). And take test function

$$u(x) = (1 + x + x^2)e^{-x}\cos x.$$

Table 1. The condition number

N	LM	CLM
15	3.997	1.040e2
25	4.189	2.747e2
35	4.269	5.260e2
45	4.313	8.583e2

Table 2. The logE_N in L^2

N	LM	CLM
15	−2	−1
25	−3.3	−0.2
35	−5	0.3
45	−6.2	1

In table 1 condition number of system (4.1)
and (5.1) have been compared. As you see in classical approach condition number of the syste m is too big and increasing amount of N, makes it even much bigger which leads to more errors in solving system. In system (5.1) choosing proper basis functions has made a reduction in condition number of the system (5.1) and increasing amount of
N, make no marvelous increase in that. In table 2 logE_N

of both approaches have been compared. In classical approach increasing N, increase error which has two main reasons:
1) Improper basis functions
2) Big system condition number

By choosing proper basis functions in Laguerre approach, we have overcome above problems. Comparing two approaches shows that classical approach is not a good one to solve the model. Although approximation space in both approaches is too close bu t using linear combination of Laguerre functions has efficient numerical results which gives acc eptable solutions, unlike classical approach.

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