

## Euler-Lagrange Equation

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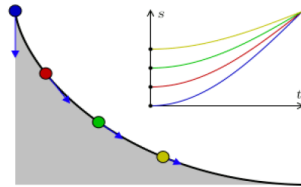
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**Abstract.** In this paper we examine the Euler- Lagrange equation and by expressing the fundamental thermo of calculus of variations, we calculate the Euler- Lagrange equation for the simplest problem of calculus of variations and by offering an example we will discuss the specific modes of Euler - Lagrange equation.

**Keywords:** Euler-Lagrange equation, the fundamental lemma of calculus of variations, optimal control, functional changes, extreme functional

### 1. INTRODUCTION

In 1750s, Euler and Lagrange discovered the Euler – Lagrange equation when they were working on the tautochrone problem. The tautochrone is about how we can find a curve on which we drop a ball and the time to get the ball down the curve will be constant regardless the height (Figure 1). Lagrange solved this problem in 1755 and sent the solution to Euler. Both further developed Lagrange's method and applied it to mechanics, which led to the formulation of Lagrangian mechanics. Their correspondence ultimately led to the calculus of variations, a term coined by Euler himself in 1766.



**Figure 1.** Four dots from four different positions on the cycloid are released, but they all reach the bottom at the same time. Blue arrows shows points' acceleration along the curve. Above, the time-space diagram is shown.

### 2. HISTORY

Joseph-Louis Lagrange (born 25 January 1736 in Turin, Piedmont-Sardinia; died 10 April 1813 in Paris) was an Italian Enlightenment Era mathematician and astronomer. He made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics.

In 1766, on the recommendation of Euler and d'Alembert, Lagrange succeeded Euler as the director of mathematics at the Prussian Academy of Sciences in Berlin, Prussia, where he stayed for over twenty years, producing volumes of work and winning several prizes of the French Academy of Sciences. Lagrange's treatise on analytical mechanics (*Mécanique Analytique*, 4.

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ed., 2 vols. Paris: Gauthier-Villars et fils, 1888–89), written in Berlin and first published in 1788, offered the most comprehensive treatment of classical mechanics since Newton and formed a basis for the development of mathematical physics in the nineteenth century.

In 1787, at age 51, he moved from Berlin to Paris and became a member of the French Academy. He remained in France until the end of his life. He was significantly involved in the decimalisation in Revolutionary France, became the first professor of analysis at the École Polytechnique upon its opening in 1794, founding member of the Bureau des Longitudes and Senator in 1799.

Leonhard Euler (15 April 1707 – 18 September 1783) was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function. He is also renowned for his work in mechanics, fluid dynamics, optics, astronomy, and music theory.

Euler is considered to be the pre-eminent mathematician of the 18th century and one of the greatest mathematicians to have ever lived. He is also one of the most prolific mathematicians; his collected works fill 60–80 quarto volumes. He spent most of his adult life in St. Petersburg, Russia, and in Berlin, Prussia.

### 3. CALCULUS OF VARIATIONS

The Problems of the calculus of variations are considered in the following general form.

$$\text{Min } J[x(t)] = \int_{t_0}^{t_f} g(t, x(t), \dot{x}(t)) dt \quad (1)$$

$$x(t_0) = x_0 \quad x(t_f) = x_f$$

One-functional changes:

$$J: \Omega \rightarrow R$$

$$\Delta J[x, \delta x] = J[x + \delta x] - J[x] \quad (2)$$

If  $\Delta J$  can be written in the following form

$$\Delta J[x, \delta x] = \delta J(x, \delta x) + g(x, \delta x) \|\delta x\| \quad (3)$$

In which  $\delta J$  is linear to  $\delta x$

$$\lim_{\|\delta x\| \rightarrow 0} g(x, \delta x) = 0 \quad (4)$$

In this case  $J$  is differentially supported in  $x$ , and  $\delta x$  is  $J$ 's differential.

Extrema function:  $J$  function with the defined range  $\Omega$  in  $x^*(t)$  has a relative Extrema curve if

$$\exists \varepsilon > 0; \forall x(t) \in \Omega \quad \|x(t) - x^*(t)\| < \varepsilon \Rightarrow \quad (5)$$

$J$  increment should always have a sign. In this case if

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$$\Delta J = J[x(t)] - J[x^*(t)] \geq 0 \Rightarrow J[x^*(t)] \quad \text{local minimum} \quad (6)$$

$$\Delta J = J[x(t)] - J[x^*(t)] \leq 0 \Rightarrow J[x^*(t)] \quad \text{local maximum} \quad (7)$$

Fundamental theorem of arithmetic

Assume  $x$  as Vector function of  $t$  and a member of  $\Omega$ , and  $J$  is a differentially supported function of  $x$ . assume functions have no restriction in  $\Omega$ . If  $x^*$  is a Curve extrema,  $J$  changes on  $x^*$  must be zero, that is

$$\forall \delta x \text{ admissible}, \delta J(x^*, \delta x) = 0 \quad (8)$$

Fundamental Lemma of calculus of variations:

If  $h$  is continuous and for each continuous  $\delta x$  in  $[t_0, t_f]$  range we have

$$\int_{t_0}^{t_f} h(t) \delta x(t) dt = 0 \quad (9)$$

Then  $h$  must be zero in all areas of  $[t_0, t_f]$

Using the fundamental theorem and Lemma of calculus of variations, we can achieve the Euler - Lagrange equation obtained from the simplest problem of calculus of variations.

### 4. THE SIMPLEST PROBLEM OF CALCULUS OF VARIATIONS

$$J[x(t)] = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t) dt$$

$$x(t_0) = x_0 \quad x(t_f) = x_f$$

$g$  has a continuous first and second order partial derivatives with respect to any of its components.

$$\|y(t) - x(t)\| < \varepsilon \quad (10)$$

$\Delta x$  is a curve which is due to

$$y(t) = x(t) + \delta x(t)$$

$$\begin{cases} x(t_0) = x_0 & x(t_f) = x_f \\ y(t_0) = x_0 & y(t_f) = x_f \end{cases} \Rightarrow \delta x(t_0) = 0, \delta x(t_f) = 0 \quad (11)$$

If all acceptable curves pass through  $(t_f, x_f)$ ,  $(t_0, x_0)$  then the above relation could be reached

$$y(t) = x(t) + \delta x(t)$$

$$\Delta J(x, \delta x) = J(x + \delta x) - J(x)$$

$$\begin{aligned}
 &= \int_{t_0}^{t_f} g(x(t) + \delta x(t), \frac{d}{dt}(x(t) + \delta x(t)), t) dt - \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t) dt \\
 &= \int_{t_0}^{t_f} [g(x(t) + \delta x(t), \dot{x}(t) + \delta \dot{x}(t), t) - g(x(t), \dot{x}(t), t)] dt \\
 &= \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t) + \left( \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) \right) \delta x(t) \\
 &\quad + \left( \frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t) \delta \dot{x}(t) + O((\delta x(t))^2, (\delta \dot{x}(t))^2) \right) - g(x, \dot{x}, t) dt \quad (12)
 \end{aligned}$$

When  $\delta x, \delta \dot{x} \rightarrow 0$

$$O((\delta x(t))^2, (\delta \dot{x}(t))^2) \rightarrow 0 \quad (13)$$

On the other hand

$$\Delta J(x, \delta x) = \delta J(x, \delta x) + h(x, \delta x) \|\delta x\|$$

Hence

$$\begin{aligned}
 \delta J(x, \delta x) &= \int_{t_0}^{t_f} \left( \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) \right) \delta x(t) dt + \int_{t_0}^{t_f} \left( \frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t) \delta \dot{x}(t) \right) dt \\
 &= \left( \frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t) \right) \delta x(t) \Big|_{t_0}^{t_f} \\
 &\quad + \int_{t_0}^{t_f} \left[ \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) \delta x(t) - \frac{d}{dt} \left( \frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t) \right) \delta x(t) \right] dt \\
 &= \int_{t_0}^{t_f} \left[ \frac{\partial g}{\partial x}(x, \dot{x}, t) - \frac{d}{dt} \left( \frac{\partial g}{\partial \dot{x}}(x, \dot{x}, t) \right) \right] \delta x(t) dt \quad (14)
 \end{aligned}$$

From the fundamental theorem of calculus of variations

$\forall \delta x$

$$\delta J(x^*, \delta x) = 0$$

$$\delta J(x, \delta x) = \int_{t_0}^{t_f} \left[ \frac{\partial g}{\partial x}(x^*, \dot{x}^*, t) - \frac{d}{dt} \left( \frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t) \right) \right] \delta x(t) dt = 0 \quad (15)$$

According to The fundamental lemma calculus of variations

$$\frac{\partial g}{\partial x}(x^*, \dot{x}^*, t) - \frac{d}{dt} \left( \frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t) \right) = 0 \quad (16)$$

Therefore answer of

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$$\text{Min or Max } J[x(t)] = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t) dt$$

$$x(t_0) = x_0 \quad x(t_f) = x_f$$

Can be obtained from the Euler – Lagrange equation.

$$\frac{\partial g}{\partial x} - \frac{d}{dt} \left( \frac{\partial g}{\partial \dot{x}} \right) = 0$$

$$x(t_0) = x_0 \quad x(t_f) = x_f$$

now we obtain extrema functional curve

Ex.1: [1]

$$J(x) = \int_0^1 [x^2(t) + \dot{x}^2(t)] dt ; \quad x(0) = 0 \quad x(1) = 1$$

Solution

$$g(x, \dot{x}, t) = x^2(t) + \dot{x}^2(t)$$

$$\frac{\partial g}{\partial x} = 2x(t) \quad , \quad \frac{\partial g}{\partial \dot{x}} = 2\dot{x}(t) \rightarrow \frac{d}{dt} \left( \frac{\partial g}{\partial \dot{x}} \right) = \frac{d}{dt} (2\dot{x}(t))$$

By pasting in the Euler - Lagrange equation

$$\frac{\partial g}{\partial x} - \frac{d}{dt} \left( \frac{\partial g}{\partial \dot{x}} \right) = 0 \Rightarrow 2x(t) - 2\dot{x}(t) = 0 \rightarrow \ddot{x}(t) - x(t) = 0$$

The above equation is a differential equation of order two homogeneous with constant coefficients that with solving it as follows we have

$$a^2 - 1 = 0 \rightarrow a_1 = 1, \quad a_2 = -1 \rightarrow x(t) = c_1 e^t + c_2 e^{-t}$$

$$x(0) = 0 \rightarrow c_2 = -c_1$$

$$x(1) = 1 \rightarrow c_1 e^1 + c_2 e^{-1} = 1 \xrightarrow{c_2 = -c_1} c_1 = \frac{e}{e^2 - 1}$$

$$x(t) = \frac{e(2\sinh t)}{e^2 - 1}$$

Ex.2: [2]

$$\int_a^b \frac{\dot{y}^2}{x^3} dx$$

Solution

$$g(y, \dot{y}, x) = \frac{\dot{y}^2}{x^3}$$

$$\frac{\partial g}{\partial y} = 0 \quad , \quad \frac{\partial g}{\partial \dot{y}} = \frac{2}{x^3} \dot{y} \quad , \quad \frac{d}{dx} \left( \frac{\partial g}{\partial \dot{y}} \right) = \frac{d}{dx} \left( \frac{2}{x^3} \dot{y} \right)$$

$$\rightarrow \frac{\partial g}{\partial y} - \frac{d}{dx} \left( \frac{\partial g}{\partial \dot{y}} \right) = 0 \rightarrow \ddot{y} = \frac{3}{x} \dot{y}$$

By solving the differential equation we have

$$y(x) = c_1 x^4 + c_2$$

## 5. THE SPECIFIC MODES OF EULER - LAGRANGE EQUATION

(1) F is independent of x or (  $\frac{\partial F}{\partial x} = 0$  )

$$\rightarrow 0 - \frac{d}{dt} (F_{\dot{x}}) = 0 \rightarrow \frac{d}{dt} (F_{\dot{x}}) = 0 \rightarrow \frac{\partial F}{\partial \dot{x}} = \text{constant}$$

(2) F is independent of t or  $F(x, \dot{x})$

$$\frac{d}{dt} F(x, \dot{x}, t) = F_x \frac{dx}{dt} + F_{\dot{x}} \frac{d\dot{x}}{dt} + \frac{\partial F}{\partial t}$$

(3) Because F is independent of time, hence

$$\frac{d}{dt} F(x, \dot{x}, t) = \dot{x} F_x + \ddot{x} F_{\dot{x}}$$

$$\xrightarrow{F_x - \frac{d}{dt} F_x = 0} \dot{x} \left( \frac{d}{dt} (F_{\dot{x}}) \right) + \ddot{x} F_{\dot{x}} = \frac{d}{dt} (\dot{x} F_{\dot{x}}) \Rightarrow F = \dot{x} F_{\dot{x}} + C$$

As a result, when the function is not a dependent integrand to t, the Euler – Lagrange equation is

$$F - \dot{x} F_{\dot{x}} = C$$

In which c is a constant.

For example, from the latter case, the problem can be noted.

## 6. LAGRANGE – EQUATION APPLICATION

From NYU, Kyle Kranmer states:” although the equation is abstract but it clarifies many substantive issues. This equation has created a big revolution in quantum mechanics. In this equation, L stands for Lagrange which is used for measuring the system energy. There is another theory within this equation called the neutral theorem that says if you have a symmetry in your system, there will be a better chance for energy storage. This theory is used for mechanical design of the spacecraft.

## 7. CONCLUSION

## Euler-Lagrange Equation

For functional extrema with proved initial and final conditions, the necessary requirement is to be applied in Euler -Lagrange equations. In fact the Euler -Lagrange equations provide the optimally condition of the system.

Or in detail

$$\frac{\partial g}{\partial x} - \frac{d}{dt} \left( \frac{\partial g}{\partial \dot{x}} \right) = 0 \quad \text{,} \quad i = 1, 2, 3, \dots, n.$$

### REFERENCES

- [1] Kirk, Donald E., “ Optimal Control Theory: An Introduction”,. 1970, Englewood Cliffs, N.J.: Prentice-Hall
- [2] Gelfand, I.M. and Fomin, S. V., “Calculus of Variations”, 1963. Englewood Cliffs, N.J: Prentice-Hall
- [3] Lagrange, Joseph Louis: Columbia Encyclopedia, Sixth Edition, Copyright(c) 2005
- [4] A Biography of Joseph Louis Lagrange
- [5] Lagrange, Joseph Louis de: The Encyclopedia of Astrobiology, Astronomy and Space Flight
- [6] <http://fa.wikipedia.org>
- [7] <http://tebyan.net>