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Introducing Skew Discrete Laplace

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Abstract. In this paper, an appropriate substitution was introduced for distributing skew Laplace which had been achieved from rupturing continuous skew discrete Laplace. This distribution has been flexible and posses a closed form for probability function, distribution function, moment-generating function, characteristic function of probability, and other distribution features such as high expectation and variance. Here we deal with distribution properties like estimating parameters based on maximum likelihood, moments, moments modified and ratio method. We will determine CI for the parameters based on fisher and logic information matrix and then we will analyze necessary inference and hypothesis testing. We will use Monte Carlo stimulation method.

Keywords: Skew discrete Laplace distribution, Fisher information matrix, Monte Carlo stimulation, maximum likelihood method

1. INTRODUCITON

According to significance of modeling by discrete distribution, too many models have been presented for rupturing a positive continuous distribution so far, but rupturing of continuous distribution on total R was less noticed. For example, negative binomial distribution is discrete form of gamma distribution [7] which is applied for modeling non-negative discrete data. Maybe the only possible development on total R has been rupturing normal distribution [14] and Laplace distribution [8]. Skew discrete Laplace distribution is another flexible model which was raised by Barbiero [3] and it was mentioned as a substitution for Laplace distribution introduced by Inusah. S, Kozubowski. T.J. It has relatively interesting properties and it this research we mostly focus on it. This skewed discrete distribution possesses a closed from for probability density function, distribution function and other distribution properties such as mathematic expectation and variance and it is widely applied in practice and especially in discrete lifetime data defined on integers. Reviewing various forms mentioned for skew Laplace distribution, in this paper first we will point out previous methods for constructing skewed discrete Laplace distribution. Then we will provide a new method according to survival function for establishing a new form of skewed discrete Laplace distribution. Their property will be investigated and then parameters will be dealt with based on maximum likelihood, torques, torques modified and ratio method. CI for the parameters will be determined based on Fisher and logic information matrices. Next, we will analyze necessary inference and hypothesis testing. We will use Monte Carlo stimulation method.

2. INTRODUCING SKEWED LAPLACE DISTRIBUTION

Since main aim of the current paper is rupturing skew Laplace distribution among different forms of Laplace distribution, researchers mostly concentrate on rupturing one of the best forms because of their top and simple properties in calculations and in practice they own more efficiency, so we will deal with them.

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Following is the form introduced for distributing skew Laplace which is mostly noticed by researchers:

In which $0 < p, q < 1$

$$f(x; p, q) = \frac{-\log(p)\log(q)}{\log(pq)} \begin{cases} p^x & x \geq 0 \\ q^{-x} & x < 0 \end{cases} \tag{1}$$

Also, its remaining function is as follows:

$$S(x; p, q) = \begin{cases} \dfrac{\log(q)}{\log(pq)} p^x & x \geq 0 \\ 1 - \dfrac{\log(p)}{\log(pq)} q^{-x} & x < 0 \end{cases} \tag{2}$$

3. RUPTURING SKEW LAPLACE DISTRIBUTION

This method was introduced by Barbiero which devoted the main discussion to itself in this paper. Using discretization instruction-which is regarded as difference between survival functions-we will have followings according to conventional forms of p and q.

$$\begin{aligned} \emptyset(x) = p(X_d = x) = p(x \leq X < x) &= p(X < x+1) - p(X < x) \\ &= S(x) - S(x+1) \qquad x \epsilon Z \end{aligned} \tag{3}$$

$$\emptyset(x; p, q) = \frac{1}{\log(\mathrm{pq})} \begin{cases} \log(p)\left[q^{-(x+1)}(1-q)\right] & x = \cdots, -2, -1 \\ \log(q)\left[p^x(1-p)\right] & x = 0,1,2, \ldots \end{cases} \tag{4}$$

Above form introduces a skew discrete Laplace distribution with parameters $0 < p, q < 1$ and from now on we consider it as follows where figure 1 displaces sections (a), (b) and (C) of the distribution for p.q

$X_d \sim ADSLaplace(p, q)$

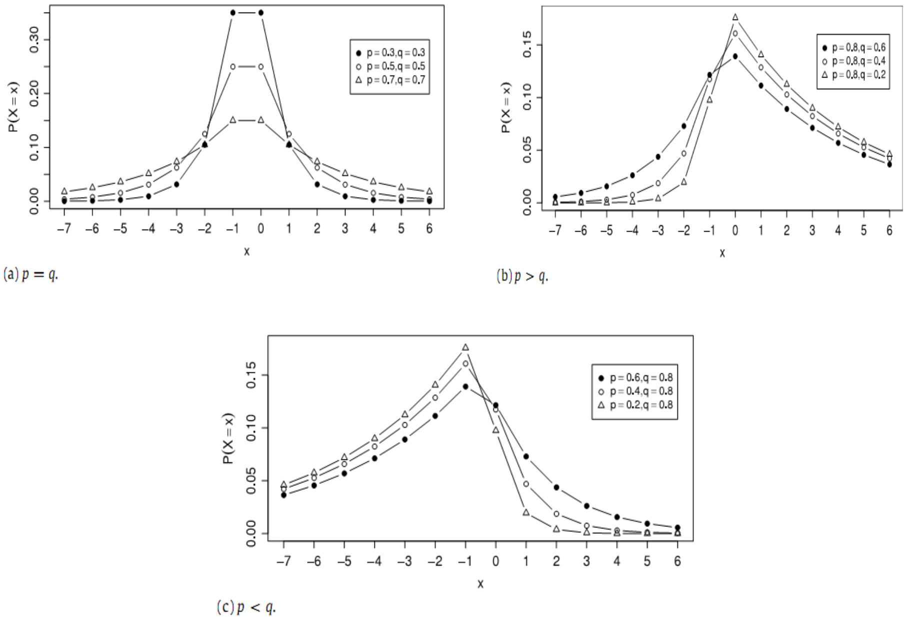

Figure1: skew discrete Laplace distribution for various forms of p,q

$$F(x;p,q)=\begin{cases}\dfrac{\log(p)}{\log(pq)}q^{-([x]+1)} & x\geq 0\\ 1-\dfrac{\log(q)}{\log(pq)}p^{[x]+1} & x<0\end{cases} \tag{5}$$

4. Skewed discrete Laplace distribution moments

Moment-generating function in skew discrete Laplace distribution can be achieved by easy calculations as follows by which we can produce raw moments in this distribution.

$$M(t;p,q)=E[e^{tX}]=\frac{1}{\log(pq)}\left(\log(q)\frac{1-p}{1-pe^{t}}+\log(p)\frac{(1-q)e^{-t}}{1-qe^{-t}}\right) \tag{6}$$

Meantime, this function is gained as follows which is a more general case of moment-generating function:

$$C(it;p,q)=E\left[e^{itX}\right]=\frac{1}{\log(pq)}\left(\log(q)\frac{1-p}{1-pe^{it}}+\log(p)\frac{(1-q)e^{-it}}{1-qe^{-it}}\right) \tag{7}$$

For example, first and second order moments and their variance is calculated as follows:

$$E(X^2)=\frac{\log(q)}{log(pq)}.\frac{p(1+p)}{(1-p)^2}+\frac{log(p)}{log(pq)}.\frac{1+q}{(1-q)^2}$$

$$\mu = E(X) = \frac{log(q)}{log(pq)} . \frac{p}{1-p} - \frac{log(p)}{log(pq)} . \frac{1}{1-q}$$

$$Var(X) = \frac{(1-p)^2 q (\log(p))^2 + (1+p+q-6pq+p^2q+pq^2+p^2q^2)\log(p)\log(q) + p(1-q)^2(\log(q))^2}{(log(pq))^2 (1-p)^2 (1-q)^2}$$

Variance is a function of q and p and it can be observed in contour plot of figure 2 for parameters q and p.

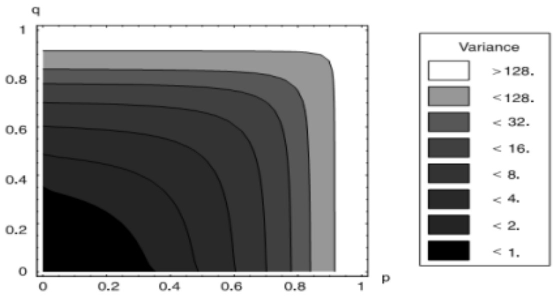

Figure 1. Contour plot for variance of skew discrete Laplace distribution based on parameters q and p

5. SKEWNESS AND SKEWED DISCRETE LAPLACE DISTRIBUTION

Defining formulas $Sk(x) = \frac{E(x-\mu)^3}{\sigma^3}$ for skewness and $Ku(x) = \frac{E(x-\mu)^4}{\sigma^4}$ for strain, we can calculate skewness and strain easily for skewed discrete Laplace distribution. So that for skewness we will have: $Sk(x; p, q) = -Sk(x; q, p)$

And for (Ku)strain we have: $Ku(x; p, q) = Ku(x; q, p)$

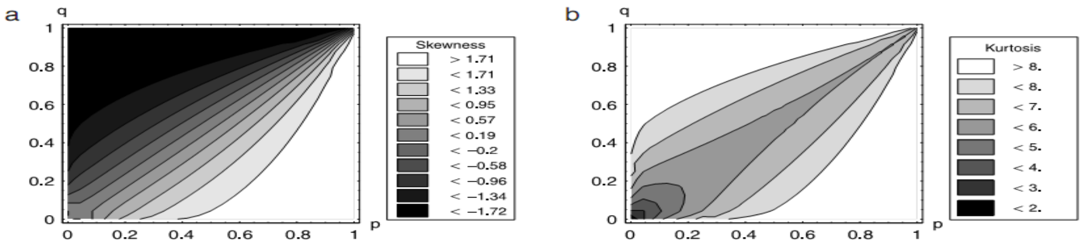

Figure3. Contour plot for skewness and skewed elongation Laplace distribution

6. ESTIMATING PARAMETERS:

6.1. Maximum likelihood method

Likelihood logarithm function in discrete Laplace model which was calculated based on independent and distribution samples $\boldsymbol{X} = (X_1, \dots, X_n)$ is as follows:

$$l(p,q;\boldsymbol{X}) = log\prod_{i=1}^{n}\emptyset(p,q;X_i)$$
$$= s^{-}\log\left(\frac{\log(p)}{\log(pq)}\right) + s^{+}\log\left(\frac{\log(q)}{\log(pq)}\right) - -\left(s^{-} + \sum_{x_i<0} x_i\right) loq(q)$$
$$+ s^{-}\log(1-q) + \sum_{x_i\geq 0} x_i \log(p) + s^{+}\log(1-p) \qquad (8)$$

In which s^{-} and s^{+} with definitions $s^{-} = \sum_{i=1}^{n} 1_{x_i<0}$ *and* $s^{+} = \sum_{i=1}^{n} 1_{x_i\geq 0}$indicate values of possible samples with negative and positive values, respectively.

Also, first-order derivative for likelihood logarithm function is calculated as follows:

$$\frac{\partial l(p,q;x)}{\partial p} = s^{-}\frac{1}{plog(p)} - n\frac{1}{plog(pq)} + \sum_{x_i\geq 0}\frac{x_i}{p} - \frac{s^{+}}{1-p}$$

$$\frac{\partial l(p,q;x)}{\partial q} = s^{+}\frac{1}{qlog(q)} - n\frac{1}{qlog(pq)} - \sum_{x_i<0}\frac{x_i}{q} - \frac{s^{-}}{q(1-q)}$$

Answer of above equations can lead to estimation of maximum likelihood in the model parameter which is shown by $(\hat{p}_{ML}, \hat{q}_{ML})$. It is quite vivid that there is no close form for answering these equations and estimating maximum likelihood. Hence, to achieve them, we have to apply numerical methods.

6.2. Moment method

It is easy to achieve two first moments in skewed discrete Laplace distribution, however moment method can easily produce parameters q and p. In fact, equal placing of $E(X)$ and $E(X^2)$ with equivalent sample moments can lead to non-linear equation based on q and p which can only be solved by numerical methods.

6.3. Modified moment estimators

This method is mostly noticed in discrete distributions is in this state where we can consider non-negative and negative samples separately as a replacing method for estimating moments. To do this, assume$x_i^{+} = x_i 1_{x_i\geq 0}$, and $x_i^{-} = x_i 1_{x_i<0}$. Then following equation and mathematical expectations will be gained:

$$E[X|X\geq 0] = \frac{\sum_{i=1}^{n} x_i^{+}}{s_i^{+}} \quad and \quad E[X|X<0] = \sum_{i=1}^{n} x_i^{-}/s_i^{-}$$

Recalling relations mentioned for conditional expectations, estimators of parameters q and p are achieved as follows:

$$\hat{p}_{MM} = \frac{\sum_{i=1}^{n} x_i^{+}}{s_i^{+} + \sum_{i=1}^{n} x_i^{+}} \qquad and \qquad \hat{q}_{MM} = 1 + \frac{s_i^{-}}{\sum_{i=1}^{n} x_i^{-}} \qquad (9)$$

6.4. Ratio method

Under skewed discrete Laplace model we have $P(X=0)=p_0=(1-p)\log\frac{q}{\log(pq)}$ and $P(X\geq 0)=p_+=\log\frac{q}{\log(pq)}$. By estimating method of Khan et al [9] we consider these two relations and possess a simple calculation which is as follows:

$$\hat{P}_p = 1 - p_0/p_+ \qquad \textit{and} \qquad \hat{q}_p = p^{\frac{p_+}{1-p_+}} \tag{10}$$

7. ASYMPTOTIC CONFIDENCE INTERVALS OF PARAMETERS

In this section we achieve asymptotic confidence intervals for the parameters with the help of Fisher information matrix so that observed Fisher information matrix $\hat{I}(n)$ in independent samples and distribution $\boldsymbol{X}$ can be defined as follows:

$$\hat{I}_{(n)} = -\begin{pmatrix} \partial^2 l(p,q;x)/\partial p^2 & \partial^2 l(p,q;x)/\partial p \partial q \\ \partial^2 l(p,q;x)/\partial p \partial q & \partial^2 l(p,q;x)/\partial q^2 \end{pmatrix} \tag{11}$$

So Fisher information matrix can be defined as follows:

$$(p,q) = -E\begin{pmatrix} \partial^2 log\emptyset(p,q;x_i)/\partial p^2 & \partial^2 log\emptyset(p,q;x_i)/\partial p \partial q \\ \partial^2 log\emptyset(p,q;x_i)/\partial p \partial q & \partial^2 log\emptyset(p,q;x_i)/\partial q^2 \end{pmatrix} \tag{12}$$

It is worth to mention that if accurate values of q and p are not available, then we can use values $\hat{p}_{ML}$ and $\hat{q}_{ML}$ instead of q and p, respectively. This way we will have maximum likelihood estimators by applying asymptotic normal properties:

$$\sqrt{n}\begin{pmatrix} \hat{p}_{ML} \\ \hat{q}_{ML} \end{pmatrix} \to N(0, \boldsymbol{I}^{-1}(p,q))$$

In other words, with the help of Fisher information matrix we can calculate sample big CI $(1-\alpha)$ percent for parameters q and p as follows:

$$\hat{p}_{ML} \mp z_{1-\frac{\alpha}{2}}\sqrt{\hat{I}^{-1}_{(n)11}} \qquad \textit{and} \qquad \hat{q}_{ML} \mp z_{1-\alpha/2}\sqrt{\hat{I}^{-1}_{(n)22}} \tag{13}$$

You should notice that instead of using $\hat{I}^{-1}_{(n)}$ in above relations, you may apply $\hat{I}^{-1}/n$ and achieve CI. Such CI may have lower boundary less than zero (more than 1). If sample size is small, in these cases we have to achieve CI from left and then we can cut in zero (from right in one). This way modified CI can be gained. We can also gain another CI by applying variance stabilizing transformations such as logit conversion, since q and p will gain their values in interval (0,1).

Now by defining $y = log(\frac{p}{1-p})$ and $\hat{y} = log(\frac{\hat{p}_{ML}}{1-\hat{p}_{ML}})$ we can gain fine CI for y which is as follows:

$$(y_L, y_U) = \hat{y} \mp z_{1-\alpha/2}\frac{\sqrt{\hat{I}^{-1}_{(n)11}}}{\hat{p}_{ML}(1-\hat{p}_{ML})}$$

Then appropriate CI for p is gained with photo converting logit which is as follows:

$$(p_L, p_U) = (\frac{exp(y_L)}{1 + exp(y_L)}, \frac{exp(y_U)}{1 + exp(y_U)})$$

We may apply a similar method for q and gain CI which is more trustable than interval created by Fisher information matrix and without converting logit. It is worth to mention that this interval for $\hat{p}_{ML}$ and $\hat{q}_{ML}$ is a symmetric interval.

8. HYPOTHESIS TEST

To test hypothesis $H_0: (p, q) = (p_0, q_0) \in (0,1) \times (0,1)$ against $H_1: (p, q) \neq (p_0, q_0)$, statistic T can be used as follows:

$$T = -2(l_0 - \hat{l}) \quad (15)$$

In which $\hat{l}$ is maximum likelihood logarithm calculated in areas $\hat{p}_{ML}$ and $\hat{q}_{ML}$ under the title of maximum likelihood estimators. Also l_0 is the same function in areas (p_0, q_0). Under H_0‘ T has k score distribution with two degrees of freedom asymptotically. Therefore, in a simple asymptotic test based on T in $(1 - \alpha)$ percent significance level, hypothesis H_0 is rejected, if we have $t > \chi^2_{1-\alpha}$in which t is sample value of statistic T.

9. STIMULATION STUDIES

Now to assess performance of estimations, asymptotic CI and introduced asymptotic test, in the previous section we will assess topics and estimations mentioned in the third section by using Monte Carlo and various states for q and p, especially 0/25, /5 and 0/75 and also sample sizes of 50, 100 and 1000. In other words, estimations of areas such as moments method (M), modified moment method (M), ratio method (MM) and maximum likelihood method (P) will be determined and assessed. Meantime, (CLs) 95 percnet of CI and T test statistic will be assessed for the samples took.

As it can be observed from table 1, moment method can produce area estimators with the highest level of diagonal, of course this level is negative under investigating different states for q and p. Modified method of estimators can somehow reduce absolute value of this level and it produces more optimum estimators, although this value is still negative. Ratio method produces estimators with the least diagonal which is greatly close to maximum likelihood method.

After argument over the-spot estimators, we will follow interval estimators which are followed in table 2 based on analyzed and observed Fisher information matrix for logit ratio. As it can be observed there is no major difference between converge rate and average width of CI calculated based on table values. In other words, there does not exist a significant difference between CI calculated for Fisher information matrix and observed one. As you can observe, converge rate for CI is always bigger than 80/0 and even sometime it is more than pre-assumption value (0/98 percent confidence level). Average width is usually small and an acceptable value, although these values somehow improve with logit ratio in Fisher information matrix.

Table 1. Diagonal level and standard deviation for estimators

			M				***MM***				***P***				***ML***			
p	p	n	$bias(\hat{p})$	Sd($\hat{p}$)	$bias(\hat{q})$	Sd($\hat{q}$)	$bias(\hat{p})$	Sd($\hat{p}$)	$bias(\hat{q})$	Sd($\hat{q}$)	$bias(\hat{p})$	Sd($\hat{p}$)	$bias(\hat{q})$	Sd($\hat{q}$)	$bias(\hat{p})$	Sd($\hat{p}$)	$bias(\hat{q})$	Sd($\hat{q}$)
0.25	0.25	50	-0.015	0.061	-0.014	0.069	-0.009	0.091	-0.008	0.089	0.005	0.099	0.003	0.089	-0.001	0.091	-0.018	0.089
0.25	0.25	100	-0.014	0.069	-0.012	0.049	-0.005	0.123	-0.003	0.079	0.001	0.061	-0.001	0.060	-0.001	0.061	-0.011	0.079
0.25	0.25	1000	-0.011	0.021	-0.009	0.261	-0.001	0.251	-0.018	0.138	-0.002	0.033	-0.001	0.049	-0.000	0.041	-0.008	0.069
0.25	0.5	50	-0.021	0.099	-0.026	0.069	-0.016	0.161	-0.015	0.119	0.008	0.085	0006	0.089	-0.009	0.099	-0.021	0.088
0.25	0.5	100	-0.019	0.072	-0.024	0.041	-0.010	0.041	-0.011	0.087	0.002	0.071	0.001	0.070	-0.011	0.081	-0.012	0.072
0.25	0.5	1000	-0.013	0.161	-0.021	0.065	-0.007	0.291	-0.013	0.229	-0.002	0.054	-0.001	0.058	-0.009	0.071	-0.009	0.059
0.25	0.75	50	-0.032	0.169	-0.016	0.309	-0.029	0.213	-0.009	0.023	0.050	0.089	0.043	0.085	-0.008	0.088	-0.023	0.099
0.25	0.75	100	-0.027	0.081	-0.010	0.190	-0.017	0.091	-0.002	0.189	0.010	0.074	0.009	0.076	-0.007	0.061	-0.016	0.073
0.25	0.75	1000	-0.008	0.011	-0.005	0.125	-0.006	0.097	-0.011	0.165	0.00	0.051	-0.001	0.059	-0.013	0.051	-0.008	0.053
0.5	0.25	50	-0.042	0.0161	-0.029	0.045	-0.023	0.191	-0.007	0.082	0.001	0.096	0.00	0.088	-0.011	0.079	-0.018	0.079
0.5	0.25	100	-0.038	0.0111	-0.027	0.029	-0.021	0.099	-0.003	0.123	-0.001	0.056	-0.002	0.059	-0.012	0.061	-0.014	0.059
0.5	0.25	1000	-0.029	0.0147	-0.016	0.169	-0.015	0.191	-0.012	0.049	-0.001	0.049	-0.005	0.041	-0.008	0.051	-0.010	0.041
0.5	0.5	50	-0.032	0.0264	-0.024	0.099	-0.030	0.166	-0.009	0.287	0.002	0.099	-0.006	0.087	-0.011	0.089	-0.024	0.088
0.5	0.5	100	-0.025	0.0361	-0.017	0.124	-0.022	0.191	-0.007	0.122	-0.002	0.079	0.001	0.067	-0.012	0.070	-0.012	0.063
0.5	0.5	1000	-0.018	0.092	-0.011	0.109	-0.016	0.235	-0.012	0.189	-0.002	0.051	-0.003	0.061	-0.006	0.059	-0.009	0.039
0.5	0.75	50	-0.026	0.241	-0.014	0.169	-0.009	0.141	-0.012	0.115	0.001	0.086	0.002	0.088	-0.014	0.086	-0.021	0.094
0.5	0.75	100	-0.021	0.298	-0.011	0.127	-0.006	0.021	-0.009	0.119	0.00	0.063	0.001	0.077	-0.013	0.081	-0.019	0.078
0.5	0.75	1000	-0.016	0.071	-0.008	0.251	-0.003	0.076	-0.004	0.181	-0.001	0.043	0.004	0.059	-0.011	0.059	-0.014	0.054
0.75	0.25	50	-0.047	0.261	-0.034	0.132	-0.031	0.123	-0.021	0.111	0.001	0.092	-0.004	0.095	-0.010	0.086	-0.019	0.092
0.75	0.25	100	-0.042	0.069	-0.029	0.139	-0.023	0.191	-0.016	0.294	0.001	0.087	-0.003	0.089	-0.017	0.061	-0.014	0.056
0.75	0.25	1000	-0.039	0.231	-0.018	0.168	-0.019	0.061	-0.013	0.137	0.00	0.058	-0.001	0.046	-0.014	0.051	-0.011	0.041
0.75	0.5	50	-0.019	0.098	-0.014	0.049	-0.011	0.191	-0.008	0.059	-0.002	0.073	-0.004	0.084	-0.013	0.099	-0.026	0.089
0.75	0.5	100	-0.015	0.232	-0.012	0.288	-0.009	0.081	-0.007	0.039	-0.001	0.061	-0.001	0.071	-0.011	0.061	-0.018	0.069
0.75	0.5	1000	-0.007	0.081	-0.009	0.088	-0.004	0.028	-0.005	0.027	0.001	0.039	0.001	0.038	-0.012	0.085	-0.012	0.059
0.75	0.75	50	-0.025	0.161	-0.019	0.139	-0.023	0.321	-0.017	0.231	0.001	0.098	-0.001	0.089	-0.011	0.081	-0.013	0.091
0.75	0.75	100	-0.022	0.095	-0.017	0.123	-0.019	0.231	-0.012	0.129	-0.002	0.082	0.00	0.070	-0.012	0.061	-0.011	0.072
0.75	0.75	1000	-0.014	0.089	-0.013	0.099	-0.008	0.121	-0.008	0.109	-0.001	0.076	0.00	0.039	-0.013	0.033	-0.008	0.050

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Table 2: Coverage rate and average width of confidence interval for the parameters

			Fisher				*Fisher Logit*				*Fisher Observation*				*Fisher Logit Observation*			
p	p	n	Cr(p)	Cr(q)	Aw(p)	Aw(q)	Cr(p)	Cr(q)	Aw(p)	Aw(q)	Cr(p)	Cr(q)	Aw(p)	Aw(q)	Cr(p)	Cr(q)	Aw(p)	Aw(q)
0.25	0.25	50	0.903	0.921	0.321	0.332	0.921	0.933	0.341	0.305	0.911	0.932	0.332	0.334	0.933	0.936	0.348	0.306
0.25	0.25	100	0.912	0.911	0.296	0.386	0.916	0.902	0.299	0.375	0.917	0.918	0.292	0.381	0.910	0.913	0.295	0.377
0.25	0.25	1000	0.971	0.903	0.341	0.243	0.982	0.911	0.364	0.237	0.976	0.911	0.349	0.240	0.987	0.915	0.365	0.232
0.25	0.5	50	0.981	0.966	0.306	0.349	0.974	0.972	0.315	0.351	0.989	0.967	0.308	0.341	0.971	0.973	0.313	0.359
0.25	0.5	100	0.871	0.865	0.277	0.297	0.808	0.867	0.264	0.288	0.874	0.863	0.282	0.294	0.809	0.860	0.269	0.293
0.25	0.5	1000	0.814	0.891	0.312	0.419	0.809	0.866	0.362	0.455	0.819	0.896	0.310	0.420	0.813	0.861	0.368	0.459
0.25	0.75	50	0.919	0.990	0.299	0.288	0.990	0.922	0.253	0.219	0.912	00.994	0.294	0.283	0.791	0.928	0.258	0.218
0.25	0.75	100	0.923	0.955	0.314	0.354	0.964	0.938	0.331	0.378	0.927	0.953	0.317	0.356	0.968	0.930	0.339	0.373
0.25	0.75	1000	0.876	0.897	0.255	0.315	0.855	0.832	0.241	0.319	0.870	0.891	0.252	0.311	0.851	0.845	0.242	0.310
0.5	0.25	50	0.920	0.922	0.241	0.423	0.982	0.987	0.254	0.444	0.907	0.928	0.243	0.429	0.923	0.988	0.256	.0455
0.5	0.25	100	0.818	0.892	0.251	0.324	0.848	0.879	0.266	0.346	0.816	0.899	0.255	0.320	0.845	0.876	0.263	0.345
0.5	0.25	1000	0.976	0.905	0.233	0.265	0.921	0.944	0.261	0.239	0.976	0.905	0.233	0.265	0.921	0.944	0.261	0.239
0.5	0.5	50	0.865	0.877	0.245	0.488	0.863	0.841	0.209	0.483	0.865	0.877	0.245	0.488	0.863	0.841	0.209	0.483
0.5	0.5	100	0.915	0.932	0.321	0.231	0.926	0.910	0.341	0.225	0.915	0.932	0.321	0.231	0.826	0.911	0.341	0.225
0.5	0.5	1000	0.823	0.881	0.342	0.356	0.863	0.874	0.378	0.309	0.823	0.881	0.342	0.356	0.863	0.874	0.378	0.309
0.5	0.75	50	0.924	0.909	0.341	0.376	0.939	0.918	0.351	0.333	0.924	0.909	0.341	0.376	0.939	0.918	0.351	0.333
0.5	0.75	100	0.808	0.893	0.234	0.304	0.821	0.847	0.254	0.344	0.808	0.893	0.234	0.304	0.821	0.847	0.254	0.344
0.5	0.75	1000	0.901	0.962	0.361	0.236	0.965	0.917	0.333	0.382	0.908	0.967	0.363	0.328	0.961	0.918	0.329	0.389
0.75	0.25	50	0.816	0.921	0.354	0.298	0.865	0.942	0.317	0.263	0.817	0.926	0.357	0.291	0.863	0.941	0.314	0.269
0.75	0.25	100	0.903	0.982	0.352	0.356	0.951	0.941	0.348	0.392	0.912	0.989	0.353	0.360	0.958	0.947	0.340	0.396
0.75	0.25	1000	0.820	0.911	0.451	0.345	0.864	0.962	0.439	0.391	0.832	0.910	0.459	0.357	0.861	0.966	0.431	0.395
0.75	0.5	50	0.912	0.965	0.256	0.312	0.948	0.931	0.209	0.374	0.918	0.968	0.254	0.323	0.949	0.939	0.211	0.370
0.75	0.5	100	0.872	0.808	0.239	0.330	0.833	0.842	0.217	0.314	0.874	0.806	0.230	0.307	0.836	0.845	0.218	0.316
0.75	0.5	1000	0.911	0.945	0.291	0.316	0.951	0.923	0.246	0.382	0.919	0.944	0.297	0.311	0.958	0.924	0.249	0.385
0.75	0.75	50	0.841	0.866	0.304	0.294	0.839	0.818	0.342	0.265	0.842	0.869	0.311	0.293	0.835	0.810	0.344	0.266
0.75	0.75	100	0.931	0.977	0.310	0.311	0.989	0.906	0.372	0.393	0.938	0.975	0.311	0.314	0.988	0.905	0.377	0.392
0.75	0.75	1000	0.871	0.889	0.308	0.313	0.838	0.825	0.329	0.354	0.873	0.886	0.311	0.324	0.830	0.826	0.322	0.358

10. BAYESIAN ANALYSIS

Let $\theta = (p, q)$ be the parameters of the discrete skewed Laplace distribution with the prior distribution $\pi(\theta) = \pi(p)\pi(q|p)$. Note that we assume p and q are independent, so $\pi(\theta) = \pi(p)\pi(q)$. Also, the prior distribution for p and q are the non-informative prior $U(0,1)$, uniform distribution. If $f(x|\theta)$ be the desired distribution, then the posterior distribution of θ given x is as follows:

$$\pi(\theta|x) = \frac{f(x|\theta)}{\int_\theta \; f(x|\theta)d\theta},$$

Note that under square integrable loss function, the Bayes estimator for $\theta = (p, q)$ is as follow

$$\theta^B = \frac{\int_\theta \; \theta \prod_{i=1}^n f(x_i|\theta)\pi(\theta)d\theta}{\int_\theta \; \prod_{i=1}^n f(x_i|\theta)\pi(\theta)d\theta}$$

That leads to

$$\theta^B = \frac{E[\theta \prod_{i=1}^n f(x_i|\theta)\pi(\theta)]}{E[\prod_{i=1}^n f(x_i|\theta)\pi(\theta)]}$$

Now, if $\theta_1, \theta_2, \dots, \theta_m$ be m iid sample from prior distribution $\pi(\theta)$, so we have

$$\frac{1}{m}\sum_{j=1}^{m} \theta_j \prod_{i=1}^{n} f(x_i \,|\theta_j) \xrightarrow{a.s.} E\left[\theta \prod_{i=1}^{n} f(x_i \,|\theta)\right], m \to \infty$$

and

$$\frac{1}{m}\sum_{j=1}^{m} \prod_{i=1}^{n} f(x_i \,|\theta_j) \xrightarrow{a.s.} E\left[\prod_{i=1}^{n} f(x_i \,|\theta)\right], m \to \infty$$

So the empirical Bayes estimators θ^{EB} is as follows

$$\theta^{EB} = \frac{\frac{1}{m}\sum_{j=1}^{m} \theta_j \prod_{i=1}^{n} f(x_i \,|\theta_j)}{\frac{1}{m}\sum_{j=1}^{m} \prod_{i=1}^{n} f(x_i \,|\theta_j)}$$

and finally the Bayes estimator $\hat{p}_{EB}$ and $\hat{q}_{EB}$ can be easily found.

10.1. Simulation study

Now, to validate our estimation method presented in this paper, we simulate 1000 sample for different combination of the (p, q) and compare ML estimator with empirical Bayes method. Note that R package /DiscreteLaplace/ from website

http://CRAN.R-project.org/package=DiscreteLaplace (Barbiero, 2014) can be easily achieved. As Table 1 shows, the differences between considered amount for (p, q) and their empirical Bayes estimator are less than maximum likelihood method.

Table1. Simulation study for empirical Bayes and maximum likelihood method.

(p, q)	$\hat{p}_{ML}$	$\hat{q}_{ML}$	$\hat{p}_{EB}$	$\hat{q}_{EB}$
(0.25,0.25)	0.211	0.229	0.231	0.238
(0.25,0.5)	0.221	0.472	0.231	0.495

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(0. 5,0.25)	0. 482	0.232	0. 491	0.239
(0.75,0.75)	0.724	0.722	0.731	0.739

11. CONCLUSION

Skewed discrete Laplace distribution is a flexible distribution which is defined on total absolute value space. Rupturing this distribution led to distribution definition which can be defined on total absolute value. It was observed that in analyses there were so significant and could be applied for modeling difference between numerical data. Despite previous skewed distribution which was not defined on total absolute value, this distribution possessed a closed form of probability function, distribution function, moment-generating function, characteristic function of probability, mathematical expectation and variance and statistical analyses could be conducted much easier. The presented paper investigates Bayesian analysis for discrete skewed Laplace distribution and compares it to the classical estimation method, maximum likelihood estimator. The BIC criteria show that the empirical Bayes estimators are preferable.

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