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Harry-Dym Hierarchy: Hereditary Operator and Ablowitz-Ramani-Segur Conjecture

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Abstract. Using Lax representation, hereditary operator related to the Harry-Dym (HD) hierarchy is constructed. Also Lie analysis and Lax pair are used to drive the conserved quantities for the HD hierarchy. Lie symmetry method is also applied to study Painlev´e test for the family of HD hierarchy. It is found that for this family the Painlev´e property can be at most sufficient for integrability, but not necessary. This fact has been proved earlier for just first member of this family.

Keywords: Lie analysis, conservation laws, hereditary operator, Painlev´e property, ARS conjecture

1. INTRODUCTION

One of the remarkable integrable equations is Harry-Dym (**HD**) equation

$$u_t \quad u^3 u_{xxx} \tag{1.1}$$

which has a large number of diverse applications of physical and mathematical disciplines. The integrality of (1.1) in soliton theory is a well known fact. Its cusp solutions followed by construction the Gel'fand-Levitan-Marchenko integral equation has successfully been found by Wadati *et.al* (1980) [2].

This equation has a strong link with Korteweg-de Vries (**KdV**) equation. For example, Ibragimov (1985) [4] and Hereman *et.al* (1989) [11] show how (1.1) is transformed to **KdV** equation where its applications were found in several physical problems including hydrodynamics, for example see Vosconcelos and Koclanoff (1991)[10].

Lax representation of an nonlinear integrable equation is the first step to find the general *n*-soliton solutions via an standard procedure so-called Inverse Scattering Transform (**IST**) where an solution obtained at a nonzero time based on an initial solution. This method has successfully been employed to construct the soliton solutions of several integrable equations, *e.g.*, nonlinear Schrodinger (**NLS**) equation Yang (2010) [1], and recently matrix complex modified Korteweg-de Vries equation by Ahmadi zeidabadi and Hoseini (2013) [3]. This linear formulation allows us to construct most general hierarchy related to the integrable equation. The procedure is also called as Ablowitz-Kaup-Newll-Segur (**AKNS**) hierarchy.

Lax pair also provides the means to construct a recursive relation to find the conservation laws. Using a method due to Wadati *et al* [9], the recursion of the conserved quantities for NLS equation can easily be found without any knowledge of IST, *i.e.*, see Yang (2010) [1]. It is well known fact that HD related Lax representation is associated with Sturm-Lioville operator.

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Lie Analysis is a strong tools to construct the vector fields and therefore symmetries for the ordinary and partial differential equations (**ODE**, **PDE**)'s which are essential for checking the Painlev´e property, for more detail of the procedure we refer the reader to [7].

Painlev´e property deals with ODEs with no critical and movable singularities. In other words, an **ODE** satisfies the property if all its critical and movable singularities are absent. In some sense, it has been shown that this property came to be synonymous to integrability, for more details see 5.

The connection between the Painlev´e property and when integrability is considered in terms of Lax representation (*i.e.* IST procedure) is known as AblowitzRamani-Segur (**ARS**) conjecture [13]. **ARS** conjecture states that integrability of an **PDE** is related to the Painlev´e property of its ODE reductions. At least for (1.1), it has been shown that this relation is necessary, but not sufficient.

The paper is organized as: 2 describes the construction of the HD hierarchy based on the Lax pair formulation. In 3 we apply the important concepts of Lie symmetry analysis to find the symmetry groups for HD hierarchy. 4 deals with the reduction HD hierarchy to some ODE's which are essential to study the Painlev´e property. As a conclusion, in 5 we show that the reduced ODE's related to HD hierarchy do not satisfy in Painlev´e property

2. LAX PAIR AND CONSERVATION LAWS

The present section consists of construction of the HD hierarchy where its Lax representation is well known. The procedure is an standard method in soliton theory known as **AKNS** hierarchy. This hierarchy was firstly found for **NLS** and similar methods have been employed for other integrable systems *i.e.* see Yang (2010) [1]. One of the main results of the present section is to construct a strong symmetry and therefore a hereditary operator for HD family using the Lax pair structure.

2.1 HD hierarchy and hereditary operator

It is well known that the expressing a PDE in the Lax pair is a sufficient condition for integrability, *i.e.*, IST method can be applied to find the most general soliton solutions. The IST has been successfully applied for several integrable system of equations. As one of the remarkable results of extracting the soliton solutions via IST, the conservation quantities including the fluxes and density can be constructed. The most general solitary wave solutions for HD equation (1.1) have been studied via IST successfully by Wadati *et.al* (1980) [2]. Here we summarize some facts and more results which can be used to construct the conservation laws. As the first step the HD equation should be considered as the compatibility condition for a pair of the first-order PDE's. The Lax matrix representation for HD is

$$Y_x \quad MY, \qquad Y_t \quad NY. \tag{2.1}$$

where the zero curvature $M_t - N_x + [M, N] = 0$, is the same as the HD equation (1.1). The explicit forms of the matrices *M* and *N* are

$$M = \begin{pmatrix} 0 & 1 \\ \lambda/u^2 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} -2\lambda u_x & 4\lambda u \\ \frac{4\lambda^2}{u} - 2\lambda u_{xx} & 2\lambda u_x \end{pmatrix},$$

(2.2)

where λ is the (complex) spectral parameter. Analogous to AKNS hierarchy to NLS, one can construct new integrable higher-order hierarchies for HD equation. These equations can also be solved via IST, knowing the Lax pair.

For this purpose we consider the Lax pair (2.1) in more general from

$$M = \begin{pmatrix} 0 & 1 \\ \lambda/u^2 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.3}$$

where the functions in N are functions of u and its x-derivatives. Applying the compatibility condition yields

$$A = \frac{1}{2}(D_0 - B_x), \quad C = \frac{\lambda}{u^2}B - \frac{1}{2}B_{xx}, \quad A = \frac{1}{2}(D_0 + B_x), \tag{2.4}$$

where $\boldsymbol{D_0}$ is a constant and the evolution equation is

$$\frac{u_t}{u^3} = -\frac{B_x}{u^2} + \frac{1}{4\lambda}B_{xxx} + \frac{u_x}{u^3}B. \tag{2.5}$$

As the general form (2.5) is independent of the constant D_0 then we can simply set $D_0 = 0$. Expanding the function B in the λ power series

$$B = -4u\sum B_j(-\lambda)^{n-j-1}, \tag{2.6}$$

Determines

$$u_t = u^3(uB_n)_{xxx}, \tag{2.7}$$

where the coefficients $\mathbf{B_j}$ satisfy the recursive relation

$$B_{j+1} = \int_{-\infty}^{x} u\,(uB_j)_{xxx}dx, \quad j = 1, \dots, n-1, \tag{2.8}$$

and $\boldsymbol{B_1} = 1$. As for instance

$$u_t = u^3 u_{xxx}, \tag{2.9}$$

$$u_t = -\frac{1}{4}u^3\left(u^2 u_{xx} - \frac{1}{2}u u_x^2\right)_{xxx} \tag{2.10}$$

are constructed for $n = 1$ and $n = 2$. Note that the choice

$$B = -4\sum_{j=1}^{n} B_j(-\lambda u)^{n-j+1}, \tag{2.11}$$

yields the same hierarchy (2.7) and (2.8).

Interestingly enough, combining (2.7) and (2.8) gives the *strong* symmetry

$$u_t = \Phi^n u_x, \quad \Phi = -\frac{1}{4}\left(u^2D^2 - uu_xD + uu_{xx} + u^3u_{xxx}D^{-1}\frac{1}{u^2}\right). \tag{2.12}$$

where the operator D is normal derivative respect to the independent variable x, and

$$(D^{-1}f)(x) = \int_{-\infty}^{x} f(x)\,dx, \tag{2.13}$$

where the asymptotic condition $u(\pm\infty) = 1$ has been utilized. Indeed Φ is a *hereditary* operator. For more details and the proof refer to [6], where the operator (2.12) has been constructed by the Lie-Bu¨ckland symmetry approach.

More general form of (2.5) is found when the spectral parameter is considered as 0 a function of the temporal variable t ($\lambda'(t) \neq 0$) *i.e.*

$$\frac{u_t}{u^3} = -\frac{1}{2}u^2\frac{\lambda_t}{\lambda} - \frac{B_x}{u^2} + \frac{1}{4\lambda}B_{xxx} + \frac{u_x}{u^3}B\,. \tag{2.14}$$

By a similar procedure and

$$\lambda_t = \sum_{j=0}^{n-1}\kappa_j(t)\,(-\lambda)^{n-j+1}, \tag{2.15}$$

where the recursive formulation for $\kappa_j(t)$ can easily be yielded.

For (2.10) the linearized operator of $K(u) = -\frac{1}{4}u^3(u^2u_{xx} - \frac{1}{2}uu_x^2)_{xxx}$ is defined by

$$\begin{aligned} K'[u]\sigma = \frac{1}{8}u^2[&10u^2u_{xxxxx} + 15u_x^2u_{xxx} + 40uu_{xxxx}u_x + 40uu_{xxx}u_{xx} \\ &+(10u^2u_{xxxx} + 10uu_xu_{xxx})D + 10u^2u_{xxx}D^2 \\ &+(10u^2u_{xx} + 5uu_x^2)D^3 + 10u^2u_xD^4 + 12u^3D^5]\sigma, \end{aligned} \tag{2.16}$$

which is the Gateaux derivation of K at u in the direction σ *i.e.*

$$K'[u]\sigma = \frac{\partial}{\partial\varepsilon}K(u+\varepsilon\sigma)|_{\varepsilon=0}. \tag{2.17}$$

2.2. Conservation Laws

Analogous to the novel method due to Wadati to construct an infinite number of conserved quantities for an integrable equation, we will extract the conservation laws for the HD family (2.7) from its Lax pair, requiring to solve a *Ricatti* equation.

Denoting $Y = (y_1,y_2)^T$ and $\mu = y_1/y_2$, Lax pair (2.1) and (2.3) are simplified to

$$(\ln y_2)_x = \frac{\lambda}{u^2}\mu, \tag{2.18}$$
$$(\ln y_2)_t = C\mu + D. \tag{2.19}$$

Cross-differentiating with respect t and x, respectively, yields

$$\left(\frac{\lambda}{u^2}\mu\right)_t = (C\mu)_x + D_x. \tag{2.20}$$

On the other hand, the first equation of Lax pair gives the Riccati equation

$$\mu_x + \frac{\lambda}{u^2}\mu^2 = 1\,.$$

(2.21)

Substituting Laurant series

$$\mu = u^2\sum_{i=-1}^{\infty}\mu_i\lambda^{-\frac{i}{2}-1}, \tag{2.22}$$

in (2.20)-(2.21) and zero asymptotic conditions, the conservation law is determined as

$$\frac{d}{dt}\int_{-\infty}^{\infty}\mu_{2s-1}dx = [-2u^2(B_n)_{xx}\mu_{2s-1} + 4(-1)^n uB_1\mu_{2(n+s)-1} +$$
$$2u\sum_{j=s}^{n+s-2}(-1)^{j+s}\mu_{2j+1}(u(B_{n+s-j-1})_{xx} - 2B_{n+s-j})]|_{-\infty}^{+\infty}, \tag{2.23}$$
$$\frac{d}{dt}\int_{-\infty}^{\infty}\mu_{2s}dx = [-2u^2(B_n)_{xx}\mu_{2s} + 4(-1)^n uB_1\mu_{2(n+s)}$$
$$2u\sum_{j=s}^{n+s-2}(-1)^{j+s}\mu_{2(j+1)}(u(B_{n+s-j-1})_{xx} - 2B_{n+s-j})]|_{-\infty}^{+\infty},\ s = 0,1,...,$$

(2.24)

And

$$\mu_{-1} = \frac{1}{u},$$
$$\mu_j = \frac{u}{2}\left[(\mu_{j-1})_x + \sum_{i=0}^{j-1}\mu_i\mu_{j-i-1}\right] - u_x\mu_{j-1},\ j = 0,1,\ldots, \tag{2.25}$$

which gives a recursion for μ_j. By inspection, it is not difficult to show from (2.25) that the asymptotic conditions at infinities for μ_j except for μ_{-1} are zero. For μ_{-1} the second terms (2.23)-(2.24) are constants and therefore vanished. Thus right hand sides of (2.23)-(2.24) are zero which means that μ_j are indeed densities.

3. LIE ALGEBRA AND PAINLEV´E ANALYSIS

Recently, several authors studied Lie algebra analysis and its applications for PDEs including HD equation (1.1). It is a well known fact that (1.1) does not possess the Painlev´e property, which indicates that the Painlev´e property can be at most sufficient for integrability, but not necessary. This is so-called ARS conjecture. Here in this section we will show that the same phenomena occurs for HD hierarchy (2.7). To start with, we use the Lie algebra to find the reduced ODEs for (2.7). We refer the reader to see the complete discussion in reference [7] and the references therein.

3.1. Symmetry groups for HD Hierarchy

Symmetry Lie analysis can be performed to find the vector fields and therefor the corresponding one-parameter group related to the general HD equation (2.7). To facilitate a clear understanding of the analysis we demonstrate the procedure for the first-order HD (2.9), whenever its needed.

We begin with the following proposition which can be proved by induction initiated from

$$B_1 = 1,\quad B_2 = \frac{1}{8}u_x^2 - \frac{1}{4}uu_{xx}. \tag{3.1}$$

Proposition:The recursive operator B_j, defined by (2.8) is homogenous of order $2(j-1)$ respect to u and x, that is;

a) $B_j(\alpha u(x,t)) = \alpha^{2(j-1)}B_j(u(x,t))$,

b) $B_j(u(\alpha x,t)) = \alpha^{2(j-1)}B_j(u(x,t))$, $\alpha \in R$.

b) shows that the derivative maximum-order respect to the space variable x in (2.7) is $2n+1$, and hence the vector field related to symmetries of HD hierarchy (2.7) is

$$v = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}, \tag{3.2}$$

where the functions ξ,τ and ϕ are determined so that $pr^{(2n+1)}v[4(x,u^{(2n+1)})] = 0$, where $4(x,u^{(2n+1)})$ is the single PDE defined by (2.7). Consequently, the corresponding one-parameter group$exp(\epsilon v)$ will be a symmetry group of the HD equation (2.7). Here the vector field is defined on $X \times U = R^2 \times R$. Applying $pr^{(2n+1)}v$ to

(2.7), the infinitesimal criterion is explicitly determined. For (2.9), this criterion is

$$\phi^t - u^3\phi^{xxx} - 3u^2 u_{xxx} = 0, \tag{3.3}$$

where the coefficients ϕ^t and ϕ^{xxx} are

$$\begin{aligned} \phi^t &= D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt} \\ &= \phi_t + (\phi_u - \tau_t)u_t - \xi_t u_x - \xi_u u_x u_t - \tau_u u_t^2, \\ \phi^{xxx} &= D_x^3(\phi - \xi u_x - \tau u_t) + \xi u_{xxxx} + \tau u_{xxxt}. \end{aligned} \tag{3.4}$$

Substituting the general formulae (3.4) into (3.3), replacing u_t by (2.7) whenever it occurs, and equating the confinements of the various monomials in the first and higher order partial derivatives of u, we find the equations for the symmetry group in a general form to be

$$\tau = c_1 t + c_2, \quad \xi = \frac{1}{2}c_3 x^2 + c_4 x + c_5, \quad \phi = c_3 x u + c_4 u - \frac{c_1 u}{2n+1}, \tag{3.5}$$

where the constants c_i are arbitrary and then the linearly independent vector fields for (2.7) as

$$v_1 = \partial_t, \ v_2 = \partial_x, \ v_3 = -(2n+1)t\partial_t, \ v_4 = x\partial_x + u\partial_u. \tag{3.6}$$

Moreover, the extra vector field $v_5 = \frac{1}{2}x^2\partial_x + xu\partial_u$ is also obtainable among (3.6), which is not applicable in Lie analysis here. The one-parameter groups G_i generated

by the v_i, determined via $\exp(\epsilon v_i)(x,t,u) = (\tilde{x},\tilde{t},\tilde{u})$ are

$$\begin{aligned} &G_1: (x, t+\varepsilon, u), && G_2: (x+\varepsilon, t, u), \\ &G_3: \left(x, te^{-(2n+1)\varepsilon}, ue^{\varepsilon}\right), && G_4: (xe^{\varepsilon}, t, ue^{\varepsilon}). \end{aligned}$$

(3.7)

Therefor more extended solutions of (2.7) will be in the forms

$$\begin{aligned} &u^{(1)} = f(x, t+\varepsilon), && u^{(2)} = (x+\varepsilon, t), \\ &u^{(3)} = \varepsilon f\left(x, te^{-(2n+1)\varepsilon}\right), && u^{(4)} = \varepsilon^{-1}(x\varepsilon, t). \end{aligned} \tag{3.8}$$

if $u = f(x,t)$ is a solution of (2.7), where is any real number. Importantly, although the extended solution (3.8) has been constructed by inspection, above proposition can be easily used to show that it is a solution of (2.7) for any natural number n.

The most convenient way to display the structure of a given Lie algebra is to write it in tabular form. The commutator table for (3.6) will be 4×4 table, whose (i,j)- th entry expresses the Lie bracket $[v_i,v_j]$. Note that the table is always skew-symmetric since $[v_i,v_j] = -[v_j,v_i]$; in particular, the diagonal entries are all zero;

Table 1. The commutator table for (3.6).

[,]	v_1	v_2	v_3	v_4
v_1	0	0	$-(2n+1)v_1$	0
v_2	0	0	0	v_2
v_3	$(2n+1)v_1$	0	0	0
v_4	0	$-v_2$	0	0

Thus, for example,

$$[v_1,v_3] = \{v_1(-(2n+1)t)\partial_t + v_1(u)\partial_u\} - \{v_3(1)\partial_t\} = -(2n+1)v_1.$$

Note that due to the obstacle of the integral in hereditary Φ in (2.12), the general form for vector fields for HD can not explicitly determined, as found for KdV equation in [8]

These solutions for (2.9) are the time- and space-invariant $u^{(1)}$ and $u^{(2)}$ and

$$u^{(3)} = \epsilon f(x, t\epsilon^3), \quad u^{(4)} = f(x\epsilon, t\epsilon^3),$$

The groups G_4 and G_3 demonstrate the time- and space-invariance of the equation, The well-known scaling symmetry turns up in G_2, and G_1 represents a kind of Galilean boost. The most general one-parameter group of symmetries is obtained by considering a general linear combination $c_1v_1 + \cdots + c_4v_4$ of the given vector fields; we can represent an arbitrary group transformation g as the composition of transformations in the various one- parameter subgroups $G_1,\dots,G_4$. In particular, if g is near the identity, it can be represented uniquely in the form

$$g = exp(\epsilon v_1) \cdot exp(\epsilon v_2) \cdot exp(\epsilon v_3) \cdot exp(\epsilon v_4).$$

4. SIMILARITY REDUCTIONS OF THE HD EQUATION

In this section we use the method of characteristics to determine the invariants and reduced ODEs corresponding to each subalgebra given in (3.6).

Similarity reduction corresponding to the symmetry generator $v_4 - v_3$, for an example, is obtained by solving the characteristic equations

$$\frac{dx}{x} = \frac{dt}{-(2n+1)t}$$

Integration of these ordinary differential equations yields

$$u = F(\zeta), \qquad \zeta = xt^{-\frac{1}{2n+1}},$$

and satisfies the ordinary differential equation

$$\zeta F' + (2n+1)F^3(FB_n)''' = 0. \tag{4.1}$$

Where

$$B_1 = 1, \quad B_{n+1} = \int_{-\infty}^{x} F(FB_n)'''. \tag{4.2}$$

We summarize the results of similarity reductions related to the vector fields in (3.6) in the following table;

5. PAINLEV´E ANALYSIS AND ARS CONJECTURE

An ODE is called to possess Painlev´e property if all its critical (which are multivalued) and movable (*i.e.* depend on initial condition) are absent. It has been shown [12] that the construction of the solution of the Painlev´e equations (with Painlev´e property) can be very complicated, but can importantly be done. In other words, they can be integrated; they are integrable in the sense that can be solved in closed form. The connection between the property and other integrability contexts, including algebraic integrability and linearisability (construction a system of linear

Table 2. The similarity reductions for (3.6).

Generator	Point transformations	Invariant solution	Similarity reduction
v_1 :	$\tilde{x} = x, \tilde{t} = t + \epsilon$, $\tilde{u} = u$	$F(t)$	$F^{0} = 0$
v_2 :	$\tilde{x} = x + \epsilon, \tilde{t} = t$, $\tilde{u} = u$	$F(x)$	$F3(FBn)000 = 0$
v_3 :	$\tilde{x} = x, \tilde{t} = te^{-(2n+1)\epsilon}$, $\tilde{u} = ue^{\epsilon}$	no invariant solution	
v_4 :	$\tilde{x} = xe^{\epsilon}, \tilde{t} = t$, $\tilde{u} = ue^{\epsilon}$	no invariant solution	
$v_4 - v_3$:	$\tilde{x} = xe^{\epsilon}, \tilde{t} = te^{-(2n+1)\epsilon}$, $\tilde{u} = u$	$F(\xi)$	$\xi F^{0} + (2n + 1)F^{3}(FB_n)^{000} = 0$

equation from a given system via local transformations) has been studied by several authors.

Is there any connection between an integrable in terms of IST technique PDE and Painlev´e property? Recently, Ablowitz, Ramani and Segur [13] conjectured that ODE obtained by an exact reduction of an integrable equation may satisfy in Painlev´e property, and vice versa. For several integrable equations it has been shown that the ARS conjecture holds in both directions. But several authors explained that the Painlev´e property can be at most sufficient for integrability. They refer to the HD equation (1.1) as a familiar example in this context. We shall apply a similar procedure for the HD hierarchy (2.7) in next subsection, using the leading-order analysis.

5.1. The leading-order analysis

The ARS algorithm proceeds in three steps, dealing with the dominant behaviors, the resonances and the compatibility conditions at resonances, respectively [13]. In the leading-order analysis, it is sufficient to substitute

$$F(\zeta) = \beta g(\zeta)^{\alpha}, \tag{5.1}$$

where β is a constant, in (4.2) to determine the leading exponent α. It can be substituted by $g(\zeta) = (\zeta - \zeta_0)$ for convenience, where ζ_0 is a pole. In the resulting polynomial system, equating every two or more possible lowest exponents of $g(\zeta)$ in each equation gives a linear system for α. The proposition in 3.1 or a direct induction illustrates that

$$B_{j+1}[\beta(\zeta - \zeta_0)^{\alpha}] = A_{j+1}(\zeta - \zeta_0)^{\gamma}, \quad \gamma = 2j(\alpha - 1), \tag{5.2}$$

where A_{j+1} is a function of α and β. Now, applying the same procedure for (4.1) and equating the exponents results

$$\alpha = \frac{2n}{2n+1}, \tag{5.3}$$

which is a positive non-integer number. The traditional Painlev´e test requires that all the α's are integers and that at least one is negative. Therefore the equation (2.7) does not pass the Painlev´e test. Meanwhile, the package PDEPtest in *Mathematica* Baldwin and Hereman (2006) [15] has been used to confirm the above results. However, a suitable change of variables in (4.1), we can use the "weak" Painlev´e test[13].

Furthermore, it is also noted that Clarkson and Kruskal [13] proposed a direct method to find some new similarity reductions which can not be tracked by the standard Lie group method. But here in this paper, the reduction needed for Painlev´e analysis has been found through the regular prolongation theory based on the Lie algebra.

6. CONCLUSION

The Harry-Dym (HD) hierarchy is studied from the Lax pair and Lie symmetry analysis view points. The hereditary operator as a generator to construct the most general form of HD hierarchy is also found. Based on the prolongation theory, the vector field related to symmetries of HD hierarchy is defined on $R^2 \times R$ and determined via standard procedure. These vector fields are used to find the similarity reductions which are essential for Painlev´e analysis.

The main results of the present paper is to show that the Ablowitz-Ramani-Segur (ARS) conjecture is at most sufficient for integrability for HD hierarchy. This fact previously has been shown for just first member of the family.

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