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Lie Symmetry Method for Solutions of Differential Equations with Applications in Physics

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Abstract. A mathematical method in pure mathematics (differential geometry) for finding solutions of differential equations is considered. The method is based on constructing a Lie algebra associated to a given system of differential equation, called Lie algebra of the symmetries of the given system. This Lie algebra is a vector space which maps a given solution, such as a constant solution, to another solution, it is a significant tool for finding new solution for system of differential equation specially partial differential equations. Then we will apply it to some differential equations in fluid mechanics and physics.

Keywords: Differential equations, Fluid Mechanics, boundary layers, Newtonian fluid, Flows of vector fields, heat transfer equation.

1. INTRODUCTION

System of differential equations, specially partial differential equations have a vast applications in study all of engineering sciences, physics, biology, applied and pure mathematics and etc., in other words differential equations discuss how a phenomenon has been done [[1],[3]]. When we have a system of differential equations the last and important aim is to finding the solutions of the given system. If the system is ordinary i.e., we have one variable, there are many methods for finding solutions, such as Riccati equation, complete equation, homogeneous equation and etc., but a lot of problem appeared when the given system is partial kind. Usually in this situation there is no any rule for finding the solutions explicitly. This paper introduces a method, an algorithmic method, to solve the system by an initial solution such as boundary solutions [[5]]. As we said, this method is based on pure mathematics, thus it needs a lots of mathematical and geometric foundations. Here we use a clear explanation to describe it with out of any details [[6],[7],[8]]. The main object is vector fields which is constructed in differential geometry and has many applications in physics and engineering sciences. In this paper we work on a special kind of vector fields called symmetry of differential equations that give us a large set of solutions of a given system of differential equations. But we should know that the symmetries of differential equations do not limited to this kind of symmetries; which is called *point symmetry*. We have almost contact symmetry, mirror symmetry, potential symmetry, classical symmetry, generalized symmetry and…, but point symmetries is more applicable than the others. The symmetry set of a system of differential equations is the largest set of transformations acting on the independent and dependent variables of the system with the property that it transform solutions of the system to other solutions. The main goal of this paper is to introduce a useful, systematic, computational method that will explicitly determine the symmetry of given system of differential equations. Applications in some mechanical important partial differential equation are presented in the sequel [[2](Bluman J.W., Kumei S, 1989)].

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2. PROBLEM FORMULATION

First we begin by a definition of a system of differential equations. An n-th order of a system of m-differential equations with with p-independent variables and q-dependent variable

can be written in the form of

$$\Delta_\gamma\left(x, u^{(n)}\right) = 0, \qquad \gamma = 1, \ldots, m, \tag{1}$$

where $u^{(n)}$ denote the derivatives of u respect to x up to order n . For example consider the equation for the conduction of heat in a one-dimensional rod $u_t = u_{xx}$ As the definition above this equation will be written such as $\Delta(x,t,u;u_x,u_t,u_{xx},u_{xt},u_{tt}) = u_t - u_{xx} = 0.$

Suppose

$$X = \sum_{i=1}^{p} \alpha^i(x,u)\frac{\partial}{\partial x^i} + \sum_{j=1}^{q} \varphi_j(x,u)\frac{\partial}{\partial u^j}, \tag{2}$$

is a vector field corresponds to the equation (1), where α^i 's and φ_j 's are some differentiable function of independent and dependent variables.The n-th prolongation of this vector field is a vector field

$$X^{(n)} = X + \sum_{j=1}^{q}\sum_{J} \varphi_j^J(x,u^{(n)})\frac{\partial}{\partial u_J^j}, \tag{3}$$

where $J = (j_1,\ldots,j_k)$ is a multi indices and $1 \le j_k \le p, 1 \le k \le n$ [6]. The differential term φ_j^J ,called coefficients of the prolongation, are obtained by the formula

$$\varphi_j^J(x,u^{(n)}) = D_J\left(\varphi_j - \sum_{i=1}^{p} \alpha^i u_{J,i}^j\right), \tag{4}$$

Where $u_i^j = \dfrac{\partial u^j}{\partial x^i}$ and $u_{J,i}^j = \dfrac{\partial u_J^j}{\partial x^i}$. The operator $D_J = D_{x^{i_1}} \ldots D_{x^{i_k}}$ is a a differential operator defined as

$$D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{j=1}^{q}\sum_{J} u_{J,i}^j \frac{\partial}{\partial u_J^j} \tag{5}$$

For example for the one-dimensional heat transfer equation the operators (5) are:

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + u_{xxx}\frac{\partial}{\partial u_{xx}} + u_{xxt}\frac{\partial}{\partial u_{xt}} + u_{xtt}\frac{\partial}{\partial u_{tt}} \tag{6}$$

$x = (x^1, \ldots, x^p)$

$u = (u^1, \ldots, u^q)$

$$D_t = \frac{\partial}{\partial t} + u_x\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + u_{xxt}\frac{\partial}{\partial u_{xx}} + u_{xtt}\frac{\partial}{\partial u_{xt}} + u_{ttt}\frac{\partial}{\partial u_{tt}} \tag{7}$$

The first step for finding symmetries of differential equations is to calculate the prolongation of the vector field (2). Consider a vector field $X = u\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial u}$ defined on plane. If u is a function of x we have

$$X^{(2)} = X + \varphi^x \frac{\partial}{\partial u_x} + \varphi^{xx} \frac{\partial}{\partial u_{xx}}, \tag{8}$$

where φ^x and φ^{xx} are coefficients of the prolongation obtained from (4). These coefficients are

$\varphi^x = D_x(\varphi + uu_x) - uu_x = 1 + u_x^2,\ \varphi^{xx} = D_x^2(\varphi + uu_x) - uu_{xxx} = 3u_x u_{xx},$ thus,

$$X^{(2)} = u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_x} + 3u_x u_{xx}\frac{\partial}{\partial u_{xx}}.$$

According to a theorem in mathematics [6], vector field (2) is a symmetry of the system (1) if

$$X^{(n)} = \left\{ \Delta_\gamma \left(x, u^{(n)}\right)\right\} = 0, \quad \gamma = 1, \ldots, m. \tag{9}$$

After expanding this equations and eliminating any dependencies among the derivatives of the u^i caused by the system itself (since (9) need only hold on solutions of the system), we can then equate the coefficients of the remaining unconstrained partial derivatives of to zero. This will result in a large number of elementary partial differential equations for the coefficients functions α^i , φ_j of the vector field (2), called the *defining equations* or *determining equations* for the symmetry set of the given system. The general transformations of the solutions itself can then be found by exponentiating the given vector fields. The process will become clearer in the sequel.

3. FLOW OF A VECTOR FIELD

Suppose $x = (x^1, \ldots, x^p)$ be a coordinate on the space M. A general form of a vector field on this space can be written as

$$X = \alpha_1(x)\frac{\partial}{\partial x^1} + \ldots + \alpha_p(x)\frac{\partial}{\partial x^p}. \tag{10}$$

The flow of this vector field is a curve, a differentiable vector function respect to parameter ε, which obtained by solving the following system of ordinary differential equations with an initial value,

$$\frac{dx^i}{d\varepsilon} = \alpha^i, \quad i = 1, \ldots, p$$
$$x(0) = x_0. \tag{11}$$

For example the flow of the vector field $X = u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}$ is the curve, $(x(\varepsilon), u(\varepsilon)) = (x\cos\varepsilon - u\sin\varepsilon, x\sin\varepsilon + u\cos\varepsilon)$, which is a solution of the system $\frac{dx}{d\varepsilon} = u, \qquad \frac{du}{d\varepsilon} = -x.$

It is clear that this curve represents the parametric equation of a circle centered at origin in a plane. In other words vector field $X = u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}$ is a rotating vector field. Now consider the ordinary differential equations $u' = -x/u$. A simple calculations shows that the circle $x^2 + u^2 = c^2$ is the solution of this equation. It means if we move on the flow of the vector field $X = u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}$ then we are on the graph of the solutions of the equation $u' = -x/u$. thus X is a symmetry of this equation, and its flow is the transformation that transforms the solutions of this equation to another solutions [6,8]. In the next sections we will apply these results to some differential equations in fluid mechanics.

3. EULER EQUATIONS

As a first illustration of the basic method of computing symmetry sets, we consider the system of Euler equations for the motion of an inviscid, incompressible ideal fluid in three dimensional domain [6]. Here there are four independent variables, $\mathrm{x} = (x, y, z)$ being spatial coordinates and t the time, together with four dependent variables, the velocity field $\mathrm{x} = (x, y, z)$ and the pressure p. (The density ρ is normalized to be 1.) In vector notation, the system has the form

$$\frac{\partial \mathrm{u}}{\partial t} + \langle \mathrm{u}, \nabla \mathrm{u} \rangle = -\nabla p, \qquad (12)$$
$$\langle \nabla, \mathrm{u} \rangle = 0,$$

in which the components of the nonlinear terms $\langle \mathrm{u}, \nabla \mathrm{u} \rangle$ are $(uu_x + vu_y + wu_z, uv_x + vv_y + wv_z, uw_x + vw_y + ww_z)$.

Here $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the gradient vector respect to $\mathrm{x} = (x, y, z)$ and $\langle\ \rangle$ is the inner product notation.

A symmetry vector field of the system (12) will be vector field of the form

$$X = \alpha^1 \frac{\partial}{\partial x} + \alpha^2 \frac{\partial}{\partial y} + \alpha^3 \frac{\partial}{\partial z} + \alpha^4 \frac{\partial}{\partial t} + \varphi_1 \frac{\partial}{\partial u} + \varphi_2 \frac{\partial}{\partial v} + \varphi_3 \frac{\partial}{\partial w} + \varphi_4 \frac{\partial}{\partial p} \qquad (13)$$

where α^i 's and φ_j 's are some differentiable function of $\mathrm{x}, \mathrm{t}, \mathrm{u}$ and p. Applying the first prolongation (13) to the Euler equations (12), (because the system is of order one.) we find the following system of the determining equations

$$\varphi_1{}^t + u\varphi_1{}^x + v\varphi_1{}^y + w\varphi_1{}^z + u_x\varphi_1 + u_y\varphi_2 + u_z\varphi_3 = 0, \qquad (14)$$

$$\varphi_2{}^t + u\varphi_2{}^x + v\varphi_2{}^y + w\varphi_2{}^z + v_x\varphi_1 + v_y\varphi_2 + v_z\varphi_3 = 0, \qquad (15)$$

$$\varphi_3{}^t + u\varphi_3{}^x + v\varphi_3{}^y + w\varphi_3{}^z + w_x\varphi_1 + w_y\varphi_2 + w_z\varphi_3 = 0, \qquad (16)$$

$$\varphi_1{}^x + \varphi_2{}^y + \varphi_3{}^z = 0, \qquad (17)$$

which must be satisfied whenever u and p satisfy (12). Here $\varphi_1{}^t, \varphi_2{}^x$, etc. are the coefficients of the first order derivatives $\partial/\partial u_t$, $\partial/\partial v_x$, etc. appearing in $X^{(1)}$, typical expression for these functions follow from the prolongation formula (3), so $\varphi_1{}^t = D_t\varphi_1 - u_x D_t\alpha^1 - u_y D_t\alpha^2 - u_z D_t\alpha^3 - u_t D_t\alpha^4$,

and $\varphi_2{}^x = D_x\varphi_2 - v_x D_x\alpha^1 - v_y D_x\alpha^2 - v_z D_x\alpha^3 - v_t D_x\alpha^4$. Since (14) – (17) need only hold on solutions of (12), we can substitute for p_x, p_y, p_z and w_z wherever they occur in (14) – (17) using their expression from the four equations (12). We may then equate all the coefficients of the remaining first order derivatives of u and p in (14) – (17) and solve the resulting system of determining equations for α^i 's and φ_j 's. This system is

$$(\varphi_1)_p = \alpha_x^4, \quad a_p^1 = u\alpha_p^4, (\varphi_2)_p = \alpha_y^4, \quad \alpha_p^2 = v\alpha_p^4, (\varphi_3)_p = \alpha_z^4, \quad \alpha_p^3 = w\alpha_p^4,$$

$$\alpha_{yt}^1 = \alpha_{zt}^1 = \alpha_{xt}^2 = \alpha_{zt}^2 = \alpha_{xt}^3 = \alpha_{yt}^3 = 0,\ \alpha_{xt}^1 = \alpha_{yt}^2 = \alpha_{zt}^3 = \alpha_{tt}^4,\ \alpha_{tt}^1 = -(\varphi_4)_x,\ \alpha_{tt}^2 = -(\varphi_4)_y,\ \alpha_{tt}^3 = -(\varphi_4)_z, \quad (18)$$

Solve this system leads us to the coefficients of the vector field (13):

$$\begin{aligned}
&\alpha^1 = \delta_t x + c_1 y - c_2 z + \mu,\ \alpha^2 = -c_1 x + \delta_t y + c_3 z + \beta,\\
&\alpha^3 = c_2 x - c_3 y + \delta_t z + \gamma, \alpha^4 = 2\delta + c_4 t + c_5,\\
&\varphi_1 = -(\delta_t + c_4)u + c_1 v - c_2 w + \mu_t, \varphi_2 = -c_1 u - (\delta_t + c_4)v + c_3 w + \beta_t,\\
&\varphi_3 = c_2 u - c_3 v + (\delta_t + c_4)w + \gamma_t, \varphi_4 = -2(\delta_t + c_4)p - \frac{1}{2}\delta_{tt}(x^2 + y^2 + z^2) -\\
&\mu_{tt} x - \beta_{tt} y - \gamma_{tt} z + \theta,
\end{aligned} \quad (19)$$

in which $\mu, \beta, \gamma, \delta$ and θ are functions of t, and c_1, c_2, c_3, c_4, c_5 are arbitrary constants. Finally the divergence-free condition (17) imposes the further restriction that $\delta_{tt} = 0$, so $\delta = c_6 t + c_7$. We have thus shown that the symmetry set of the Euler equations in three dimension is generated by the vector fields

$$X_\mu = \mu\frac{\partial}{\partial x} + \mu_t\frac{\partial}{\partial u} - \mu_{tt}\frac{\partial}{\partial p}, \quad (20)$$

$$X_\beta = \beta\frac{\partial}{\partial y} + \beta_t\frac{\partial}{\partial v} - \beta_{tt}\frac{\partial}{\partial p}, \quad (21)$$

$$X_\gamma = \gamma\frac{\partial}{\partial z} + \gamma_t\frac{\partial}{\partial w} - \gamma_{tt}\frac{\partial}{\partial p}, \quad (22)$$

$$X_t = \mu\frac{\partial}{\partial t}, \quad (23)$$

$$X_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}, \quad (24)$$

$$X_2 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} - 2p\frac{\partial}{\partial p}, \quad (25)$$

$$X_{xy} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \tag{26}$$

$$X_{zx} = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} + u\frac{\partial}{\partial w} - w\frac{\partial}{\partial u}, \tag{27}$$

$$X_{yz} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}, \tag{28}$$

$$X_{\theta} = \theta\frac{\partial}{\partial p}. \tag{29}$$

The corresponding flows of symmetries of the Euler equations are then:

1. Transformation to an arbitrarily moving coordinate system:
$T_{\mu} : (x,t,u,p) \mapsto \left(x + \varepsilon\mu(t), t, u + \varepsilon\mu_t, p - \varepsilon\langle x, \mu_{tt}\rangle - \frac{1}{2}\varepsilon^2\langle \mu, \mu_{tt}\rangle \right),$
2. where $\mu = (\mu, \beta, \gamma)$ and T_{μ} is generated by the linear combination $X_{\mu} = X_{\mu} + X_{\beta} + X_{\gamma}$, of the first three vector fields.
3. The translation $T_t : (x,t,u,p) \mapsto (x, t+\varepsilon, u, p),$
4. Scale transformations:

$$T_1 : (x,t,u,p) \mapsto (\lambda x, \lambda t, u, p), T_2 : (x,t,u,p) \mapsto (x, \lambda t, \lambda^{-1}u, \lambda^{-2}p),$$

where $\lambda = e^{\varepsilon}$ is a multiplicative parameter.

5. If R is a rotational matrix in three dimension domain (The set of these matrices denoted by $SO(3) = \{ R_{3\times 3} : RR^T = I_3, \det R = 1\}$ where R^T is transpose of matrix R), then the flow of the vector fields (26) – (27) simultaneously is $SO(3) : (x,t,u,p) \mapsto (Rx, t, Ru, p),$
6. Pressure changes $T_{\theta} : (x,t,u,p) \mapsto (x, t, u, p + \varepsilon\theta(t))$. The corresponding action of these transformations on solutions of the Euler equations says that if $u = f(x,t)$ and $p = g(x,t)$ are solutions, so are

$$T_{\mu} : u = f(x - \varepsilon\mu(t), t) + \varepsilon\mu_t, \quad p = g(x - \varepsilon\mu(t), t) - \varepsilon\langle x, \mu_{tt}\rangle - \frac{1}{2}\varepsilon^2\langle \mu, \mu_{tt}\rangle, \quad T_t : u = f(x-\varepsilon), = g(x - \varepsilon\mu(t), \quad p = g(x, t-\varepsilon),$$

$$T_1 : u = f(\lambda x, \lambda t), \quad p = g(\lambda x, \lambda t), T_2 : u = \lambda f(x, \lambda t), \quad p = \lambda^2 g(x, \lambda t),$$

$$SO(3) : u = Rf(R^{-1}x, t), \quad p = g(R^{-1}x, t), T_{\theta} : u = f(x, t), \quad p = g(x,t) + \varepsilon\theta(t).$$

Note that in our change to a moving coordinate system T_{μ} , we must adjust the pressure according to the inherently assumed acceleration $\varepsilon\mu_{tt}$. The final transformation T_{θ} results from the fact that the pressure p is only defined up to the addition of an arbitrary function of t . This complete the list of symmetries and solutions of Euler equations.

Thus with an initial solution of the system (12) such as constant solutions $u = (c_1, c_2, c_3)$ and $p = c_4$ we have some following new solutions:

$$T_{\mu} : u = (c_1, c_2, c_3) + \varepsilon\mu_t, \quad p = c_4 - \varepsilon\langle x, \mu_{tt}\rangle - \frac{1}{2}\varepsilon^2\langle \mu, \mu_{tt}\rangle,$$

$T_t : u = (c_1, c_2, c_3), \quad p = c_4, \ T_1 : u = (c_1, c_2, c_3), \quad p = c_4$
$T_2 : u = \lambda(c_1, c_2, c_3), \quad p = \lambda^2 c_4, SO(3) : u = R(c_1, c_2, c_3), \quad p = c_4$
$T_\theta : u = (c_1, c_2, c_3), \quad p = c_4 + \varepsilon\theta(t).$

4. SOURCELESS HEAT TRANSFER EQUATION

Consider the sourceless heat transfer equation which is represented in the form of:

$$\frac{\partial u}{\partial t} = a\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right), \tag{30}$$

where a is a constant. In this equation u is a function of (r, t), so that the symmetry of the equation (30) is a vector filed of the form

$$X = \alpha^1 \frac{\partial}{\partial r} + \alpha^2 \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial u}, \tag{31}$$

and we must find the coefficients of this vector field to obtain the symmetries. Because of the order of this equation, we should prolong (31) up to order two and by a same process as Euler equations we obtain the following determining equations for (30):

$$\begin{aligned} &\alpha_u^1 = \alpha_u^2 = \varphi_{uu} = 0, \alpha_r^1 - \alpha_t^2 = \alpha_t^1 - a\alpha_r^2 = \alpha_{tt}^1 - a\alpha_{rr}^2 = \alpha_t^1 + r\varphi_{tu} = 0, \\ &\alpha^1 - r\alpha_r^1 - r^2\varphi_{ru} = r\varphi_{tt} - 2ar\varphi_r - ar\varphi_{rr} = 0, \end{aligned} \tag{32}$$

solving this equations leads us to the symmetries:

$$X_1 = \frac{\partial}{\partial t}, \tag{33}$$

$$X_2 = u\frac{\partial}{\partial u}, \tag{34}$$

$$X_3 = \frac{r}{2}\frac{\partial}{\partial r} + t\frac{\partial}{\partial t}, \tag{35}$$

$$X_4 = \frac{\partial}{\partial r} - \frac{4}{r}\frac{\partial}{\partial u}, \tag{36}$$

$$X_5 = -t\frac{\partial}{\partial r} + \frac{(2at + r^2)u}{2ar}\frac{\partial}{\partial u}, \tag{37}$$

$$X_6 = \frac{rt}{2}\frac{\partial}{\partial r} + \frac{t^2}{2}\frac{\partial}{\partial t} - \frac{(6at + r^2)u}{8a}\frac{\partial}{\partial u}. \tag{38}$$

Solutions of the system (11) give us the flow of the vector fields (33) – (38) as follows:

$$\mathrm{T}_1:(r,\mathrm{t},u)\mapsto(\mathrm{r},\mathrm{t}+\varepsilon,u),\mathrm{T}_2:(r,\mathrm{t},u)\mapsto(\mathrm{r},t,e^{\varepsilon}\mathrm{u}),\mathrm{T}_3:(r,\mathrm{t},u)\mapsto(e^{\varepsilon}r,e^{\varepsilon}\mathrm{t},u),\mathrm{T}_4:(r,\mathrm{t},u)\mapsto\left(\mathrm{r}+\varepsilon,\mathrm{t},\frac{\mathrm{ru}}{\mathrm{r}+\varepsilon}\right),$$

$$\mathrm{T}_5:(r,\mathrm{t},u)\mapsto\left(-\varepsilon t+r,t,-\exp\left\{\frac{2\varepsilon r-\varepsilon^2 r}{4a}\right\}\frac{ur}{\varepsilon t-r}\right),\mathrm{T}_6:(r,\mathrm{t},u)\mapsto\left(\frac{2r}{2-\varepsilon t},\frac{2t}{2-\varepsilon t},u\left(\frac{\varepsilon t-2}{2}\right)^{3/2}\exp\left\{\frac{\varepsilon r^2}{4a(\varepsilon t-2)}\right\}\right),$$

If is $u=f(r,t)$ a solution of the equation (30), so are:

$$\mathrm{T}_1:u{=}f(\mathrm{r},\mathrm{t}-\varepsilon),\mathrm{T}_2:u{=}e^{\varepsilon}f(\mathrm{r},\mathrm{t}),\mathrm{T}_3:u{=}f\left(e^{-\varepsilon/2}\mathrm{r},e^{-\varepsilon}\mathrm{t}\right),\mathrm{T}_4:u{=}\frac{r-\varepsilon}{r}f(r-\varepsilon,\mathrm{t}),$$

$$\mathrm{T}_5:u{=}\frac{r+\varepsilon t}{r}\exp\left\{\frac{2\varepsilon r+\varepsilon^2 t}{4a}\right\}f(\varepsilon t+\mathrm{r},\mathrm{t}),\mathrm{T}_6:u{=}\left(\frac{-2}{2+\varepsilon t}\right)^{3/2}\exp\left\{\frac{-\varepsilon r^2}{8a(\varepsilon t+2)}\right\}f\left(\frac{2r}{2+\varepsilon t},\frac{2t}{2+\varepsilon t}\right).$$

Suppose $u=c$ is a constant solution of (30), then some interesting solutions obtained from this simple one respect to $\mathrm{T}_4,\mathrm{T}_5$ and T_6 are:

$$u=c\frac{r-\varepsilon}{r},u=c\frac{r+\varepsilon t}{r}\exp\left\{\frac{2\varepsilon r+\varepsilon^2 t}{4a}\right\},\text{ and } u=c\left(\frac{-2}{2+\varepsilon t}\right)^{3/2}\exp\left\{\frac{-\varepsilon r^2}{8a(\varepsilon t+2)}\right\}.$$

5. NEWTONIAN INCOMPRESSIBLE FLUID’S EQUATIONS FLOW IN TURBULENT BOUNDARY LAYARS

In physics and fluid mechanics, a boundary layer is that layer of fluid in the immediate vicinity of a bounding surface. In the Earth's atmosphere, the planetary boundary layer is the air layer near the ground affected by diurnal heat, moisture or momentum transfer to or from the surface. On an aircraft wing the boundary layer is the part of the flow close to the wing. The boundary layer effect occurs at the field region in which all changes occur in the flow pattern. The boundary layer distorts surrounding non-viscous flow. It is a phenomenon of viscous forces. This effect is related to the Reynolds number (In fluid mechanics and heat transfer, the Reynold's number is a dimensionless number that gives a measure of the ratio of inertial forces to viscous and, consequently, it quantifies the relative importance of these two types of forces for given flow conditions). Laminar boundary layers come in various forms and can be loosely classified according to their structure and the circumstances under which they are created. The thin shear layer which develops on an oscillating body is an example of a Stokes boundary layer, whilst the Blasius boundary layer refers to the well-known similarity solution for the steady boundary layer attached to a flat plate held in an oncoming unidirectional flow. When a fluid rotates, viscous forces may be balanced by the Coriolis effect, rather than convective inertia, leading to the formation of an Ekman layer. Thermal boundary layers also exist in heat transfer. Multiple types of boundary layers can coexist near a surface simultaneously. The deduction of the boundary layer equations was perhaps one of the most important advances in fluid dynamics. Using an order of magnitude analysis, the well-known governing Navier-Stokes equations of viscous fluid flow can be greatly simplify within the boundary layer. Notably, the characteristic of the partial differential equations becomes parabolic, rather than the elliptical form of the full Navier-Stokes equations. This greatly simplifies the solution of the equations. By making the boundary layer approximation, the flow is divided into an inviscid portion (which is easy to solve by a number of methods) and the boundary layer, which is governed by an easier to solve partial differential equations. Flow and heat transfer of an incompressible viscous fluid over a stretching sheet appear in several manufacturing processes of industry such as the extrusion of polymers, the cooling of metallic plates, the aerodynamic extrusion of plastic sheets, etc. In the glass industry, blowing, floating or spinning of fibres are processes, which involve the flow due to a stretching surface. Mahapatra and Gupta studied the steady two-dimensional stagnation-point flow of an incompressible viscous fluid over a flat deformable sheet when the sheet is stretched in its own plane with a velocity proportional to the distance from the stagnation-point. They concluded that, for a fluid of small kinematic viscosity, a boundary layer is formed when the stretching velocity is less than the free stream velocity and an inverted boundary layer is formed when the stretching velocity exceeds the free stream velocity. Temperature distribution in the boundary layer is determined when the surface is held at constant temperature giving the so called surface heat flux. In their analysis, they used the finite-differences scheme along with theThomas algorithm to solve the resulting system of ordinary differential equations. The treatment of turbulent boundary layers is far more difficult due to the time-dependent variation of the flow properties. One of the most widely used techniques in which turbulent flows are tackled is to apply Reynolds decomposition. Here the instantaneous flow properties are decomposed into a mean and fluctuating component. Applying this technique to the boundary layer equations gives the full turbulent boundary layer equations

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}=0,$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\upsilon\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{v}}{\partial y^2}\right)-\frac{\partial \overline{u'v'}}{\partial y}-\frac{\partial \overline{u'^2}}{\partial x}, \tag{39}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y}+\upsilon\left(\frac{\partial^2 \bar{v}}{\partial x^2}+\frac{\partial^2 \bar{v}}{\partial y^2}\right)-\frac{\partial \overline{u'v'}}{\partial x}-\frac{\partial \overline{u'^2}}{\partial y},$$

where ρ is the density, p is the pressure, υ is the kinematic viscosity of the fluid at a point and $\bar{u}$ and $\bar{v}$ are average of the velocity components in Reynold decomposition. Here u' and v' are the velocity fluctations such that; $u=\bar{u}+u'$ and $v=\bar{v}+v'$ .By using the *scale analysis* (a powerful tool used in the mathematical sciences forthe simplification of equations with many terms), it can be shown that the system (39) reduce to the classical form

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}=0, \bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\upsilon\frac{\partial^2 \bar{u}}{\partial y^2}-\frac{\partial \overline{u'v'}}{\partial y}, \frac{\partial \bar{p}}{\partial y}=0, \tag{40}$$

The term $\overline{u'v'}$ in the system (40) called *Reynolds shear stress*, a tensor that conventionally written

$$\mathrm{R}_{\mathrm{ij}} \equiv \rho\overline{u'v'}. \tag{41}$$

The divergence of this stress is the force density on the fluid due to the *turbulent fluctuations*. Using Navier�Stokes equations for a fluid whose stress versus rate of strain curve is linear and passes through the origin (*Newtonian fluid*) the tensor(41) reduces to $\mathrm{R}_{\mathrm{ij}} \equiv \mu\frac{\partial \bar{u}_i}{\partial x_j}$, where μ is the fluid viscosity, thus the last term inthe second equation of (40) is $\mu \partial\bar{u}/\partial y$. According to the system (40) the general form of the symmetry for this system is

$$X=\alpha^1\frac{\partial}{\partial x}+\alpha^2\frac{\partial}{\partial y}+\varphi_1\frac{\partial}{\partial \bar{u}}+\varphi_2\frac{\partial}{\partial \bar{v}}+\varphi_3\frac{\partial}{\partial \bar{p}}. \tag{42}$$

After prolonging this vector field up to order two determining equations obtained as follows:

$$\alpha^1_y = \alpha^1_{\bar u} = \alpha^1_{\bar v} = \alpha^1_{\bar p} = \alpha^1_{xx} = \alpha^2_x = \alpha^2_{\bar u} = \alpha^2_{\bar v} = \alpha^2_{\bar p} = \alpha^2_{yy} = \varphi_{1\bar u} = \varphi_{1\bar v} = \varphi_{1\bar p} = \varphi_{1x} = \varphi_{1y} = 0$$

$$\varphi_{1\bar p} = 2\alpha^1_x - 4\alpha^2_y, \qquad \varphi_2 = \bar u(\alpha^1_x - 2\alpha^2_y), \qquad \varphi_3 = -\alpha^2_y \bar v,$$

and consequently symmetries are

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial \bar p}, \qquad X_3 = x\frac{\partial}{\partial x} + \bar u \frac{\partial}{\partial \bar u} + 2\bar p \frac{\partial}{\partial \bar p}, \qquad X_4 = y\frac{\partial}{\partial y} - 2\bar u \frac{\partial}{\partial \bar u} - \bar v \frac{\partial}{\partial \bar v} - 4\bar p \frac{\partial}{\partial \bar p}.$$

For finding the solutions we should fine flows of these vector fields;

$$\begin{aligned} &\mathrm{T}_1 : (x, y, \bar u, \bar v, \bar p) \mapsto (\mathrm{x} + \varepsilon, y, \bar u, \bar v, \bar p), \mathrm{T}_2 : (x, y, \bar u, \bar v, \bar p) \mapsto (x, y, \bar u, \bar v, \bar p + \varepsilon), \\ &\mathrm{T}_3 : (x, y, \bar u, \bar v, \bar p) \mapsto (\lambda \mathrm{x}, y, \lambda \bar u, \bar v, \lambda^2 \bar p), \mathrm{T}_4 : (x, y, \bar u, \bar v, \bar p) \mapsto (\mathrm{x}, \lambda y, \lambda^{-2} \bar u, \lambda^{-1} \bar v, \lambda^{-4} \bar p), \end{aligned} \tag{43}$$

where $\lambda = e^{\varepsilon}$. If $\mathrm{u} = (\bar u, \bar v), \mathrm{u} = f(x, y)$ and $\bar p = g(x, y)$ be solutions of the system, so the transformations of the solutions are

$$\begin{aligned} &\mathrm{T}_1 : \mathrm{u} = \mathrm{f}(x - \varepsilon, y), \qquad p = g(x - \varepsilon, y), \mathrm{T}_2 : \mathrm{u} = \mathrm{f}(x, y), \qquad p = g(x, y) + \varepsilon, \\ &\mathrm{T}_3 : \bar u = \lambda f(\lambda^{-1} x, y), \quad \bar v = f(\lambda^{-1} x, y), \quad \bar p = \lambda^{-2} g(\lambda^{-1} x, y), \\ &\mathrm{T}_4 \bar u = \lambda^{-2} f(x, \lambda^{-1} y), \quad \bar v = \lambda^{-1} f(x, \lambda^{-1} y), \quad \bar p = \lambda^{-4} g(x, \lambda^{-1} y). \end{aligned}$$

6. CONCLUSION

The illustrated method is due to a conception in pure mathematics based on the transformations that transforms given solutions to another solutions which is a powerful tools for differential equations and has a lot of applications in applicable sciences. Some times when the variables are more that three or four we need to use some calculating soft wares such as Maple and Mathematica. It is useful to say that this method has not limited and could be applied to any kind of differential equations.

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