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L^p Solutions of G-Backward Stochastic Differential Equations with Continuous Coefficients

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Abstract. In this paper, we study G-backward stochastic differential equations with continuous coefficients. We give existence and uniqueness results for G-backward stochastic differential equations, when the generator f is uniformly continuous in (y, z), and the terminal value $\xi \in L_G^p(\mathcal{F})$ with $1 < p \leq 2$.
We consider the G-backward stochastic differential equations driven by a G-Brownian motion $(B_t)_{t\geq 0}$ in the following form:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)d\langle B\rangle_s - \int_t^T Z_s dB_s - (K_T - K_t) \quad (1)$$

where Y, Z and K are unknown and the random function f, called the generator, and the random variable ξ, called terminal value, are given. Our main result of this paper is the existence and uniqueness of a solution (Y, Z, K) for (1) in the G-framework.

Keywords: G-expectation, G-Brownian motion, G-martingale, G-Backward stochastic differential equations, L^p solution.

1. INTRODUCTION

Let $(\Omega, \mathfrak{F}, P)$ be a probability space carrying a 1-dimensional Brownian motion $(B_t)_{t\geq 0}$, and $(\mathfrak{F}_t)_{t\geq 0}$ be the filtration generated by $(B_t)_{t\geq 0}$. It is well known that $(B_t)_{t\geq 0} = \omega_t$ for $\omega \in \Omega$, is a standard Brownian motion under P. A classical Backward Stochastic Differential Equation (BSDE) is an equation of the following type

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad (1.1)$$

where g is a given function, called the generator of (1.1) and ξ is a given $\mathfrak{F}_T$-measurable random variable called the terminal condition. The solution of (1.1) consists of a pair of adapted processes (Y, Z).

Note that the above classical BSDE is based on a probability space framework. Recently, there is one motivation to drive BSDEs and the corresponding time-consistent nonlinear expectations to develop ahead beyond the probability space framework. It is that the classical BSDE can only provide a probabilistic interpretation of a PDE for quasi linear but not for fully nonlinear cases.

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In order to overcome the above shortcomings of classical BSDEs, Peng systemically established a time-consistent fully nonlinear expectation theory. The notion of time-consistent fully nonlinear expectations was first introduced in [2, Peng 2004] and [3, Peng 2005].

As a typical case, Peng (2006) introduced G-expectation (see [4] and the references therein). Under G-expectation framework (G-framework), a new type of Brownian motion called G-Brownian motion was constructed and the corresponding stochastic calculus of Ito's type was established. The existence and uniqueness of solution of a SDE driven by G-Brownian motion can be proved in a way parallel to that in the classical SDE theory. But the BSDE driven by G-Brownian motion $(B_t)_{t\geq 0}$ becomes a challenging problem.

Just as in the classical case, the G-martingale representation theorem is the key to solve a BSDE in this G-framework. For a dense family of G-martingales, Peng [5] obtained the following result: a G-martingale M has the form

$$M_t = M_0 + \bar{M}_t + K_t,$$

$$\bar{M}_t := \int_0^t Z_s dB_s, \qquad K_t := \int_0^t \eta_s \langle B \rangle_s - \int_0^t 2G(\eta_s) ds.$$

Here M is decomposed into two incompatible G-martingales. The first one $\bar{M}$ is called the symmetric G-martingale. That is, $-\bar{M}$ is also a G-martingale. The second one K is quite unusual since it is a decreasing process.

Due to the above G-martingale representation theorem, a natural formulation of a BSDE driven by G-Brownian motion consists of a triple of processes (Y, Z, K), satisfying (1).

We review some basic notions and results of G-expectation and the related spaces of random variables. The readers may refer to [4,5,6,7,8] for more details.

2. NONLINEAR EXPECTATIONS

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real functions defined on Ω containing 1, namely $\mathcal{H}$ is a linear space such that $1 \in \mathcal{H}$ and that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. $\mathcal{H}$ is a space of random variables. We assume the functions on $\mathcal{H}$ are all bounded.

Definition 2.1 [8]. A nonlinear expectation $\mathbb{E}$ is a functional $\mathcal{H} \to \mathbb{R}$ satisfying the following properties

a) Monotonicity: if $X, Y \in \mathcal{H}$ and $X \geq Y$ then $\mathbb{E}[X] \geq \mathbb{E}[Y]$,
b) Preservation of constants: $\mathbb{E}[c] = c$,
c) Subadditivity: $\mathbb{E}[X] - \mathbb{E}[Y] \leq \mathbb{E}[X - Y]$, $\forall X, Y \in \mathcal{H}$,
d) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda\, \mathbb{E}[X]$, $\forall \lambda \geq 0, X \in \mathcal{H}$,
e) $\mathbb{E}[X + c] = \mathbb{E}[X] + c$.

Let n is a positive integer, we denote by $lip(\mathbb{R}^n)$ the space of all bounded and Lipschitz real functions on $\mathbb{R}^n$. In this section we consider $\Omega = \mathbb{R}$ and $\mathcal{H} = lip(\mathbb{R})$.

In classical linear situation, a random variable X with standard normal distribution, i.e., $X \sim \mathcal{N}(0,1)$, can be characterized by

$$\mathcal{E}[\phi(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \phi(x) dx, \qquad \forall \phi \in lip(\mathbb{R}).$$

We know from Bachelier 1900 and Einstein 1950 that $\mathcal{E}[\phi(X)] = u(1,0)$ where $u = u(t,x)$ is the solution of the heat equation

$$\partial_t u = \frac{1}{2} \partial_{xx}^2 u,$$

with Cauchy condition $u(0,x) = \phi(x)$.

In this paper we set $G(a) = \frac{1}{2}(a^+ - \sigma_0^2 a^-), a \in \mathbb{R}$ where $\sigma_0 \in [0,1]$ is fixed.

Definition 3.1 [6]. $X \in \mathcal{H}$ with **G-normal distribution** (with mean at $x \in \mathbb{R}$ and variance $t > 0$) is characterized by its G-expectation defined by

$$\mathbb{E}[\phi(x + \sqrt{t}X)] = P_G^1 \phi(x + \sqrt{t}X) := u(t,x),$$

Where $\phi \in lip(\mathbb{R})$ and $u = u(t,x)$ is a bounded continuous function on $[0,\infty) \times \mathbb{R}$ which is the solution of the following G-heat equation:

$$\partial_t u - G(\partial_{xx}^2 u) = 0, \quad u(0,x) = \phi(x).$$

We denote

$$P_G^t(\phi)(x) := P_G^1\left(\phi\left(x + \sqrt{t} \times .\right)\right) = u(t,x), \quad (t,x) \in [0,\infty) \times \mathbb{R}.$$

3. 1–DIMENSIONAL G-BROWNIAN MOTION

In this section we use some notations and definitions of [6].

Let $\Omega = C_0(\mathbb{R}^+)$ be the space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$ with $\omega_0 = 0$. For any $\omega^1, \omega^2 \in \Omega$ we define

$$\rho(\omega^1,\omega^2) = \sum_{i=1}^{\infty} 2^{-i}\left[\left(\max_{t\in[0,i]}|\omega_t^1 - \omega_t^2|\right)\wedge 1\right].$$

We set, for each $t \in [0,\infty)$

$W_t := \{\omega_{\wedge t} : \omega \in \Omega\}$,

$\mathcal{F}_t := \mathcal{B}_t(W) = \mathcal{B}(W_t)$,

$$\mathcal{F}_{t^+} := \mathcal{B}_{t^+}(W) = \bigcap_{s>t} \mathcal{B}_s(W),$$

$\mathcal{F} = \vee_{s>t}\mathcal{F}_s = \sigma(\cup_{s>t}\mathcal{F}_s)$.

Then $(\Omega, \mathcal{F})$ is the canonical space with the natural filtration. This space is used throughout the rest of this paper.

For each fixed $T \geq 0$, we consider the following space of random variables

$$l_{ip}^0(\mathcal{F}_t) := \left\{X(\omega) = \phi\left(\omega_{t_1}, \ldots, \omega_{t_m}\right), \forall m \geq 1, \quad t_1, \ldots, t_m \in [0,T], \forall \phi \in lip(\mathbb{R}^m)\right\}.$$

Obviously, it holds $l_{ip}^0(\mathcal{F}_t) \subseteq l_{ip}^0(\mathcal{F}_T)$, for any $t \leq T < \infty$. We further define

$l_{ip}^0(\mathcal{F}) := \cup_{n=1}^{\infty} l_{ip}^0(\mathcal{F}_n)$.

We will consider the canonical space and set $B_t(\omega) = \omega_t, t \in [0,\infty)$, for $\omega \in \Omega$.

Definition 3.1 [6]. The canonical process B is called a G–Brownian motion under a nonlinear expectation $\mathbb{E}$ defined on $l_{ip}^0(\mathcal{F})$ if for each $T > 0$, $m = 1, 2, \ldots$, and for each $\phi \in lip(\mathbb{R}^m)$, $0 \leq t_1 < \cdots < t_m \leq T$, we have

$$\mathbb{E}\left[\phi\left(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}\right)\right] = \phi_m,$$

where $\phi_m \in \mathbb{R}$ is obtained via the following procedure:

$$\phi_1(x_1, \dots, x_{m-1}) = P_G^{t_m - t_{m-1}}(\phi(x_1, \dots, x_{m-1}, .)),$$

$$\phi_2(x_1, \dots, x_{m-2}) = P_G^{t_{m-1} - t_{m-2}}(\phi_1(x_1, \dots, x_{m-2}, .)),$$

$$\vdots$$

$$\phi_{m-1}(x_1) = P_G^{t_2 - t_1}(\phi_{m-2}(x_1, .)),$$

$$\phi_m = P_G^{t_1}(\phi_{m-1}(.)).$$

The related conditional expectation of $X = (B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})$ under $\mathcal{F}_{t_j}$ is defined by

$$\mathbb{E}\left[X|\mathcal{F}_{t_j}\right] = \mathbb{E}\left[\phi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})|\mathcal{F}_{t_j}\right]$$

$$= \phi_{m-j}\left(B_{t_1}, \dots, B_{t_j} - B_{t_{j-1}}\right).$$

It is proved in [10] that $\mathbb{E}[.]$ consistently defines a nonlinear expectation on the vector lattice $l_{ip}^0(\mathcal{F}_T)$ as well as on $l_{ip}^0(\mathcal{F})$ satisfying (a)–(e) in Definition 2.1. It follows that $\mathbb{E}[|X|]$ where $X \in l_{ip}^0(\mathcal{F}_T)$ (resp. $l_{ip}^0(\mathcal{F})$) forms a norm and that $l_{ip}^0(\mathcal{F}_T)$ (resp. $l_{ip}^0(\mathcal{F})$) can be continuously extended to a Banach space, denoted by $L_G^1(\mathcal{F}_T)$ (resp. $L_G^1(\mathcal{F})$).

Definition 3.2 [6]**.** The expectation $\mathbb{E}[.] : L_G^1(\mathcal{F}) \to \mathbb{R}$ introduced through above procedure is called G-expectation. The corresponding canonical process B is called a G–Brownian motion under $\mathbb{E}[.]$.

For a given $p > 1$, we also denote $L_G^p(\mathcal{F}) = \{X \in L_G^1(\mathcal{F}), |X|^p \in L_G^1(\mathcal{F})\}$. $L_G^p(\mathcal{F})$ is also a Banach space under the norm $\| X \|_p := (\mathbb{E}[|X|^p])^{1/p}$.

4. ITO INTEGRAL FOR G-BROWNIAN MOTION

Definition 4.1 [8]. Let $M_G^{p,0}(0,T)$ be the collection of processes in the following form: for a given partition $\pi_T = \{t_0, \dots, t_N\}$ of $[0,T]$,

$$\mu_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathrm{I}_{[t_j, t_{j+1})}(t),$$

where $T \in \mathbb{R}^+$, $p \geq 1$ and $\xi_j \in L_G^p\left(\mathcal{F}_{t_j}\right)$, are given**.**

Definition 4.2 [6]**.** For an $\eta \in M_G^{1,0}(0,T)$ with $\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathrm{I}_{[t_j, t_{j+1})}(t)$, the related Bochner integral is

$\int_0^T \eta_t(\omega)dt = \sum_{j=0}^{N-1}\xi_j(\omega)(t_{j+1}-t_j)$.

Let $\| \eta \|_{H_G^p} = \left[\mathbb{E}\left[\left(\int_0^T |\eta_s|^2 ds\right)^{p/2}\right]\right]^{1/p}$, $\| \eta \|_{M_G^p} = \left[\mathbb{E}\left[\int_0^T |\eta_s|^p ds\right]\right]^{1/p}$ and denote by $H_G^p(0,T)$, $M_G^p(0,T)$ the

completions of $M_G^{p,0}(0,T)$ under the norms $\| \eta \|_{H_G^p}$, $\| \eta \|_{M_G^p}$ respectively.

Let $S_G^{p,0}(0,T) = \{h(t, B_{t_1\wedge t}, \dots, B_{t_n\wedge t}):\ t_1, \dots, t_n \in [0,T],\ h \in lip(\mathbb{R}^{n+1})\}$. For $p \geq 1$ and $\eta \in S_G^{p,0}(0,T)$, set $\| \eta \|_{S_G^p} = \left[\mathbb{E}[sup_{t\in[0,T]}|\eta_t|^p]\right]^{1/p}$. Denote by $S_G^p(0,T)$ the completion of $S_G^{p,0}(0,T)$ under the norm $\| \eta \|_{S_G^p}$.

We call $L_G^p(\mathcal{F}_T)$, $M_G^p(0,T)$, $H_G^p(0,T)$ and $S_G^p(0,T)$ the spaces of the G-framework.

Definition 4.3 [1]. A process $\{M_t\}$ with values in $L_G^1(\mathcal{F}_T)$ is called a G-martingale if $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ for any $s \leq t$.

Definition 4.4 [6]. For each $\eta \in M_G^{2,0}(0,T)$ with the form $\eta_t(\omega) = \sum_{j=0}^{N-1}\xi_j(\omega)\mathrm{I}_{[t_j,t_{j+1})}(t)$, we define

$$\mathbf{I}(\eta) = \int_0^T \eta(s)dB_s := \sum_{j=0}^{N-1}\xi_j\left(B_{t_{j+1}} - B_{t_j}\right).$$

Lemma 4.5 [8]. The mapping $\mathrm{I}: M_G^{2,0}(0,T) \to L_G^2(\mathcal{F}_T)$ is a linear continuous mapping and thus can be continuously extended to $\mathrm{I}: M_G^2(0,T) \to L_G^2(\mathcal{F}_T)$.

In fact we have

$$\mathbb{E}\left[\int_0^T \eta(s)dB_s\right] = 0, \tag{4.1}$$

$$\mathbb{E}\left[\left(\int_0^T \eta(s)dB_s\right)^2\right] \leq \int_0^T \mathbb{E}\left[(\eta(s))^2\right]ds. \tag{4.2}$$

Definition 4.6 [8]. We define, for a fixed $\eta \in M_G^2(0,T)$, the stochastic integral

$\int_0^T \eta(s)dB_s := \mathbf{I}(\eta)$.

It is clear that (4.1), (4.2) still hold for $\eta \in M_G^2(0,T)$.

5. EXISTENCE OF THE SOLUTIONS

For simplicity, we consider the G-expectation space $(\Omega, L_G^1(\mathcal{F}_T), \mathbb{E})$. We consider the following type of G-BSDEs for simplicity

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t), \tag{5.1}$$

where $f(t,\omega,y,z) : [0,T] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$.

In this section, we shall use the following assumptions:

There exists some $p > 1$ such that

(H1) for any y, z, $f(.,.,y,z) \in M_G^p(0,T)$,

(H2) $|f(t,\omega,y,z) - f(t,\omega,y',z')| \leq L(|y - y'| + |z - z'|)$ for some $L > 0$,

(H3) $|f(t,\omega,y,z)| \leq K(1 + |y| + |z|)$, where K is a positive constant,

(H4) $\forall (t,\omega) \in [0,T] \times \Omega$, $f(t,y,z)$ is continuous in (y,z).

For simplicity, we denote by $\mathfrak{Z}_G^q(0,T)$ the collection of processes (Y,Z,K) such that $Y \in S_G^q(0,T)$, $K \in H_G^q(0,T)$ is a decreasing G-martingale with $K_0 = 0$ and $K_T \in L_G^p(\mathcal{F}_T)$.

Definition 5.1 [1]. Let $\xi \in L_G^p(\mathcal{F}_T)$ and f satisfy (H1) and (H2) for some $p > 1$. A triplet of processes (Y,Z,K) is called a solution of Eq. (5.1) if for some $1 < q \leq p$ the following properties hold:

(a) $(Y,Z,K) \in \mathfrak{Z}_G^q(0,T)$,

(b) $Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dB_s - (K_T - K_t)$.

Theorem 5.2. [1] Assume that $\xi \in L_G^p(\mathcal{F}_T)$ and (H1) and (H2) are satisfied by f for some $p > 1$. Then Eq. (5.1) has a unique solution (Y,Z,K). Moreover, for any $1 < q < p$ we have $Y \in S_G^q(0,T)$, $Z \in H_G^q(0,T)$ and $K_T \in L_G^q(\mathcal{F}_T)$.

Let $p \in (1,2]$ is a given constant in the rest of this paper. In the following theorem we provide the existence of L^p solution for $p = 2$.

Theorem 5.3. If $\xi \in L_G^2(\mathcal{F}_T)$ and (H1), (H3) and (H4) are satisfied by f, then (5.1) has a solution $(Y,Z,K) \in S_G^2(0,T) \times H_G^2(0,T) \times L_G^2(\mathcal{F}_T)$.

6. UNIQUENESS OF THE SOLUTIONS

Now we turn to the uniqueness of the solution of Equation (5.1) when f is uniformly continuous with respect to (y,z).

In this section we consider the following hypothesis on the generator $f: [0,\infty] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$.

(H5) f is uniformly continuous in y uniformly with respect to (t,ω,z), i.e., there exist a continuous non-decreasing function $\psi: \mathbb{R}^+ \to \mathbb{R}^+$, satisfying

$\psi(0) = 0$,

$0 < \psi(x) \leq A(x+1), \ \forall x > 0$,

where A is a positive constant,

$\int_0^\infty [\psi(x)]^{-1} dx = \infty$,

Such that

$|f(t,y,z) - f(t,y',z)| \leq \psi(|y-y'|), \quad \forall t,y,y',z,$ a.s.,

(H6) f is uniformly continuous in z uniformly with respect to (t,ω,y), that is, there exists a continuous function $\varphi: \mathbb{R}^+ \to \mathbb{R}^+$, satisfying

$\varphi(0) = 0$,

$0 < \varphi(x) \leq B(x+1), \quad \forall x > 0$,

where B is a positive constant,

Such that

$|f(t,y,z) - f(t,y,z')| \leq \varphi(|z-z'|), \quad \forall t,y,z,z',$ a.s., (H7) The function $z \to f(t,y,z)$ is Lipschitz continuous uniformly with respect to (t,ω,y).

Theorem 6.1. Let $\xi \in L_G^2(\mathcal{F}_T)$. Suppose $\{f(t,0,0)\}_{t\in[0,T]} \in H_G^2(\mathcal{F}_T)$, If (H5)-(H7) are satisfied by f, then the G-BSDE associated with (f,ξ) has a unique solution $(Y,Z,K) \in S_G^2(0,T) \times H_G^2(0,T) \times L_G^2(\mathcal{F}_T)$.

Theorem 6.2. Suppose (H1), (H3) and (H6) are satisfied by $f: [0,\infty] \times \Omega \times \mathbb{R}^2 \to \mathbb{R}$, if $\xi \in L_G^2(\mathcal{F}_T)$, G-BSDE associated (f,ξ) has a unique solution $(Y,Z,K) \in S_G^2(0,T) \times H_G^2(0,T) \times L_G^2(\mathcal{F}_T)$.

Theorem 6.3. Let $\xi \in L_G^p(\mathcal{F}_T)$. If (H1), (H3), (H5) and (H6) are satisfied by f, there exists a unique solution $(Y,Z,K) \in S_G^p(0,T) \times H_G^p(0,T) \times L_G^p(\mathcal{F}_T)$ which solves eq. (5.1).

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REFERENCES

[1] Hu, M., Ji, S., Peng, S., & Song, Y. (2014). Backward stochastic differential equations driven by G-Brownian motion. Stochastic Processes and their Applications, 124(1), 759-784.

[2] S. Peng, Filtration consistent nonlinear expectations and evaluations of contingent claims, Acta Math. Appl. Sin. 20 (2) (2004) 1–24.

[3] S. Peng, Nonlinear expectations and nonlinear Markov chains, Chin. Ann. Math. B 26 (2) (2005) 159–184.

[4] S. Peng, Nonlinear expectations and stochastic calculus under uncertainty, 2010. arXiv:1002.4546v1 [math.PR].

[5] S. Peng, G-Brownian motion and dynamic risk measure under volatility uncertainty, 2007. arXiv:0711.2834v1 [math.PR].

[6] S. Peng, G-expectation, G-Brownian motion and related stochastic calculus of Ito type, in: Stochastic Analysis and Applications, in: Abel Symp., vol. 2, Springer, Berlin, 2007, pp. 541–567.

[7] S. Peng, Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation, Stochastic Process. Appl. 118 (12) (2008) 2223–2253.

[8] S. Peng, A new central limit theorem under sublinear expectations, 2008. arXiv:0803.2656v1 [math.PR].

[9] Oksendal, B.(2003). Stochastic differential equations (pp20-26).Springer Berlin Heidelberg.

[10] Cheridito, P., Soner, H.M., Touzi, N. and Victoir, N., Second order backward stochastic differential equations and fully non-linear parabolic PDEs, Preprint (pdf-file available in arXiv: math.PR/0509295 v1 14 Sep 2005).