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G- Brownian motion and Its Applications

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Abstract. The concept of G-Brownian motion and G-Ito integral has been introduced by Peng. Also Ito isometry lemma is proved for Ito integral and Brownian motion. In this paper we first investigate the Ito isometry lemma for G-Brownian motion and G-Ito Integral. Then after studying of $M_G^{2,0}$-class functions [4], we introduce Stratonovich integral for G-Brownian motion,say G- Stratonovich integral. Then we present a special construction for G-Stratonovich integral.

Keywords: G-expectation, G-Brownianmotion, Characterization, Ito integral, G-Stratonovich.

1. INTRODUCTION

The concept of G-Brownian motion is a very important concept in financial mathematics. With G-Brownian motion, G-Ito integral for $M_G^{2,0}$ calassfunction has been introduced in [2,3,4,5]. In this paper we introduce G-Stratonovich integral for $M_G^{2,0}$-Class functions. In the sequel we present a characterization for G-Stratonovich in integral which we define.

2. NONLINEAR EXPECTATIONS

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real functions defined on Ω containing 1, namely $\mathcal{H}$ is a linear space such that $1 \in \mathcal{H}$ and that $X \in \mathcal{H}$ implies $|X| \in \mathcal{H}$. $\mathcal{H}$ is a space of random variables. We assume the functions on $\mathcal{H}$ are all bounded.

Definition 2.1. [4] A non linear expectation $\mathbb{E}$ is a functional $\mathcal{H} \to \mathbb{R}$ satisfying the following properties

a) Monotonicity: if $X, Y \in \mathcal{H}$ and $X \geq Y$ then $\mathbb{E}[X] \geq \mathbb{E}[Y]$,
b) Preservation of constants: $\mathbb{E}[c] = c$,
c) Subadditivity $\mathbb{E}[X] - \mathbb{E}[Y] = \mathbb{E}[X - Y]$, $\forall X, Y \in \mathcal{H}$,
d) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall \lambda \geq 0$, $X \in \mathcal{H}$.
e) $\mathbb{E}[X + c] = \mathbb{E}[X] + c$.

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3. G-NORMAL DISTRIBUTIONS

For a given positive integer n, we will denote by (x, y) the scalar product of $x, y \in \mathbb{R}^n$ and by $|x| = (x, x)^{1/2}$ the Euclidean norm of x. We denote by lip($\mathbb{R}^n$) the space of all bounded and Lipschitz real functions on$\mathbb{R}^n$. We introduce the notion of nonlinear distribution– G–normal distribution. A G–normal distribution is a nonlinear expectation defined on lip($\mathbb{R}^d$) (here $\mathbb{R}^d$ is considered as Ω and lip$\mathbb{R}^d$) as $\mathcal{H}$):

$$P_1^G(\emptyset) = u(1,0): \emptyset \in Lip(\mathbb{R}^d) \to \mathbb{R}$$

where u = u(t, x) is a bounded continuous function on $[0,\infty) \times \mathbb{R}^d$ which is the viscosity solution of the following nonlinear parabolic partial differential equation (PDE)

$$\frac{du}{dt}\text{-}G(D^2u)=0\,,\ u(0,x)=\emptyset(x)\ ,\ (t,x)\in [0,\infty) \times \mathbb{R}^d, \qquad (1)$$

hereD^2u is the Hessian matrix of u, i.e., $D^2u=(\partial^2_{x^i x^j}u)^d_{i,j=1}$ and

$$G(A)= G_\tau(A) =\frac{1}{2}\sup \operatorname{tr}[\gamma\gamma^T A\ ],\ A=(A_{ij})^d_{i,j=1} \in \mathbb{S}_d, \qquad (2)$$

$\mathbb{S}_d$denotes the space of d × d symmetric matrices. τ is a given non empty, bounded and closed subset of $\mathbb{R}^{d\times d}$,the space of all $d \times d$ *matrices*.

3.1. Dimensional G-Brownian motion under G-expectation

In this section we use some definitions and notions of [2,4].

Let $\Omega = C_0(\mathbb{R}^+)$ be the space of all $\mathbb{R}$ -valued continuous paths $(\omega_t)_{t\in\mathbb{R}^+}$with $\omega_0 = 0$. For any

$\omega^1, \omega^2 \in \Omega$ we define

$$\rho(\omega^1,\omega^2) = \sum_{i=1}^{\infty} 2^{-i}\left[\left(\max_{t\in[0,i]}|\omega_t^1 - \omega_t^2|\right)\wedge 1\right].$$

We set, for each $t \in [0,\infty)$

$$W_t := \{\omega_{\wedge t} : \omega \in \Omega\},$$

$$\mathcal{F}_t := \mathcal{B}_t(W) = \mathcal{B}(W_t),$$

$$\mathcal{F}_{t^+} := \mathcal{B}_{t^+}(W) = \bigcap_{s>t} \mathcal{B}_s(W),$$

$$\mathcal{F} := \vee_{s>t}\mathcal{F}_s.$$

Then $(\Omega,\mathcal{F})$ is the canonical space with the natural filtration. This space is used throughout the rest of this paper.

For each fixed T ≥ 0, we consider the following space of random variables

$l_{ip}^{0}(\mathcal{F}_t) := \{X(\omega) = \phi(\omega_{t_1}, \dots, \omega_{t_m}), \forall m \geq 1, \quad t_1, \dots, t_m \in [0, T], \forall \exists \phi \in lip(\mathbb{R}^m)\}$.

Obviously, it holds $l_{ip}^{0}(\mathcal{F}_t) \subseteq l_{ip}^{0}(\mathcal{F}_T)$, for any $t \leq T < \infty$. We further define,

$l_{ip}^{0}(\mathcal{F}) := \cap_{n=1}^{\infty} l_{ip}^{0}(\mathcal{F}_n)$.

We will consider the canonical space and set $B_t(\omega) = \omega_t, t \in [0, \infty)$, for $\omega \in \Omega$.

Definition 3.1. The canonical process GB is called a (d–dimensional) G–Brownian motion under a nonlinear *expectation* $\mathbb{E}$ *defined on* $L_{ip}^{0}(\mathcal{H})$ *if*

(i) For each $s, t \geq 0$ *and* $\psi \in lip(\mathbb{R}^d)$, $GB_t$and $GB_{t+s} - GB_s$ are identically distributed:

$\mathbb{E}[\psi(GB_{t+s} - GB_s)] = \mathbb{E}[\psi(GB_t)] = P_t^G(\psi)$.

(ii)For each $m = 1,2, \dots, 0 \leq t_1 < \dots < t_m < \infty$, the increment $GB_{t_m} - GB_{t_{m-1}}$ is “backwardly” independent from $GB_{t_1}, \dots, GB_{t_{m-1}}$ in the following sense: for each $\emptyset \in lip(\mathbb{R}^{d\times m})$,

$\mathbb{E}[\emptyset(GB_{t_1}, \dots, GB_{t_{m-1}}, GB_{t_m})] = \mathbb{E}[\emptyset_1(GB_{t_1}, \dots, GB_{t_{m-1}}, GB_{t_m})]$

where $\emptyset_1(x^1, \dots, x^{m-1}) = \mathbb{E}[\emptyset(x^1, \dots, x^{m-1}, GB_{t_m} - GB_{t_{m-1}} + x^{m-1})], x^1, \dots, x^{m-1} \in \mathbb{R}^d$.

The related conditional expectation of $\emptyset(GB_{t_1}, \dots, GB_{t_m})$ under $\mathcal{H}_{t_k}$ is defined by

$\mathbb{E}[\emptyset(GB_{t_1}, \dots, GB_{t_k}, \dots, GB_{t_m})|\mathcal{H}_{t_k}] = \emptyset_{m-k}(GB_{t_1}, \dots, GB_{t_k})$,

where

$\emptyset_{m-k}(x^1, \dots, x^k) = \mathbb{E}[\emptyset(x^1, \dots, x^k, GB_{t_{k+1}} - GB_{t_k} + x^k, \dots, GB_{t_m} - GB_{t_k} + x^k)]$.

Definition 3.2.

$M_G^{p,0}(0, T)$={η:η_t(ω)=$\sum_{j=1}^{n-1} \xi_j I_{[t_j, t_{j-1})}$(t), $\forall n > 0, o \leq t_0 \leq \dots \leq t_n$, ξ_i(ω)∈ $L_G(\mathcal{F}_{t_i}), i = 0, \dots, n-1$}[4].

Definition 3.3. [1] In the sequel we assume (Ω, F, P) is a fixed probability space. f(t,ω):[0,∞)×Ω→ $\mathbb{R}$ is belongs to $\mathrm{P}_2 = \mathrm{P}_2(\mathrm{S}, \mathrm{T})$ Class functions set if and only if we have,

(i) $(t, \omega) \to f(t, \omega)$ is $\mathcal{B}\times\mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel $\sigma - \text{field on}$ [0,∞).
(ii) For t∈[0,∞),f(t,.) is $\mathcal{F}_t$-adapted.
(iii) E[$\int_S^T f^2(t, \omega) dt < \infty$, $\forall T \geq 0$.

Remark 3.1. (The It$\hat{o}$ isometry)[1] *let* $\emptyset(t, \omega) \in P_2$ *be bounded and elementary function,*

Then we have

E[($\int_S^T \emptyset(t, \omega)\, dB_t$)^2]=E[$\int_S^T \emptyset(t, \omega)^2 dt$],

where $\int_S^t \emptyset(t, \omega)\, dB_t$ is Itô intgral [1].

Remark 3.2. The isometry lemma for G-Brownian motion is not necessary holded i.e.

There is $\eta \in M_G^2(0,T)$ such that,

$$E\left[\left(\int_S^T \eta(S)\, dGB\right)^2\right] \neq E\int_S^T \eta(S)^2\, dGB$$

4. G-STRATONOVICH (STRATONOVICH INTEGRAL FOR G-BROWNIAN MOTION)

Definition 4.1. For $T \in \mathbb{R}_+$, a partition ρ of $[0,T]$ is a finite ordered subset $\rho = \{t_1, \dots, t_n\}$ such that $0 = t_0 < t_1 < \cdots < t_n = T$.

$$\mu(\rho) = \max\{|t_{j+1} - t_i|, i = 0,1,\dots,N-1\}$$

We use $\mathcal{P}_T^n = \{t_0^n < t_1^n < \cdots < t_N^n\}$ to denote a sequence of partitions of $[0,T]$ such that $\lim_{N\to\infty} \mu(\rho_T^n) = 0$.

For each $f \in M_G^{2,0}(0,T)$

We denote G-Stratonovich integral as following

$$\int_0^T f(t,\omega)\, d(GB) = \lim_{n\to\infty} \sum_{i=1}^n f(t^*,\omega)\,(GB)_{t_i} - (GB)_{t_{i-1}})$$

Where $t^* = \frac{t_j - t_{j-1}}{2}$.

In the fallowing theorem we present a characterization for G-Stratonovich integral.

Theorem 4.1. In the above definition if we choose t^* randomly with the Uniform distribution then the random sequence tends to G-Stratonovich integral when n tends to ∞.

Proof. If we choose t_i^*'s randomly with the Uniform distribution and show the resulting integral with

$$\mathbb{U}^* \int_0^T f(t,\omega)\, d(GB),$$

then it's not difficult to show that

$$\mathbb{E}(\mathbb{U}^* \int_0^T f(t,\omega)\, d(GB) - \int_0^T f(t,\omega)\, d(GB) \,|\, t_i^*\text{'s}),$$

tends to zero, where $\mathbb{E}$ is defined in definition 3.1.■

5. CONCLUSION

For G- Brownian motion, Stratonovich integral which we call it G-Stratonovich integral is definable. Also we presented a random characterization for G-Stratonovich integral.

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REFERENCES

[1] Oksendal,B.(2003). Stochastic differential equations (pp20-26). Springer Berlin Heidelberg.

[2] Peng, S. (2006) G–Expectation, G–Brownian Motion and Related Stochastic Calculus of Itˆo's type, preprint (pdf-file available in arXiv:math.PR/0601035v1 3Jan 2006), to appear in *Proceedings of the 2005, Abel Symposium*.

[3] Peng, S. (2005), Dynamically consistent nonlinear evaluations and expectations, preprint (pdf-file available in arXiv:math. PR/0501415 v1 24 Jan2005).

[4] Peng, S. (2004) Nonlinear expectation, nonlinear evaluations and risk measurs,in K. Back T. R. Bielecki, C. Hipp, S. Peng, W. Schachermayer, *Stochastic Methods in Finance Lectures*, 143–217, LNM 1856, Springer-Verlag.

[5] Peng, S. (2004) Filtration Consistent Nonlinear Expectations and Evaluations of Contingent Claims, *Acta Mathematicae Applicatae Sinica*, English Series 20(2), 1–24.