Cumhuriyet Üniversitesi Fen Fakültesi
Fen Bilimleri Dergisi (CFD), Cilt:36, No: 3 Özel Sayı (2015)
ISSN: 1300-1949

Cumhuriyet University Faculty of Science
Science Journal (CSJ), Vol. 36, No: 3 Special Issue (2015)
ISSN: 1300-1949

Results for nonlinear fractional differential equations on pseudo-metric spaces with three boundary val

Akbar AZAMI[1,*], Hakimeh MOHAMMADI[1]

[1]*Department of mathematics, Myandoab Baranch, Islamic Azad University, Myandoab, Iran*

Received: 01.02.2015; Accepted: 05.05.2015

Abstract. In this paper, we investigate the existence of solutions in Gauge spaces for the nonlinear fractional differential equation boundary value problem

$D_0^{\alpha} + u(t) + f(t, u(t)) = 0, \quad t \in [0, T]$ (1.1)

With different boundary value

$u(O) = u(T) = 0$

and

$u(0) = \beta_1 u(\mu), u(T) = \beta_2 u(\mu)$

and different order 1 < a ≤ 2 and 0 < a < 1. where 0 < /31 < /32 < T and μ ∈ [O, T]and D_0^{α}+ is the standard Riemann-Lowville differentiation with f(t, u(t)) ∈ C([O, T] x [O, ∞), R). By using some fixed-point results on gauge spaces, some existence results of positive solutions are obtained.

Keywords: Boundary value problem, Fixed point, Fractional differential equation, Green function, Positive solution.

1. INTRODUCTION

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, engine erg. For details, see [1]-[6]. It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations in terms of special functions [6]-[8]. Recently, there are some papers that deal with the existence and uniquence of solution of nonlinear initial fractional differential equation by the use of techniques of nonlinear analysis (fixed point theorems, Leray-Shauder theory, etc.); see [9]-[18].

2. THE PRELIMINARY

For the convenience of the reader, we present the necessary definitions from fractional calulus theory. These definitions can be found in the recent literaturs and books (see [8]).

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function f : (0, ∞) → R is given by

* Corresponding author. *Email address: akbar.azami1362@gmail.com*

Special Issue: The Second National Conference on Applied Research in Science and Technology

$$I^{\alpha}f(t) = \frac{1}{T(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds$$

provided the right side is pointwise defind on (0, oo).

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function f: (0, ∞) → R is given by

$$D^{\alpha}f(t) = \frac{1}{T(n-\alpha)}(\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1}f(s)ds.$$

where ti = [α] + 1, provided the right side is pointwise de find on (0, ∞). **Definition 2.3.** An altering distance function is a function: [O, ∞) → [O, ∞) which satisfies

(i) ψ is continuous and nondecreasing; (ii) ψ (t) = 0 if and only if t = 0.

Definition 2.4. Let X be a nonempty set. A map d : X x X → [O, ∞) is called a pseudo-metric in X whenever

(i) $d(x,x) = 0\, for\, all\; x \in X$;

(ii)$d(x,y) = d(y,x) for\, all\, x,y \in X$;

(iii)$d(x,y) \leq d(x,z) + d(z,y)\; for\, all\, x,y,z \in X$.

Definition 2.5. Let X be a nonempty set endowed with a pseudo-metric d. The d-ball of radius ε > 0 centered at x ∈ X is the set

$B(x;d,\varepsilon) = \{y \in Xld(x,y) < \varepsilon\}$.

Definition 2.6. A family ϒ = { $d_\lambda J\lambda$ ∈ A} of pseudo-metrics is called separating if for each pair x ≠ y, there exists a d_λ. ∈ ϒ such that d_λ.(x, y) ≠ o.

Definition 2.7. Let X be a nonempty set and ϒ = {$d_\lambda./\lambda$ ∈ A} be a separating family of pseudo-metrics on X. The topology T(ϒ) having for the subbasis the family

$B(\Upsilon) = \{B(x;\; d_\lambda.,\mathcal{E})lx \in x,, d_\lambda \in \Upsilon, \mathcal{E} > O\}$

of balls is called the topology in X induced by the family ϒ. The pair (X, T(ϒ)) is called a gauge space. Note that (X, T(ϒ)) is hausdorff because we require Υ to be separating.

Definition 2.8. Let Υ = { d_λ ./$\lambda \in$ A} be a family of pseudo-metrics on X, (X, Υ, $\leq$) is called an ordered gauge space and (X, $\leq$) is a partially ordered set.

For more details on gauge spaces, we refer the reader to [?].

Definition 2.9. Let (X, $\leq$) be a partially ordered set. We say that (X, $\leq$) is directed if every pair of elements has an upper bound, that is, for every a, b $\in$X, there exists c $\in$X such that a $\leq$ c and b $\leq$ c.

We consider the class of functions $\{\psi_\lambda\}\lambda\epsilon A$ *and* $\{\varphi_\lambda\}\lambda\epsilon A$ such that for all $\lambda \in$A, ψ_λ,φ_λ: [O,∞)→ [O, ∞) satisfy the following conditions

(c1) ψ_λ. is an altering distance function.,

(c2) φ_λ is a lower semi-continuous function with φ_λ (t) = 0 if and only if t = 0.

Theorem 2.1. (see[19}) Let (X, Υ, $\leq$)be an ordered complete gauge space and f : X → X be a no decreasing. Suppose that , ψ_λ,ψ_λ. satisfy conditions (c1, c2) and

$$\psi_{\lambda}.(d_{\lambda}.(fx,fy)) < \psi_{\lambda} d_{\lambda}.(x,y)) - \varphi_{\lambda} >.(d_{\lambda}.(x,y))$$

for all (X, Υ, $\leq$) all x, y $\in$X with x $\leq$ y. Also suppose either

(i) f is continuous or

(ii) If { Xn} $\subset$ X is a nondecreasing sequence with Xn → z $\in$ X, then Xn $\leq$ z for all n,

If there exists x0 such that x0 $\leq$ fx0, then f has a fixed point, that is, there exists z $\in$ X, such that z = f z. Moreover, if $(X, \leq)$ is directed, we obtain the uniqueness of the fixed point off.

3. MAIN RESULTS

We study the existence and uniqueness of solution for boundary value problems of nonlinear fractional differential equations in two type problems.

3.1. Existence Results for Nonlinear Fractional Differential Equations with periodic Boundary Conditions

We consider following fractional differential equation with boundary value problem

$$D_0^{\alpha} + u(t) + f(t, u(t)) = 0, \qquad t \in [0, T]$$

$$u(O) = u(T) = 0$$

where 1 < α ≤: 2, is a real number, $D_0^{\alpha}+$ is the standard RiemannLiouville differentiation and f(t, u(t)) ∈ C([O, T] x [O, ∞), R) .

Now we prove the existence of solution for the problem (1.1). We consider the space X = C([O, T], R) of real continuous functions defind on [O, T]. For each positive integer ti ≥ 1, we define the map$ll.ll_n$: X→ [O, ∞) by

$$||x||_n = \underset{0 \le t \le n}{\max}|x(t)|, \quad for\ all \quad x \in X.$$

This map is a semi- norm on X. Define now,

$$d_n(x, y) = ||x - y||_n$$

$$for\ all \quad x, y \in X.$$

Thenϒ = { d_n}n≥1 is a separating family of pseudo-metrics on X. The gauge space (X, T(ϒ)) with respect to the family ϒ is complete. Consider on X the partial order: S defind by

$$x, y \in X, x \le y <=> x(t) \le y(t)$$

$$for\ all \ \ t \ge 0.$$

For any increasing sequence { x_n} in X converging to some z ∈ X we have x_n (t) ≤z(t) for any t ≥ 0. Also, for every x, y ∈ X , there exists c(x, y) ∈X which is comparable to x, y.

We shall prove the following result

Theorem 3.1. Suppose that

(i) $f : [O, T]\, x\, [O, oo) \rightarrow R$ is continuous,$T \ \ge\ 1$,

(ii) f is increasing, for each $t \ \ge\ 0, u \ \le v$, we have $f(t, u(t)) \ \le f(t, v(t))$

(iii) for each $t \geq 0, u \leq v$, we have

$|f(t,u) - f(t,v)| \leq \gamma(t)\sqrt{ln[(v - u)^2 + 1]}$

$\sup_{t \geq 0} \int_0^T \gamma(s)ds \leq \frac{\Gamma(\alpha)}{T^{\alpha-1}}$

(iv) there exists $x_0 \in C([0,T],R)$ such that

$x_0(t) \leq \int_0^T G(t,s)f(s,u(s))$

$ds,\ for\ any\ t \geq 0.$

Then the fractional differential equation (1.1) has a unique solution $x^* \in C([0,T],R)$.

3.2 Existence Results for Nonlinear Fractional Differential Equations with nonlocal Boundary Conditions

We consider following fractional differential equation with boundary value problem

$D_0^\alpha + u(t) + f(t,u(t)), \qquad t \in [0,T], T \geq 1 \qquad (1.2)$

$u(O) = \beta_1 u(\mu), \ u(T) = \beta_2 u(\mu)$

$b < where\ 0 < \alpha < 1, is\ a\ real\ number, 0 < \beta_1 < \beta_2 < T\ and\ \mu \in [0,T], D_0^\alpha + is\ the\ standard\ Riemann - Liouville\ differentiation\ and\ f(t,u(t)) \in C([0,T] \times [0,\infty), R).$

Now we prove the existence of solution for the problem (1.1). we consider the space $X = C([0,T],R)$ for real continuous functions defined on [0,T]. for each positive integer $n \geq 1$, we define the map $||.||_n : X \to [0,\infty) by\ ||x||_n = \max_{0 \leq t \leq n} |x(t)|$

$for\ all\ x \in X.$

This map is a semi-norm on X. Define now,

$d_n(x,y) = ||x - y||_n$

$for\ all\ x, y \in X.$

Then $\Upsilon = \{d_n\}_{n\geq 1}$ Is a separating family of pseudo-metrics on X. the gauge space $(X, T(\Upsilon)$ with respect to the family Υ is complete. Consider on X the partial order $\preccurlyeq$ defined by

$x, y \in X, x \preccurlyeq y \Leftrightarrow x(t) \leq y(t)$

$for\ all\ t \geq 0.$

For any increasing sequence $\{x_n\}$ in X converging to some $z \in X$ we have $x_n(t) \leq z(t), t \geq 0\ for\ any\ t \geq 0.$ Also, for every $x, y \in X$, there exists $c(x, y) \in X$ which is comparable to x, y.

Theorem 3.2. Suppose that

(i) $f : [O, T]\ x\ [O, \infty) \rightarrow R$ is continuous ,T ≥ 1,

(ii) f is increasing, for each $t \geq 0, u \preccurlyeq v$, we have f(t,u(t)) ≤f(t,v(t))

(iii) for each $t \geq 0, u \preccurlyeq v$, we have

$|f(t, u) - f(t, v)| \leq \gamma(t)\sqrt{ln[(v - u)^2 + 1]}$

Where $\gamma: [0, \infty) \rightarrow [0, \infty)$ is continuous, the function $\int_0^T \gamma(s) ds$ is bounded on $[0, \infty)$ and

$$\sup_{t \geq 0} \int_0^T \gamma(s) ds \leq \frac{\Gamma(\alpha)}{T^{\alpha-1}}$$

(iv) there exists $x_0 \in C([0, T], R)$ such that

$x_0(t) \leq \int_0^T G(t, s) f(s, u(s))$

$ds,\ for\ any\ t \geq 0.$

ACKNOWLEDGMENTS

Research of the second and third authors was supported by Azarbaidjan University of Shahid Madani.

REFERENCES

[1] A. M. A. El-Sayed, *Nonlinear functional differential equations of arbitrary orders*, Nonlinear Analysis 33 (1998) 181-186.
[2] A. A. Kilbas, 0. I. Marichev, S. G. Samko, *Fractional integral and derivatives; theory and applications*, Gordon and Breach Science Publishers, Yverdon, 1993.
[3] A. A. Kilbas and J. J. Trujillo, *Differential equations of fractional order: methods, results and problems*, Applicable Analysis vol. 81, no. 2, pp. 435-493, 2002.
[4] A. A. Kilbas and J. J. Trujillo, *Differential equations of fractional order: methods, results and problems*, *Applicable Analysis*, vol. 78, no. 1-2, pp. 153-192, 2001.
[5] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley, New York, 1993.
[6] I. Podlubny, Fractional differential equations, *Mathematics in Science and Engineering*, Vol. 198, Academic Press, 1999.
[7] K. S. Miller, *Fractional differential equations*, J. Fract. Cale. 3 (1993) 49-57.
[8] I. Podlubny, *The Laplace transform method for linear differential equations of the fractional order*, UEF 02-94, Institute of Experimental Physics, Slovak Academy of Sciences, Kosice, Slovakia, 1994.
[9] V. Daftardar-Gejji, A. Babakhani, *Analysis of a system of fractional differential equations*, J. Math. Analysis Appl. 293 (2004) 511-522.
[10] D. Delbosco, *Fractional calculus and function spaces*, Journal of Fractional Calculus, vol. 6, pp. 45-53, 1994.
[11] R. Gorenflo, F. Mainardi, *Fractional relaxation of distributed order*, Complexus mundi, 33-42, World Sci. Publ., Hackensack, NJ, 2006.
[12] S. Q. Zhang, T*he existence of a positive solution for a nonlinear fractional differential equation*, J. Math. Analysis Appl. 252 (2000) 804-812.
[13] S. Q. Zhang, *Existence of positive solution for some class of nonlinear fractional differential equations*, J. Math. Analysis Appl. 278 (2003) 136-148.
[14] Z. Bai, H. Lu, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Analysis Appl. 311 (2005) 495-505.
[15] A. Kochubei, *Distributed order calculus and equations of ultraslow diffusion*, J. Math. Analysis Appl. 340 (2008) 252-281.
[16] M. Stojanovic, *Existenceuniqueness result for a nonlinear n-ierm fractional equation*, J. Math. Analysis Appl. 353 (2009) 244-255.
[17] M. A. Krasnoselski, *Positive solutions of operator equations*, Noordhoff. Groningen, 1964.
[18] R. W. Leggett, L. R. Williams, *Multiple positive fixed points of nonlinera operators on ordered Banach spaces*, Indiana Univ. Math. J. 28 (1979) No. 4, 673-688.
[19] M. Cherichi, B. Samet, *Fixed point theorems on ordered gauge spaces with applications to nonlinear integral equations*, Fixed point theoy and applications. vol. 2012, (2012).