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An application of fixed point theory in existence of solutions for fractional differential equations

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Abstract. In this paper, some new existense and uniqueness results for fractional differential equation are obtained by using fixed point theorems. We study the existence of solution for the nonlinear fractional differential equation boundary value problem.
$D^{\alpha}x(t) = f(t, x(t))$

With Caputo fractional derivative and Riemann-Liouville fractional derivative and different boundary value
$x(0) = x(l) = 0,$
and
$x(0) = 0, x(l) = \int_0^{\eta} x(s)\,ds, 0 < \eta < 1.$
Keywords: Boundary value problem, fixed point, Fractional differential inclusion.

1. INTRODUCTION

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications, see [2]-[8]. It should be noted that most of papers and books on fractional calculus are devoted to the solvability of Linear initial fractional differential equations on terms of special functions. Recently, there are some papers dealing with the existence of solutions of nonlinear initial value problems of fractional differential equation by using of techniques of nonlinear analysis (fixed point theorems, Lerayschauder theory, etc.) see, [8]-[16]. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, see[l 7]-[19] and the references therein.

2. PRELIMINARIES

Let us recall some basic definitions of fractional calculus, [17]-[18].

Definition 2.1. For a continuous function f :[0, ∞) →R, the Caputo derivative of fractional order a is defined as

$$^{e}D^{a}f(t) = \frac{1}{\Gamma(n-a)}\int_{o}^{t}(t-8)^{n-a-1}\int^{(n)}(8)els.\;\; n-1 < a < ti\; n = [a]+1$$

where[α] denotes the integer part of the real number α.

Definition 2.2. The Riemann-Liouville fractional derivative of order a for α continuous

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function f (t) is defined by

$$D^{a}f(t)=\frac{1}{\Gamma\,(n-a)}\,(\frac{d}{dt})^{n}\int_{o}^{t}\frac{f(s)}{(t-s)^{n-a-1}}$$

ds, n = [α] + 1providedtherighthandsideispointwisedefinedon(0, oo).

Definition 2.3. Let (X, d) be a metric space and T : X – – + X be a given mapping. we say that T is an α- ψ- contractive mapping, if there exist two functions ψ ∈ **Ψ**and

α: X× X-→[0, ∞) such that:

α(x, y)d(Tx, Ty≤ψ(d(x, y)) for all x, y ∈ X.

Definition 2.4. Let T : X –→ X and α: X xX → [0, ∞). we say that T is α- admissible if

$x, y\ \in X, \alpha(x, y)\ \geq \mathbf{1}\ =\}\ \alpha(Tx, Ty)\ \geq \mathbf{1}$.

Throughout this paper, **Ψ**is the family of nondecreasingfunctions

ψ: [0, ∞) →[0, ∞).

Theorem 2.1 (see **[1]).** Let (X, d) be a complete metric space and T: X → X be an α − ψ − contractive mapping satisfying the following conditions:

(i) T is α- admissible,

(ii) there exists x_0∈Xsuch that a(x0, Tx0) ≥ **1,**

(iii) ifXn is a sequence in X such that a(xn, Xn + 1 ≥ **1** for all n and Xn → x ∈ X as n → +∞, then a(x_n, x) ≥ **1** for all **n**,Then, T has a fixed point.

Theorem 2.2 (see **[1]).** Let For all x, y ∈ X, There exists z ∈ X such that a(x, z) ≥ **1** and a(y, z) ≥ : **1.**

Adding this condition to the hypotheses of theorem (2.1), we obtain uniqueness of the fixed point of T.

3. MAIN RESULTS

We study the existence and uniqueness of solution for boundary value problems of nonlinear fractional differential equations in three type problems.

3.1 Existence Results for Nonlinear Fractional Differential Equations with Integral Boundary Conditions

We discuss the existence and uniqueness of solution for a boundary value problem of nonlinear fractional differential equations of order a E (1, 2] with the integral boundary conditions given by

$${}^{c}D^{a}x(t)\ =\ f(t,x(t)) \qquad 0\ <t\ <1, 1\ <\alpha\leq 2 \qquad (3.1)$$

$$x(O) = 0, x(l) = \int_0^{\eta} x(s)ds, \quad 0 < \eta < 1$$

where ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative of order α,f : [O, 1] ×X →Xis continuous function.

Here, *(X,* 11-11) is a Banach space $and\ C = C([0,1], X)$ denotes the Banach space of continuous functions from [O, 1] →X endowed with a topology of uniform convergence with the norm denoted by 11-11·

Note that the boundary condition in (3.1) corresponds to the area under the curve of solutions *x(t)* from *t* = 0 *tot= η*

It is well known that such a space with the metric given by

$$d(x,y) = llx - Yll\infty = max \frac{lx(t)}{t \in I} - y(t)I$$

is a complete metric space.

We consider the following conditions:

(i)There exists a function: $R2 \to R$ such that $for\ all\ t \in I, for\ all\ a, b \in R\ with\ \xi(a,b) \geq 0$, we have

$$lf(t,a) - f(t,b)I \leq \Gamma(\alpha + 1)\psi(la - bl),$$

$$where\psi E\ \Psi.$$

(ii) There exists $x_0 \in C(I)$ such that for all $t \in I$, we have

$$\xi(xo(t), (F(x\_0)(t)) \geq 0$$

(iii) For all t ∈I, $x,y \in C(I)$,

$$\xi(x(t), y(t)) \geq 0 \rightarrow \xi((Fx)(t), (Fy)(t)) \geq 0.$$

(iv) If Xnis a sequence in $C(I)\ such\ that\ Xn \rightarrow x \in C(I)\ and\ \xi(xn, Xn + 1) \geq 0\ for\ all\ n, then\ \xi(x_n, x) \geq 0\ for\ all\ n$

Theorem 3.1. Suppose that conditions (i)-(iv) hold. Then equation (3.1) has at least one solution x*∈C(I).

Lemma 3.2. Suppose that

(I) $f : [O,1]\ x\ R \to [O,\infty)$ is continuous and nondecreasing. (II) $For\ all\ t \in I, for\ all\ a, b \in R$ with a ≤ b, we have

$$lf(t,a) - f(t,b)I \leq \Gamma(\alpha + 1)\psi(la - bl),$$

where ?)ψ ∈ Ψ.

(III) There exists $x_0 \in C(I)$ such that for all t ∈I, we have

$x_0(t) \leq Tx_0(t)$.

Then (3.1) has one and only one solution $x^* \in C(I)$.

3.2 Existence Result for Nonlinear Fractional Differential Equation with Two Point Boundary Conditions

We consider the following two-point boundary value problem of fractional order differential equation:

$$D^a x(t) = f(t, x(t)) \qquad 0 \leq t \leq 1, 1 < \alpha > 1 \quad (3.1)$$

$$x(O) = x(l) = 0$$

where$f$: [O, 1] x R →$R$is a continuous function.

$$G(t,s) = \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1} & 0 \leq s \leq t \leq 1, \\ \dfrac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)} & 0 \leq s \leq t \leq 1. \end{cases}$$

Let I=[0,1], C(I) be the space of all continuous defined on I. it is well known that such a space with the metric given by

$$d(x,y) = llx - Yll\infty = max \frac{lx(t)}{t \in I} - y(t)I$$

is a complete metric space.

We consider the following conditions:

(i)There exists a function :R^2→R such that for all $t \in I$, for all $a, b \in R$ with ξ(a, b)≥0, we have

$$lf(t,a) - f(t,b)I \leq \psi(la - bl),$$

where$\psi \in \Psi$.

(ii) There exists $x0 \in C(I)$ such that for all $t \in I$, we have

$$\xi(x0(t), \int_0^1 G(t,s) f(s, x0(s))\, ds) \geq 0.$$

(iii) For all t ∈I, x,y∈$C(I)$,

$$\xi(x(t), y(t)) \geq 0 \rightarrow \psi(\int_0^1 G(t,s) f(s, y(s)), \int_0^1 G(t,s) f(s, y(s))\, ds) \geq 0$$

(iv) If Xnis a sequence in C(I) such that Xn → x ∈ C(I) and ξ(xn, xn + 1) ≥ 0 for all n, then ξ(xn, x) ≥ 0 for all n,

Theorem 3.3. Suppose that conditions (i)-(iv) hold. Then problem (3.2) has at least one solution x*∈C (I).

Lemma 3.4.. Suppose that

(I) f : [O, 1] x R → [O, ∞) is continuous and nondecreasing. (II) For all t ∈ I, for all a, b ∈ R with a ≤ b, we have

$\mathrm{l}f(t,a) - f(t,b)\mathrm{I} \leq \psi(\mathrm{l}a - b\mathrm{l})$,

whereψ ∈W.

(III) There exists x_0∈C(I) such that for all t ∈I, we have

$$xo(t) < \int_0^1 G(t,s)f(s,xo(s))\,ds.$$

Then (3.2) has one and only one solution $x^* \in C(I)$.

3.3 Existence Result for a Nonlinear Two-Term Fractional Differential Equation

We consider the following two-point boundary value problem of fractional order differential equation:

$$D^a x(t) + D^\beta x(t) = f(t,x(t)), 0 \leq t \leq 1, 0 </3 < a < 1 \quad (3.3)$$

$$x(0) = x(l) = 0$$

where$f : [O,1]\ x\ R \to R$is a continuous function.

Recall that the green function assosiated to (3.3) is given by

$$G(t) = t^{\alpha-1}E_{\alpha-\beta}(-t^{\alpha-\beta})$$

Let *I*= [O, 1], *C(I)* be the space of all continuous functions defined on *I*. It is well known that such a space with the metric given by

$$d(x,y) = llx - Yll_{\infty} == max\ lx(t) - y(t)I$$

is a complete metric space.

We consider the following conditions:

(i)There exists a function: $R2 \to R\ such\ that\ for\ all\ t \in I, for\ all\ a,b \in R\ with\ \xi(a,b) \geq 0$, we have

$$lf(t,a) - f(t,b)I \leq \alpha\psi(la - bl).,$$

where$\psi \in \Psi$

(ii) There exists x_0∈*C(I)* such that for all t ∈I, we have

$$\xi(xo(t), \int_0^t G(t,s)f(s,xo(s))\,ds) \geq 0.$$

(iii) For all t ∈I, x,y∈$C(I)$,

$$\xi(x(t),y(t)) \geq 0 \;=>\; \psi(\int_0^t G(t,s)f(s,x(s)), \int_0^t G(t,s)f(s,y(s))\,ds) \geq 0.$$

(iv) If $Xn$is a sequence in $C(I)\ such\ that\ Xn \to x \epsilon C(I)\ and\ \xi(xn, Xn+1) \geq 0\ for\ all\ n$, then $for\ all\ n$

$$\xi(x_n, x) \geq 0$$

Theorem 3.5. Suppose that conditions (i)-(iv) hold. Then problem (3.3) has at least one solution x*∈C

(I).

Lemma 3.6.. Suppose that

(I) $f : [0, 1] \times R \to [0, \infty)$ is continuous and nondecreasing. (II) For all $t \in I$, for all $a, b \in R$ with $a \leq b$, we have

$|f(t; a) - f(t; b)| \leq \alpha\psi(|a - b|)$

where $\psi \in \Psi$.

(III) There exists $x_0 \in C(I)$ such that for all $t \in I$, we have

$x0(t) < \int_0^t G(t - s) f(s, x0(s)) \, ds.$

Then (3.3) has one and only one solution $x^* \in C(I)$.

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