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EFFICIENT CHEBYSHEV ECONOMIZATION FOR ELEMENTARY FUNCTIONS

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Abstract. This paper presents economized power series for trigonometric and hyperbolic functions. It determines the smallest range over which a function need to be computed and scales the Chebyshev polynomials accordingly. Thus reduced degree polynomials (and hence reduced computations) can be obtained while maintaining the same accuracy as those unscaled higher degree polynomials. The paper presents the Chebyshev and the power series coefficients that enable double precision accuracy for the mathematical functions addressed herein.

1. Introduction

The Chebyshev polynomials possess useful properties that render them proper for economizing transcendental functions, specifically trigonometric and hyperbolic functions, [1]-[3]. Commonly these functions use truncated Taylor series expansion. In this truncation method, the more the number of the retained terms the higher the accuracy of the approximation. However, this method suffers from the uneven distribution of errors in the approximation. The closer the evaluated point to the origin of expansion the higher the accuracy and vice versa. This means that for a desired level of accuracy the points far from the origin will need substantially more terms than those close to the origin of expansion. This problem could be alleviated by using minimization methods such as the least square (LS) algorithm. In this case the function $f(x)$ is approximated with a finite degree polynomial $\sum_{k=0}^{N-1} c_k x^k$ whose coefficients c_k are selected such that

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$$J = \int_{-1}^{1} w(x)\left(f(x) - \sum_{k=0}^{N-1} c_k x^k \right)^2 dx \tag{1.1}$$

is minimum, where $w(x)$ is an arbitrary weighting function. With no loss of generality the [-1,1] is the interval in which the function is approximated. The minimization in Eq. (1.1) yields

$$\sum_{k=0}^{N-1} c_k \int_{-1}^{1} w(x)x^k x^n \, dx = \int_{-1}^{1} w(x) f(x) x^n \, dx, \quad n = 0, 1, \cdots \tag{1.2}$$

From its construction, Eq. (1.2) is impractical to solve as it requires the computation of a full two dimensional matrix. The reason is that the function $f(x)$ is approximated with the unorthogonal power series basis $(1, x, x^2, \cdots$
This could be avoided if the function is approximated instead with an orthogonal basis. That is, if the orthogonal basis is given by $T_0, T_1, T_2, \cdots$, then the coefficients c_k are determined by

$$c_k \int_{-1}^{1} w(x) T_k^2(x) \, dx = \int_{-1}^{1} w(x) f(x) T_k(x) dx, \quad k = 0, 1, 2, \cdots \tag{1.3}$$

Using an orthogonal basis, will cause the off diagonal terms be null; thus the coefficients become the projections of the function over the members of the basis.

Our objective is to distribute the errors uniformly over the given interval. Chebyshev basis is considered to be the best choice. To demonstrate the reason for this choice consider the simple function

$$f(x) = x^n, \quad n \text{ is integer} \geq 1$$

which we desire to approximate with a polynomial p_m of degree m (less than n). The interval over which the approximation will be carried is [-1, 1]. The error is then

$$e(x) = f(x) - p_m(x), \quad -1 \leq x \leq 1$$

In general, the error is not uniform; therefore there will be a point in the interval at which the error is a maximum. It is desired that p_m is selected such that this maximum error is the least possible. It has been established, [4]-[6], that there is no polynomial $e(x)$ of the same degree and the same leading coefficient that has a smaller magnitude in [-1, 1] than the Chebyshev polynomial $T_n(x)$. Therefore if the error $e(x) = cT_n(x)$ then the desired approximation will be $p_m(x) = f(x) - e(x) = f(x) - cT_n(x)$. The constant c is selected to annul the coefficient of x^n. In our example since the leading coefficient of $f(x)$ is 1 and (as will be seen in the next section) the leading coefficient of $T_n(x)$ is 2^{n-1}, then $c=2^{-n+1}$. To generalize, let the above function be

$$f(x) = a_0 + a_1x^1 + a_2x^2 + \ldots + a_nx^n$$

Following the same steps and reasoning as above we present the above equation as

$$f(x) = a_0 + a_1x^1 + a_2x^2 + \ldots + a_n\left(x^n - cT_n(x)\right) + a_ncT_n(x)$$

Dropping the last term in the above equation, (after selecting c as before), will yield a lower degree polynomial with the least maximum error. After dropping the last term $a_ncT_n(x)$, we expand $T_n(x)$ into a power series to recover the approximated function. This example is cited to show that the lower powers had no effect on the minimization process. This process can be repeated to get a lower degree approximation as desired. Later we show the approximation process for an arbitrary function. But first let's discuss some properties of the Chebyshev polynomials.

2. The Chebyshev Polynomials

In closed form, the Chebyshev polynomials are given by the set of equations

$$T_n(x) = \cos(n\cos^{-1}x), \quad n = 0, 1, 2, \cdots \tag{2.1}$$

and with the transformation

$$x = \cos\theta \tag{2.2}$$

they could be alternatively represented by

$$T_n(\theta) = \cos(n\theta), \quad n = 0, 1, 2, \cdots \tag{2.3}$$

The Chebyshev polynomials can be generated recursively, shown in Appendix A, by

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \end{aligned} \tag{2.4}$$

Eq. (2.3) shows that the magnitudes of the basis polynomials are ≤ 1 and are equal ±1 at the extreme points ($x=\pm1$). Further, Chebyshev polynomials are orthogonal [2]

$$\begin{aligned} &\int_{-1}^{1} w(x)T_k^2(x)\,dx = \frac{\pi}{2H(k)}, \quad k = 0,1,2,\cdots \\ &\int_{-1}^{1} w(x)T_k T_n(x)\,dx = 0, \quad k \neq n \end{aligned} \tag{2.5}$$

where $H(k)$ is defined in (A.11) and $w(x)$ is the weighting function given by

$$w(x) = \frac{1}{\sqrt{1-x^2}} \tag{2.6}$$

The inverse relations of Eq. (2.1), proved in Appendix A, are given by

$$x^{2n} = \frac{1}{2^{2n-1}} \sum_{k=0}^{n} \binom{2n}{n-k} H(k) T_{2k}, \quad n = 0,1,2,\ldots \tag{2.7}$$

$$x^{2n+1} = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{n-k} T_{2k+1}, \quad n = 0,1,2,\ldots \tag{2.8}$$

From Eq. (1.3), the projections of $f(x)$ onto the Chebyshev basis results in

$$c_k = \frac{2H(k)}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}} dx, \quad k = 0,1,2,\cdots \tag{2.9}$$

The above equation is not easily amenable to digital computations. Alternatively we consider $f(x)$ to be an analytic function represented by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2.10}$$

and rather than projecting the entire function over the basis as mandated by Eq. (2.9), each of the polynomial terms is projected individually onto the Chebyshev polynomial through the use of Eqs. (2.7) and (2.8). Initially we shall consider the special cases of even and odd functions.

Expanding an even function $f(x) = f(-x)$ in terms of Chebyshev polynomials yields

$$f(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{2^{2n-1}} a_{2n} \binom{2n}{n-k} H(k) T_{2k} \tag{2.11}$$

Reversing the summation order results in

$$f(x) = \sum_{k=0}^{\infty} c_{2k} T_{2k}, \quad f(x) \text{ is even} \tag{2.12}$$

$$c_{2k} = \sum_{n=k}^{\infty} \frac{1}{2^{2n-1}} \binom{2n}{n-k} a_{2n} H(k), \quad k = 0, 1, 2, \ldots \tag{2.13}$$

Likewise, an odd function $f(x) = -f(-x)$ in terms of Chebyshev polynomials is

$$f(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{2^{2n}} a_{2n+1} \binom{2n+1}{n-k} T_{2k+1} \tag{2.14}$$

Reversing the summation order results in

$$f(x) = \sum_{k=0}^{\infty} c_{2k+1} T_{2k+1}, \quad f(x) \text{ is odd} \tag{2.15}$$

$$c_{2k+1} = \sum_{n=k}^{\infty} \frac{1}{2^{2n}} \binom{2n+1}{n-k} a_{2n+1}, \quad k = 0, 1, 2, \dots \tag{2.16}$$

Unsymmetric function can be represented as a combination of Eqs. (2.13) and (2.16), albeit with double the amount of computations needed for a symmetric function. Implementing the economization procedure presumes that the range of the argument x of interest is between ±1. If this range is scaled by a factor of s, (range=±s), then Eqs. (2.11) –(2.16) would be modified as follows.

$$f(sx) = \sum_{k=0}^{\infty} c_{2k} T_{2k}, \qquad f(sx) \text{ is even} \tag{2.17}$$

$$c_{2k} = 2\sum_{n=k}^{\infty} \left(\frac{s}{2}\right)^{2n} \binom{2n}{n-k} a_{2n} H(k), \quad k = 0, 1, 2, \dots \tag{2.18}$$

and

$$f(sx) = \sum_{k=0}^{\infty} c_{2k+1} T_{2k+1}, \qquad f(sx) \text{ is odd} \tag{2.19}$$

$$c_{2k+1} = 2\sum_{n=k}^{\infty} \left(\frac{s}{2}\right)^{2n+1} \binom{2n+1}{n-k} a_{2n+1}, \quad k = 0, 1, 2, \dots \tag{2.20}$$

Smaller scales as observed from Eqs. (2.18) and (2.20) result in smaller coefficients and consequently faster coefficients decay and using of fewer expansion terms.
Now we address three issues regarding numerical implementation: accuracy, accounting for the scales and carrying out the summation terms. Truncating the first N terms in Eqs. (2.17) and (2.19) as follows:

$$f(sx) \approx \sum_{k=0}^{N-1} c_k T_k \tag{2.21}$$

implies that the approximation error, δf, on the truncated function is (recall that $|T_i| \le 1$)

$$\delta f \le \sum_{k=N}^{\infty} |c_k| \tag{2.22}$$

Because of the fast decay of the c_k terms, the above error is dominated by c_N . Thus N is selected so c_{N-1} is just less than the tolerable errors. Next we account for the scale by one of these two methods. The first is to modify the argument so that

$$f(z) = \sum_{k=0}^{N-1} b_k x^k, \quad x = \frac{z}{s}, -s \le z \le s \tag{2.23}$$

or by modifying the coefficients of Eq. (2.23) so that

$$\begin{aligned} f(x) &= \sum_{k=0}^{N-1} d_k x^k, \quad -s \le x \le s \\ d_k &= \frac{b_k}{s^k}, \quad k = 0,1,\ldots,N-1 \end{aligned} \tag{2.24}$$

Finally, the numerical process for the summation can be carried out by converting in Eq. (2.21) into a power series which yields the power series given by Eq. (2.23)or Eq. (2.24). Alternatively, Eq. (2.21) could be evaluated by the Clenshaw recurrence [7] as follows. Substituting from Eq. (2.4) in the last term of Eq. (2.21) yields

$$\begin{aligned} f(sx) &= c_0T_0 + \ldots + c_{N-2}T_{N-2} + c_{N-1}(2xT_{N-2} - T_{N-3}) \\ &= c_0T_0 + \ldots + (c_{N-3} - c_{N-1})T_{N-3} + (c_{N-2} + 2x\,c_{N-1})T_{N-2} \end{aligned} \tag{2.25}$$

Let

$$\begin{aligned} \alpha_{N-1} &= c_{N-1} \\ \alpha_{N-2} &= c_{N-2} + 2x\alpha_{N-1} \end{aligned}$$

Again, using Eq. (2.4) in the last term in Eq. (2.25) gives

$$\begin{aligned} f(sx) &= c_0T_0 + \ldots + (c_{N-3} - \alpha_{N-1})T_{N-3} + \alpha_{N-2}T_{N-2} \\ &= c_0T_0 + \ldots + (c_{N-4} - \alpha_{N-2})T_{N-4} + (c_{N-3} + 2x\,\alpha_{N-2} - \alpha_{N-1})T_{N-3} \end{aligned} \tag{2.26}$$

Using the recursion

$$\begin{aligned} \alpha_N &= 0 \\ \alpha_{N-1} &= c_{N-1} \\ \alpha_k &= c_k + 2x\,\alpha_{k+1} - \alpha_{k+2}, \quad k = N-2, N-3, \cdots \end{aligned} \tag{2.27}$$

equation (2.27) reduces to

$$f(sx) = c_0 + x\alpha_1 - \alpha_2 \tag{2.28}$$

3. Trigonometric Functions Economization

The Chebyshev series coefficients for the sine and cosine are given in terms of the Bessel function [3]. Herein these coefficients are given explicitly.

3.1. Sine function

The Taylor series expansion of the sine function is given by

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{3.1}$$

Scaling the argument to have a range of $\pm s$, modifies the above equation to

$$\sin(sx) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (sx)^{2n+1} \tag{3.2}$$

Substituting from Eq. (3.2) into Eq. (2.20), the Chebyshev coefficients for the scaled sine function are given by

$$\sin(sx) = \sum_{k=0}^{\infty} c_{2k+1} T_{2k+1}, \quad -1 \le x \le 1 \tag{3.3}$$

$$\begin{aligned} c_{2k+1} &= 2\sum_{n=k}^{\infty} \left(s/2\right)^{2n+1} \binom{2n+1}{n-k} \frac{(-1)^n}{(2n+1)!} \\ &= 2\sum_{n=k}^{\infty} \frac{(s/2)^{2n+1}(-1)^n}{(n-k)!(n+k+1)!}, \quad k = 0, 1, 2, \dots \end{aligned} \tag{3.4}$$

Periodicity of the sine implies that range of $\pm\pi/2$ will cover all the possible values. However, scaling the argument to a range of $\pm\pi/4$ allows us to economize it with fewer terms to get the desired accuracy. The rest of the range (from $\pi/4$ to $\pi/2$) is recovered using the identity $\sin(x) = \cos(\pi/2 - x)$ and computing the economized cosine function. With $s = \pi/4$, the Chebyshev coefficients for the sine function are

$$\begin{aligned} \sin(\frac{\pi}{4}x) &= \sum_{k=0}^{\infty} c_{2k+1} T_{2k+1}, \quad -1 \le x \le 1 \\ c_{2k+1} &= 2\sum_{n=k}^{\infty} \frac{(\pi/8)^{2n+1}(-1)^n}{(n-k)!(n+k+1)!}, \quad k = 0,1,2,\dots \end{aligned} \tag{3.5}$$

Table 1 lists the Chebyshev coefficients for the sine function. Listed also are its power series coefficients for degrees up to 13 (the max for a double precision processor).

3.2. Cosine function

The Taylor series expansion of the scaled cosine function is given by

$$\cos(sx) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(sx)^{2n} \tag{3.6}$$

Substituting from Eq. (3.6) into Eq. (2.18), the Chebyshev coefficients for the scaled cosine function, $s = \pi/4$, are given by

$$\cos(sx) = \sum_{k=0}^{\infty} c_{2k} T_{2k}, \quad -1 \le x \le 1 \tag{3.7}$$

$$\begin{aligned} c_{2k} &= 2\sum_{n=k}^{\infty} (s/2)^{2n} \binom{2n}{n-k} \frac{(-1)^n}{(2n)!} H(k) \\ &= 2\sum_{n=k}^{\infty} \frac{(\pi/8)^{2n}(-1)^n}{(n-k)!(n+k)!} H(k), \quad k = 0,1,2,\dots \end{aligned} \tag{3.8}$$

A little disadvantage with the above economization is that the computed cosine at 0 is not exactly 1. This can be overcome by forcing the cosine function to be 1 for:

$$\frac{x^2}{2} = 2.10^{-16} \Rightarrow -2.10^{-8} \le x \le 2.10^{-8} \qquad (3.9)$$

The Chebyshev coefficients for the cosine function are listed in Table 2. Listed also are the power series coefficients of the cosine function for degrees 0,2…12. On a double precision digital processor, the polynomial degree is limited to 12.

3.3. Tangent function

The tangent is computed directly via the tangent expansion as given here or indirectly via the cotangent expansion along with trigonometric identity. From [9], Sec. 4.3.67

$$\tan x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n+4}(1-2^{-2n-2})B_{2n+2}}{(2n+2)!} x^{2n+1} \qquad (3.10)$$

Substituting from Eq. (3.10) into Eq. (2.20), the Chebyshev coefficients for the scaled tangent function are given by

$$\tan(sx) = \sum_{k=0}^{n} c_{2k+1}T_{2k+1}; \quad -1 \le x \le 1$$

$$\begin{aligned} c_{2k+1} &= 2\sum_{n=k}^{\infty} (s/2)^{2n+1} \binom{2n+1}{n-k} \frac{(-1)^n 2^{4n+4}(1-2^{-2n-2})B_{2n+2}}{(2n+2)!} \\ &= 4\sum_{n=k}^{\infty} (2s)^{2n+1} \frac{(-1)^n (1-2^{-2n-2})B_{2n+2}}{(n+1)(n-k)!(n+k+1)!}, \quad k = 0,1,2,\dots \end{aligned} \qquad (3.11)$$

In Eq. (3.11) and in the sequel, the sequence B_n denotes the Bernoulli numbers. With a scale factor $s = \pi/4$ one can use Eq. (3.11) to compute the tangent for any value in the interval $[-\pi/4, \pi/4]$. From the trigonometric identity $\tan(\pi/2-x)=1/\tan(x)$ one can compute the function in the rest of the range $[\pi/4,\pi/2]$. With this scaling, the coefficients in Eq. (3.11) are extremely slow to converge. Alternatively, by selecting $s = \pi/8$ one can have an economized function that achieves the maximum double precision accuracy of 10^{-16} with a polynomial of degree 19. In this case we can use the trigonometric identity $\tan(\pi/4 - x) = (1 - \tan x)/(1 + \tan x)$ to compute the function for an argument in the interval $[\pi/8, \pi/4]$. A more economized polynomial is given by the cotangent approach as described bellow.

3.4. Cotangent function

Here we economize $f(x) = x \cot x$. From [9], Sec. 4.3.70, the Taylor series expansion of the cotangent function is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{2n!} x^{2n} \tag{3.12}$$

As in the tangent function, we use the scale factor $s = \pi/8$. Substituting from Eq. (3.12) into Eq. (2.18), the scaled cotangent function coefficients are given by

$$f(sx) = \sum_{k=0}^{n} c_{2k} T_{2k}, \quad -1 \le x \le 1 \tag{3.13}$$

$$\begin{aligned} c_{2k} &= 2\sum_{n=k}^{\infty} (s/2)^{2n} \binom{2n}{n-k} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} H(k) \\ &= 2\sum_{n=k}^{\infty} \frac{(-1)^n s^{2n} B_{2n}}{(n-k)!(n+k)!} H(k), \quad k = 0,1,2,\dots \end{aligned} \tag{3.14}$$

The tangent function is determined by

$$\tan x = \frac{x}{f(x)} \tag{3.15}$$

The maximum error on the tangent function is determined by variation of Eq. (3.15)

$$\delta \tan x = -\frac{x \delta f}{f^2(x)} \tag{3.16}$$

The maximum value is attained at $x = s = \pi/8$ for which $f(s) = \pi/8 \cot(\pi/8)$ and

$$\left|\delta \tan s\right| = \frac{s\delta f}{f^2(s)} = \frac{\tan^2(\pi/8)}{\pi/8} \delta f = .4369\, \delta f \tag{3.17}$$

Chebyshev coefficients for the cotangent function are listed in Table 3. Listed also are the power series coefficients of the cotangent function for degrees 2,4…12. Using the cotangent approach we obtain the maximum double precision accuracy with an even polynomial of degree 12 rather than the odd polynomial of degree 19 with the tangent approach.

3.5. Arctangent function

The Chebyshev coefficients for this function have been derived in [5,8]. Herein we shall obtain these coefficients through a rather elegant approach, [10]. Let

$$u = k(x + i\sqrt{1-x^2}) \qquad v = k(x - i\sqrt{1-x^2}) \tag{3.18}$$

Substituting for

$$x = \cos\theta \tag{3.19}$$

in Eq. (3.18) gives

$$u = ke^{i\theta} \qquad v = ke^{-i\theta} \tag{3.20}$$

which implies that

$$u^n + v^n = k^n(e^{in\theta} + e^{-in\theta}) = 2k^n \cos n\theta = 2k^n \cos n \cos^{-1} x = 2k^n T_n(x) \tag{3.21}$$

The Taylor series expansion of the arctangent function is

$$\tan^{-1} u = u - \frac{1}{3}u^3 + \frac{1}{5}u^5 - \frac{1}{7}u^7 + \ldots$$

from which we get

$$\tan^{-1} u + \tan^{-1} v = (u+v) - \frac{1}{3}(u^3+v^3) + \frac{1}{5}(u^5+v^5) - \frac{1}{7}(u^7+v^7) + \ldots$$

Substituting from Eq. (3.21) in the above equation

$$\begin{aligned} \tan^{-1} u + \tan^{-1} v &= 2kT_1(x) - \frac{2k^3}{3}T_3(x) + \frac{2k^5}{5}T_5(x) - \frac{2k^7}{7}T_7(x) + \ldots \\ &= 2\sum_{n=0} \frac{(-1)^n k^{2n+1}}{2n+1} T_{2n+1}(x) \end{aligned} \tag{3.22}$$

Also substituting in the trigonometric identity

$$\tan^{-1}\frac{u+v}{1-uv} = \tan^{-1} u + \tan^{-1} v$$

for *u* and *v* from Eq. (3.18) in the LHS and from Eq. (3.22) in the RHS yields

$$\tan^{-1}\frac{2k}{1-k^2}x = 2\sum_{n=0}\frac{(-1)^n k^{2n+1}}{2n+1}T_{2n+1}(x) \tag{3.23}$$

Setting the parameter k to

$$k = \tan(\alpha/2)$$

implies that

$$\frac{2k}{1-k^2} = \tan\alpha$$

which upon substituting in Eq. (3.23) yields

$$\tan^{-1}(\tan\alpha)x = 2\sum_{n=0}\frac{(-1)^n \tan^{2n+1}(\alpha/2)}{2n+1}T_{2n+1}(x), \quad -1 \le x \le 1 \tag{3.24}$$

In Eq. (3.24) $\tan\alpha$ becomes a scale that naturally sets the domain of the economized function to $\pm\alpha$. Selecting

$$\alpha = \pi/8 \tag{3.25}$$

enables Eq. (3.24) to compute the arctangent function for arguments in the range $\pm\tan(\pi/8)$ ($\tan(\pi/8)=\sqrt{2}-1$). To compute the arctangent for values in the interval $[\tan(\pi/8), \tan(\pi/4)]$ we follow this procedure. Using

$$\tan^{-1} x = \theta$$

with the trigonometric identity gives

$$\tan(\frac{\pi}{4}-\theta) = \frac{1-\tan\theta}{1+\tan\theta} = \frac{1-x}{1+x}$$

which implies that

$$\tan^{-1} x = \theta = \frac{\pi}{4} - \tan^{-1}\frac{1-x}{1+x}$$

For values greater than $\tan(\pi/4)$, we use the identity $\tan x = 1/\tan(\pi/2 - x)$. The Chebyshev coefficients for the arctangent function are listed in Table 4. Listed also are the power series coefficients of the arctangent function for various degrees 1,3…21. On a double precision machine, the polynomial degree is limited to 21.

3.6. Arcsine and arccosine functions

We first discuss the direct way of economizing the arcsine function. From Eq. (2.1) we determine the Chebyshev expansion coefficients from the identity

$$\frac{\pi}{2} c_k = \int_{-1}^{1} \sin^{-1} x \frac{T_k(x)}{\sqrt{1-x^2}} dx \tag{3.26}$$

Substituting for $\theta = \cos^{-1} x$ in the above integral and performing the integration yields

$$c_{2k+1} = \frac{4}{\pi(2k+1)^2} \tag{3.27}$$

It is evident that these coefficients do not diminish rapidly. For k=50, $c_{2k+1}=O(10^{-4})$, which is an unacceptable error for such large sum of expansion terms.

A more economic approach is to compute the arcsine and arccosine via the arctangent function. For the arccosine we use the transformation $z=(1-x)/(1+x)$. Substituting for $x = \cos\theta$ in this transformation gives

$$z = \frac{1-x}{1+x} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan^2\frac{\theta}{2}$$

which implies that

$$\theta = \cos^{-1} x = 2\tan^{-1}\sqrt{\frac{1-x}{1+x}} \tag{3.28}$$

from which the arcsine function is computed using the trigonometric identity

$$\sin^{-1} x = \pi/2 - \cos^{-1} x = \frac{\pi}{2} - 2\tan^{-1}\sqrt{\frac{1-x}{1+x}} \tag{3.29}$$

4. Hyperbolic Functions Economization

The hyperbolic functions are of infinite range; thus they must be adapted to practically large finite range computations without sacrificing the desired accuracy. Herein we develop the economization for the hyperbolic sine, cosine, tangent and cotangent. Their scales are specified when we discuss the economized exponent function which will play a central role in the computations of these functions.

4.1. Hyperbolic sine function

The Taylor series expansion for a scaled hyperbolic sine is given by

$$\sinh(sx) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(sx)^{2n+1} \tag{4.1}$$

Similarity between sin and sinh functions implies, using Eqs. (3.3) and (3.4) that

$$\sinh(sx) = \sum_{k=0}^{n} c_{2k+1} T_{2k+1}, \quad -1 \le x \le 1 \tag{4.2}$$

$$c_{2k+1} = 2\sum_{n=k}^{\infty} \frac{(s/2)^{2n+1}}{(n-k)!(n+k+1)!}, \quad k = 0,1,2,\dots \tag{4.3}$$

4.2. Hyperbolic cosine function

The Taylor series expansion for a scaled hyperbolic cosine is given by

$$\cosh(sx) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}(sx)^{2n} \tag{4.4}$$

Similarity between the cos and cosh functions implies, using Eqs. (3.7) and (3.8) that

$$\cosh(sx) = \sum_{k=0}^{\infty} c_{2k} T_{2k}, \quad -1 \le x \le 1 \tag{4.5}$$

$$c_{2k} = 2\sum_{n=k}^{\infty} \frac{(s/2)^{2n}}{(n-k)!(n+k)!} H(k), \quad k = 0,1,2,\dots \tag{4.6}$$

4.3. Hyperbolic Tangent Economization

The Taylor series expansion for the hyperbolic tangent is given by

$$\tanh x = \sum_{n=0}^{\infty} \frac{2^{4n+4}(1-2^{-2n-2})B_{2n+2}}{(2n+2)!} x^{2n+1} \tag{4.7}$$

Similarity between the tan and tanh functions implies, using Eqs. (3.10) and (3.11) that

$$\tanh(sx) = \sum_{k=0}^{n} c_{2k+1} T_{2k+1}, \quad -1 \le x \le 1 \tag{4.8}$$

$$c_{2k+1} = 4\sum_{n=k}^{\infty} \frac{(2s)^{2n+1}(1-2^{-2n-2})B_{2n+2}}{(n+1)(n-k)!(n+k+1)!}, \quad k = 0,1,2,\dots \tag{4.9}$$

4.4. Hyperbolic cotangent function

The Taylor series expansion for hyperbolic cotangent $f(x) = x \coth x$, see [9], is

$$f(x) = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n} \tag{4.10}$$

Similarity between the cot and coth functions implies, using Eqs. (3.13) and (3.14) that

$$f(sx) = \sum_{k=0}^{n} c_{2k} T_{2k}, \quad -1 \le x \le 1 \tag{4.11}$$

$$c_{2k} = 2\sum_{n=k}^{\infty} \frac{s^{2n} B_{2n}}{(n-k)!(n+k)!} H(k), \quad k = 0,1,2,\dots \tag{4.12}$$

4.5. Exponent function

To perform the exponent computation on a digital computer we first note that

$$\log_2 e^x = x \log_2 e = \frac{x}{\ln 2} \qquad \Rightarrow \qquad e^x = 2^{\frac{x}{\ln 2}} \tag{4.13}$$

hence, if n and z are the nearest integer and remainder of $x/\ln 2$ respectively, *i.e.*

$$n = \text{nint}\left(x/\ln 2\right) \qquad z = x/\ln 2 - n \tag{4.14}$$

then

$$e^x = 2^{x/\ln 2} = 2^n 2^z = 2^n e^{z \ln 2} = 2^n e^{x - n\ln 2} \quad -.5 \le z < .5$$

which shows that computing the exponent of any number is reduced to computing it for a corresponding value in the interval [-0.5ln2, 0.5ln2]. Thus the proper scale for computing the exponent function is s=0.5ln2. We now discuss three methods for economizing the exponent function.

4.5.1. Sum of sinh and cosh approach

Using the scales of the sinh and cosh, the exponent function can be expressed as

$$e^{sx} = \frac{1}{2}[\sinh(sx) + \cosh(sx)] \tag{4.15}$$

However, lack of symmetry in this approach will result in a large degree polynomial.

4.5.2. Hyperbolic tangent

The exponent function is transformed into a symmetric function as follows. Let

$$e^x = \frac{1+z}{1-z} \Rightarrow z = \frac{e^x - 1}{e^x + 1} = \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} = \tanh(x/2) \Rightarrow e^x = \frac{1+\tanh(x/2)}{1-\tanh(x/2)}.$$

Hence

$$e^{\alpha x} = \frac{1+\tanh(\alpha x/2)}{1-\tanh(\alpha x/2)} \tag{4.16}$$

Using α=s/2=ln2/4 in Eqs. (4.8) and (4.9) $e^{\alpha x}$ can be computed for any real number.

4.5.3. Hyperbolic cotangent approach

The above scale α=s/2=ln2/4 can be used to compute $f(\alpha x) = \alpha x \coth \alpha x$, from which we can compute the exponent function as follows: From Eq. (4.16) we get

$$e^{\alpha x} = \frac{\coth(\alpha x/2)+1}{\coth(\alpha x/2)-1} = \frac{(\alpha x/2)\coth(\alpha x/2)+\alpha x/2}{(\alpha x/2)\coth(\alpha x/2)-\alpha x/2} \tag{4.17}$$

The Chebyshev coefficients for the hyperbolic cotangent function are listed in Table 5. Listed also are the power series coefficients of the function for degrees 2, 4…10. The maximum error on the exponent function is determined below. Substitute in Eq. (4.17) with $f(x/2) = (x/2)\coth(x/2)$. The variation of the resulting expression yields

$$\delta e^{x} = -\frac{x\delta f}{2(f(\frac{x}{2})-\frac{x}{2})^2} = -\frac{2\delta f}{x(\coth(\frac{x}{2})-1)^2}$$

The maximum error occurs at $x/2$=s=ln2/2 which gives $\delta e^{s} \approx .25\delta f$. Thus any error in computing the exponent is of the same order as that of the cotangent function.

4.6. Hyperbolic arctangent function

The hyperbolic arctangent, atanh, and the logarithm functions are mutually dependent. We will exploit this dependence to determine the scales for each of them. We derive the Chebyshev expansion of atanh by utilizing its similarity with the arctangent function. By substituting $\alpha = i\beta$ in Eq. (3.24) and employing the identities

$$\tan i\alpha = i \tanh \alpha$$
$$\tan^{-1} i\beta = i \tanh^{-1} \beta$$

we get

$$\tanh^{-1}(\tanh \beta)x = 2\sum_{n=0} \frac{\tanh^{2n+1}(\beta/2)}{2n+1} T_{2n+1}(x), \quad -1 \le x \le 1 \tag{4.18}$$

The scale $\tanh \beta$ is related to the logarithm function and is determined next.

4.7. Logarithm function

The logarithm function is represented by the Taylor series expansion

$$\ln(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \cdots \tag{4.19}$$

Lack of symmetry of Eq. (4.19) will require more expansion terms to achieve a desired accuracy than those for symmetric functions. Thus the logarithm function is transformed into a symmetric function as follows: from Eq. (4.19) we get

$$\ln(1-y) = -(y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + \frac{1}{4}y^4 + \cdots \tag{4.20}$$

Subtracting the above two equations

$$\ln\frac{1+y}{1-y} = 2\sum_{n=0} \frac{y^{2n+1}}{2n+1} = 2\tanh^{-1} y \tag{4.21}$$

To utilize Eq. (4.18) in the above, we substitute for $y = u\tanh\beta$ in Eq. (4.21) to get

$$\ln\frac{1+u\tanh\beta}{1-u\tanh\beta} = 2\tanh^{-1}(u\tanh\beta) = 4\sum_{n=0} \frac{\tanh^{2n+1}(\beta/2)}{2n+1} T_{2n+1}(u), \quad -1 \le u \le 1 \tag{4.22}$$

Since a real number w in a digital computer is represented by $w = x2^n$, n is an integer, then its logarithm is

$$\ln w = n\ln 2 + \ln x, \quad 1 \le x < 2 \tag{4.23}$$

which shows that the logarithm needs only be computed for $1 \le x < 2$. Now let

$$s\,x = \frac{1+u\tanh\beta}{1-u\tanh\beta} \Rightarrow u\tanh\beta = \frac{s\,x-1}{s\,x+1} \tag{4.24}$$

and substituting in Eq. (4.22) yields

$$\ln x = \ln\frac{1}{s} + 2\tanh^{-1}\left(u\tanh\beta\right), \quad -1 \le u \le 1, 1 \le x \le 2 \qquad (4.25)$$

The above requires that we map the interval [1,2] of x to the interval [-1,1] of u. Therefore, from Eq. (4.24), for u=-1 and x=1 we get

$$-\tanh(\beta) = \frac{s-1}{s+1}$$

and for u=1 and x=2 Eq. (4.24) gives

$$\tanh(\beta) = \frac{2s-1}{2s+1}$$

Solving the above two equations results in,

$$\tanh(\beta) = \frac{\sqrt{2}-1}{\sqrt{2}+1} \qquad s = \frac{1}{\sqrt{2}}$$

Substituting for tanh(β) and s in Eq. (4.25) gives

$$\ln x = \ln\sqrt{2} + 2\tanh^{-1}\frac{x-\sqrt{2}}{x+\sqrt{2}}, \quad 1 \le x \le 2 \qquad (4.26)$$

Chebyshev coefficients for the hyperbolic arctangent function are listed in Table 6. Listed also are its power series for degrees 1, 3…13.

5. Conclusions

The Chebyshev polynomials is a powerful tool for economizing transcendental functions that often results in minimal computations and uniform error distribution. Scaling these polynomials according to the given function can extend their usefulness as demonstrated for the trigonometric and hyper trigonometric functions. Nevertheless there are some functions that can not be efficiently economized as the arcsine/arccosine functions. Also there are other functions that can be efficiently economized via the use of other functions as in the exponent and the logarithm functions. The economization data for the sine, cosine, tangent, arc tangent, exponent and logarithm functions are provided in Tables 1-6.

Appendix

Chebyshev Recursion Equations

Rearranging the trigonometric identity

$$2\cos n\theta\cos\theta = \cos(n\theta+\theta)+\cos(n\theta-\theta) = \cos(n+1)\theta+\cos(n-1)\theta \tag{A.1}$$

yields

$$\cos(n+1)\theta = 2\cos\theta\cos n\theta - \cos(n-1)\theta \tag{A.2}$$

Substituting from Eq. (2.3) in Eq. (A.2)

$$T_{n+1}(\theta) = 2\cos\theta T_n(\theta) - T_{n-1}(\theta) \tag{A.3}$$

Using the transformation

$$x = \cos\theta \tag{A.4}$$

in Eq. (A.3) yields

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \tag{A.5}$$

A power series as function of the Chebyshev polynomials is shown below. Let

$$u = e^{i\theta}, \quad v = e^{-i\theta} \tag{A.6}$$

The binomial theorem states

$$(u+v)^n = u^n + \binom{n}{1}u^{n-1}v + \ldots + \binom{n}{n-1}uv^{n-1} + v^n \quad \Rightarrow$$

$$(u+v)^n = (u^n+v^n) + \binom{n}{1}(u^{n-1}v + uv^{n-1}) + \binom{n}{2}(u^{n-2}v^2 + u^2v^{n-2}) + \cdots$$

Substituting from Eq. (A.6) in the above gives

$$2^n \cos^n \theta = 2\cos n\theta + 2\binom{n}{1}\cos(n-2)\theta + 2\binom{n}{2}\cos(n-4)\theta + \cdots \tag{A.7}$$

Substituting from Eq. (A.4) and Eq. (2.3) in the above gives

$$2^n x^n = 2T_n(x) + 2\binom{n}{1}T_{n-2}(x) + 2\binom{n}{2}T_{n-4}(x) + \cdots \tag{A.8}$$

The last term in the RHS of Eq. (A.8) depends on whether *n* is odd or even, hence

$$2^n x^n = 2T_n(x) + 2\binom{n}{1}T_{n-2}(x) + \ldots + \binom{n}{\frac{n}{2}}T_0(x), \quad n \text{ is even}$$

$$2^n x^n = 2T_n(x) + 2\binom{n}{1}T_{n-2}(x) + \ldots + 2\binom{n}{\frac{n-1}{2}}T_1(x), \quad n \text{ is odd}$$

Equivalently the above two equations can be represented by the equations

$$x^{2n} = \frac{1}{2^{2n-1}}\sum_{k=0}^{n}\binom{2n}{n-k}T_{2k}H(k), \quad n = 0,1,2,\ldots \tag{A.9}$$

$$x^{2n+1} = \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n+1}{k}T_{2n-2k+1}(x), \quad n = 0,1,2,\cdots \tag{A.10}$$

where

$$\begin{aligned} H(k) &= 1, \quad k = 1,2,\cdots \\ &= .5, \quad k = 0 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{A.11}$$

Table 1 Chebyshev Coefficients of sin(xπ/4), -1⩽x⩽1

T_1	T_3	T_5	T_7	T_9	T_{11}	T_{13}
7.2637567669373460e-001	-1.9420029053201504e-002	1.5169292285107397e-004	-5.6058046841200102e-007	1.2053241678543556e-009	-1.6941393087095102e-012	1.6778093175966046e-015

Power Series Coefficients of sin(x), -π/4≤x≤π/4

n	x^1	x^3	x^5	x^7	x^9	x^{11}	x^{13}
1	9.2485023590022641e-001						
3	9.9902928275155212e-001	-1.6033938833998410e-001					
5	9.9994989891170432e-001	-1.6660157013147733e-001	8.1214933926091453e-003				
7	9.9999998616424701e-001	-1.6666636734719931e-001	8.3315839716068523e-003	-1.9462058181624460e-004			
9	9.9999999997624467e-001	-1.6666666589609330e-001	8.3333263328600488e-003	-1.9838672804571564e-004	2.7135316888187163e-006		
11	9.9999999999997213e-001	-1.6666666666540469e-001	8.3333333169595888e-003	-1.9841260735019872e-004	2.7554856370117920e-006	-2.4732065101915567e-008	
13	9.9999999999999989e-001	-1.6666666666666527e-001	8.3333333333083376e-003	-1.9841269821967074e-004	2.7557311570774417e-006	-2.5050482812758435e-008	1.5883056913369989e-010

Error bound on sin(x)

n	1	3	5	7	9	11	13
	1.9268895507187134e-002	1.5113354601436698e-004	5.5937683670697407e-007	1.2036317050184022e-009	1.6924628396502615e-012	1.6764678272216946e-015	1.2457436155176656e-018

Table 2 Chebyshev Coefficients of cos(xπ/4) , -1⩽x⩽1

T_0	T_2	T_4	T_6	T_8	T_{10}	T_{12}
8.5163191370480806e-001	-1.4643664439083687e-001	1.9214493118146466e-003	-9.9649684898292967e-006	2.7576595607187385e-008	-4.7399498081648420e-011	5.5495485414851792e-014

Power Series Coefficients of cos(x), -π/4≤x≤π/4

n	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}
0	8.5163191370480806e-001						
2	9.9806855809564499e-001	-4.7478829242533427e-001					
4	9.9999000740745958e-001	-4.9970778280380668e-001	4.0397956174570697e-002				
6	9.9999997237594940e-001	-4.9999856557568295e-001	4.1655020905420913e-002	-1.3585843887443270e-003			
8	9.9999999995254496e-001	-4.9999999615145652e-001	4.1666616715874336e-002	-1.3886618604994089e-003	2.4379880313551010e-005		
10	9.9999999999994449e-001	-4.9999999999351513e-001	4.1666666543960736e-002	-1.3888880396024119e-003	2.4798929171725437e-005	-2.7173456841089967e-007	
12	1.0000000000000000e+000	-4.9999999999999267e-001	4.1666666666472389e-002	-1.3888888869983292e-003	2.4801578540009612e-005	-2.7555234093295848e-007	2.0630465643317002e-009

Error Bound on cos(x)

n	0	2	4	6	8	10	12
	1.4452513251826049e-001	1.9115118725763747e-003	9.9374392382718020e-006	2.7529251557494102e-008	4.7344049693282631e-011	5.5448388365786616e-014	4.7097049065175519e-017

Table 3 Chebyshev Coefficients of xπ/8cot(xπ/8) , -1⩽x⩽1

T_0	T_2	T_4	T_6	T_8	T_{10}	T_{12}
9.7409726717287404e-001	-2.5970025310420506e-002	-6.7541716104439387e-005	-2.5020150212295250e-007	-9.7249463554838568e-010	-3.8173451207019430e-012	-1.5017576781855927e-014

Power Series Coefficients of xcot(x) , -π/8≤x≤π/8

n	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}
0	9.7409726717287404e-001						
2	1.0000672924832945e+000	-3.3680815407017939e-001					
4	9.9999975076719005e-001	-3.3330432988025988e-001	-2.2720743308627223e-002				
6	1.0000000009686922e+000	-3.3333353390088039e-001	-2.2215743042150880e-002	-2.1831349222007019e-003			
8	9.9999999999619760e-001	-3.3333333210261851e-001	-2.2222285902551153e-002	-2.1152508533200855e-003	-2.2009901470214537e-004		
10	1.0000000000000149e+000	-3.3333333334030790e-001	-2.2222221695625703e-002	-2.1164166428058939e-003	-2.1145944108258747e-004	-2.2409518727647284e-005	
12	9.9999999999999989e-001	-3.3333333333329634e-001	-2.2222222226069781e-002	-2.1164019667773465e-003	-2.1164297845290516e-004	-2.1351600229332416e-005	-2.2867104948562489e-006

Error Bound on xcot(x)

n		2	4	6	8	10	12
	2.6037818204354121e-002	6.7792893933616100e-005	2.5117782917671290e-007	9.7632705376042073e-010	3.8324182120349849e-012	1.5073091333041674e-014	5.5514551185745660e-017

Error Bound on tan(x)

1.1376098246352341e-002	2.9619172226357848e-005	1.0974128629341012e-007	4.2656386940642206e-010	1.6744093440950972e-012	6.5855351832915799e-015	-2.4254681534169045e-017

Table 4 Chebyshev Coefficients of atan(x tan(π/8)) , -1⩽x⩽1

T_1	T_3	T_5	T_7/T_{15}	T_9/T_{17}	T_{11}/T_{19}	T_{13}/T_{21}
3.9782473475931601e-001	-5.2467950438531983e-003	1.2455722454749680e-004	-3.5201766614312475e-006	1.0832870770174304e-007	-3.5068481349183947e-009	1.1740588439779745e-010
			-4.0259236096786315e-012	1.4055019102434098e-013	-4.9756558397889277e-015	1.7811816477512138e-016

Power Series Coefficients of atan(x) , -tanπ/8≤x≤tanπ/8

n	x^1	x^3	x^5	x^7/x^{15}	x^9/x^{17}	x^{11}/x^{19}	x^{13}/x^{21}
1	9.6043387010341996e-001						
3	9.9843452136500699e-001	-2.9531203542807216e-001					
5	9.9993806006897756e-001	-3.3036509848938383e-001	1.6344337884159002e-001				
7	9.9997549276643420e-001	-3.3313892658209754e-001	1.9577748863150146e-001	-1.0768971574516645e-001			
9	9.9999990303436137e-001	-3.3332184265314269e-001	1.9961549539561521e-001	-1.3751577271705970e-001	7.7261777546314608e-002		
11	9.9999999616344493e-001	-3.3333269857468339e-001	1.9996982390201859e-001	-1.4223617929487059e-001	1.0477432328425801e-001	-5.8310861036308892e-002	
13	9.9999999984820243e-001	-3.3333329991220600e-001	1.9999786271743944e-001	-1.4279648395433761e-001	1.1021714807652334e-001	-8.3689347159893165e-002	4.5512817510059506e-002
15	9.9999999999399403e-001	-3.3333333163567069e-001	1.9999985961478298e-001	-1.4285190667168293e-001	1.1100677028988537e-001	-8.9714118031158904e-002	6.8922776154370133e-002
				-3.6384863453783105e-002			
17	9.9999999999976241e-001	-3.3333333324946679e-001	1.9999999129728246e-001	-1.4285673096903709e-001	1.1110049717507468e-001	-9.0747084454347315e-002	7.5406466406530662e-002
				-5.7978986959008758e-002	2.9614064746582694e-002		
19	9.9999999999999067e-001	-3.3333333332928133e-001	1.9999999948468755e-001	-1.4285711272658866e-001	1.1111013904137161e-001	-9.0890131147222691e-002	7.6689139060161518e-002
				-6.4814154123893158e-002	4.9533201599576075e-002	-2.4441523691688023e-002	
21	9.9999999999999967e-001	-3.3333333333314108e-001	1.9999999997060344e-001	-1.4285714077811654e-001	1.1111102918781843e-001	-9.0907110559317439e-002	7.6892140631474926e-002
				-6.6346653758053542e-002	5.6626310004955688e-002	-4.2815597215545689e-002	2.0398466384482443e-002

Error Bound on atan(x)

n	1	3	5	7/15	9/17	11/19	13/21
	-5.1256530605918588e-003	1.2114198326133938e-004	-3.4152412861574160e-006	1.0493537527383149e-007	-3.3933324279115515e-009	1.1351570700684313e-010	-3.8901773909543192e-012
				1.3574621872431269e-013	-4.8039723000282888e-015	1.7168353976063915e-016	-6.4346250144822126e-018

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