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CONVERGENCE OF SOLUTIONS OF NONAUTONOMOUS NICHOLSON'S BLOWFLIES MODEL WITH IMPULSES

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Abstract. This paper deals with a nonautonomous Nicholson's blowflies model with impulses. It is shown that under the proper conditions every positive solution of the model approaches to a constant as t tends to infinity.

1. Introduction

In this paper, we consider the following delay differential equation with impulses

$$x'(t) = b(t)\left[-\delta(t)x(t) + p(t)x(t-\tau)e^{-a(t)x(t-\tau)}\right],\ t \neq t_i,\ t \geq t_0 > 0, \qquad (1)$$

$$\Delta x\left(t_i\right) = d_i x\left(t_i\right),\ i \in \mathbb{N} = \{1, 2, ...\}, \qquad (2)$$

where $b, \delta, p, a \in C(\mathbb{R}, (0, \infty))$, $\tau > 0$ is a constant, $\{t_i\}$ is a sequence of real numbers such that $0 < t_0 < t_1 < t_2 < ... < t_j < t_{j+1} < ...$, and $\lim_{j\to\infty} t_j = \infty$, $\Delta x\left(t_i\right) = x\left(t_i^+\right) - x\left(t_i^-\right)$, $x\left(t_i^+\right) = \lim_{t\to t_i^+} x\left(t\right)$, $x\left(t_i^-\right) = \lim_{t\to t_i^-} x\left(t\right)$, $d_i > -1$, $i = 1, 2, ...$, are constants.

Nicholson [1] considered the delay differential equation

$$x'\left(t\right) = -\delta x\left(t\right) + Px\left(t-\tau\right)e^{-ax(t-\tau)},\ t \geq 0$$

where a, δ, P and τ are positive constants, to model the laboratory insect populations. In this model $x\left(t\right)$ is the size of the population at time t, P is the adult recruitment rate, δ is the adult mortality rate and τ is the maturity time delay for the eggs. This model is later studied in details in [2] and referred as Nicholson's Blowflies Model. The qualitative analysis of the solutions of Nicholson's model and its generalizations attracted many mathematicians' attention during the last two decades [3]-[10].

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[11] is a detailed survey for Nicholson's blowflies model with different types of delays. Hien [10] considered the Nicholson's blowflies model in the following form:

$$x'(t) = -\alpha(t)x(t) + \sum_{j=1}^{m} \beta_j(t)x(t-\tau_j(t))e^{-\gamma_i(t)x(t-\tau_j)}$$

where m is a given positive integer, α , β_j, γ_j, τ_j, ($j \in \underline{m} := \{1, 2, ..., m\}$) are continuous functions on R^+, $\alpha(t) > 0$, $\gamma_j(t) > 0$, $\beta_j(t) \geq 0$ and $\tau_j(t) \geq 0$ for all $t \in R^+$, $j \in \underline{m}$, and derived the sufficient condition for the existence and global exponential convergence of positive solutions of this model.
In 2010 [12], convergence of solutions of the following nonimpulsive Nicholson's model was studied:

$$x_i'(t) = b_i(t)\left[\sum_{j=1}^{n} \overline{a_{ij}}(t)x_j(t) + \sum_{j=1}^{m} \overline{\beta_{ij}}(t)x_i(t-\tau_{ij}(t))e^{-x_i(t-\tau_{ij}(t))} - \overline{d}(t)x_i(t)\right],$$

with the assumption $\int_{t_0}^{\infty} b_i(t)dt = +\infty$. One of the main motivations of this paper is to answer the question "Is it possible for the solutions of a Nicholson's model to converge to a constant if $\int_{t_0}^{\infty} b_i(t)dt < +\infty$? ".

On the other hand, it is well known that in real world, impulses may appear in several phenomena. Qualitative analysis of the solutions of delay differential equations with impulses are studied in details in [13], [14]. But, as we know, there are only a few papers on Nicholson's model with impulses ([8], [15], [16], [17]). Our aim in this paper is to obtain sufficient conditions for the convergence of solutions of equation (1)-(2) with the assumption $\int_{t_0}^{\infty} b_i(t)dt < +\infty$.

Definition 1. A real valued piecewise left continuous function $x(t)$ is said to be the solution of equation (1)-(2) if the following conditions are satisfied:
(a) $x(t)$ is continuous everywhere except at the points t_i, $i \in \mathbb{N}$,
(b) $x(t_i^+)$ and $x(t_i^-)$, $i \in \mathbb{N}$, exist and $x(t_i^-) = x(t_i)$,
(c) $x(t)$ satisfies differential equation (1) almost everywhere in $[t_0, \infty)$ and satisfies impulse conditions at $t = t_i$, $i \in \mathbb{N}$.

Let $C^+ = C^+([-\tau, 0], \mathbb{R}^+)$, $\mathbb{R}^+ = [0, \infty)$, be the space of nonnegative continuous functions equipped with the usual supremum norm $\|.\|$. Because of the biological meaning of the equation (1)-(2), we consider only positive solutions. Therefore, the initial condition is given by

$$x_{t_0} = \varphi \in C_0^+, \tag{3}$$

where $C_0^+ = \{\varphi \in C^+ : \varphi(0) > 0\}$, and x_t is defined as $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-\tau, 0]$.

Denote by $PLC\,(R, R^{+})$ the space of all piecewise left continuous functions $f : R \to R^{+}$ with points of discontinuity of the first kind at $t = t_i,\ i \in N$.

For the proof of Theorem 1 we shall use the following well known lemma [18].

Lemma 1. Suppose that for $t \geq t_0$ the inequality

$$u(t) \leq c + \int_{t_0}^{t} b(s)u(s)ds + \sum_{t_0 \leq \tau_k < t} \beta_k u(\tau_k)$$

holds, where $u(t) \in PLC(R, R^{+}),\ b(t) \in PLC(R, R^{+})$ and $\beta_k \geq 0,\ k \in N$ and $c \geq 0$ are constants, and τ_k are the first kind discontinuity points of the function $u(t)$.

Then for $t \geq t_0$

$$u(t) \leq c \prod_{t_0 \leq \tau_k < t} (1 + \beta_k) \exp\left(\int_{t_0}^{t} b(s)ds \right).$$

2. Main Results

In this section we shall prove our main result. First, by using Theorem 2.1 in [10] we show the existence of positive solutions.

Proposition 1. *For any $t_0 \in \mathbb{R}^{+},\ \varphi \in C_0^{+}$, the solution $x(t, t_0, \varphi)$ of the equation (1),(2) satisfies $x(t, t_0, \varphi) > 0$ for all $t \in [t_0, \infty)$.*

Proof. From Theorem 2.1 in [10] it is clear that the solution of the initial value problem (1),(3) without impulses is positive on each interval $[t_i, t_{i+1}],\ i = 0, 1, \ldots.$ On the other hand, from impulse conditions $x(t_i^{+}) = (1 + d_i)x(t_i) > 0$ can be calculated. So, by using method of steps, existence and uniqueness of the positive solution is obtained for all $t \in [t_0, \infty)$. □

We should note that a straightforward verification shows that the solution $x(t, t_0, \varphi)$ $= x(t)$ of the initial value problem (1)-(3) satisfies the following integral equation

$$x(t) = \begin{cases} \varphi(t - t_0), & t_0 - \tau \leq t \leq t_0, \\ \varphi(0) - \int_{t_0}^{t} b(s)\delta(s)x(s)ds + \int_{t_0}^{t} b(s)p(s)x(s - \tau)e^{-a(s)x(s-\tau)}ds & \\ \qquad + \sum_{t_0 \leq t_i < t} d_i x(t_i),\ t \geq t_0. & \end{cases} \tag{4}$$

Theorem 2. *Assume that the following conditions are satisfied:*

(i): $\int_{0}^{\infty} b(s)ds \leq K_1 < \infty,$

(ii): $\prod_{0 \leq t_i < t} (1 + |d_i|) \leq K_2 < \infty,$

(iii): $\sup_{0\le s<t} \delta(s) = \delta > 0, \ \sup_{0\le s<t} p(s) = p > 0.$

Then any solution $x(t)$ of equation (1)-(2) satisfies $\lim_{t\to\infty} x(t) = K$ (constant).

Proof. Let $x(t)$ be a solution of equation (1)-(2). The proof will be given in two steps.
Step 1. Showing the boundedness of $x(t)$:
From (4), for $t \ge t_0$ we have

$$\begin{aligned} |x(t)| &\le \varphi(0) + \int_{t_0}^{t} |b(s)|\,|\delta(s)|\,|x(s)|\,ds + \int_{t_0}^{t} |b(s)|\,|p(s)|\,|x(s-\tau)|\,e^{-a(s)x(s-\tau)}ds \\ &\quad + \sum_{t_0\le t_i<t} |d_i|\,|x(t_i)|\,. \end{aligned}$$

This inequality can be written as

$$\begin{aligned} |x(t)| &\le \varphi(0) + \int_{t_0}^{t} |b(s)|\,|\delta(s)|\,|x(s)|\,ds \\ &\quad + \sup_{t_0-\tau\le s\le t_0} \varphi(s-t_0) \int_{t_0-\tau}^{t_0} |b(s+\tau)|\,|p(s+\tau)|\,ds \\ &\quad + \int_{t_0}^{t} |b(s+\tau)|\,|p(s+\tau)|\,|x(s)|\,ds + \sum_{t_0\le t_i<t} |d_i|\,|x(t_i)| \\ &\le L_1 + \int_{t_0}^{t} \left(|b(s)|\,|\delta(s)| + |b(s+\tau)|\,|p(s+\tau)|\right)|x(s)|\,ds + \sum_{t_0\le t_i<t} |d_i|\,|x(t_i)| \end{aligned}$$

where $L_1 = \varphi(0) + \|\varphi\| \int_{t_0-\tau}^{t_0} |b(s+\tau)|\,|p(s+\tau)|\,ds$. Applying Lemma 1, it follows that

$$x(t) \le L_1 \prod_{t_0\le t_i<t} (1+|d_i|) \exp \int_{t_0}^{t} \left(|b(s)|\,|\delta(s)| + |b(s+\tau)|\,|p(s+\tau)|\right) ds$$

for $t \ge t_0$.
So, by (i), (ii), and (iii), we obtain that

$$x(t) \le L_1 K_2 e^{(\delta+p)K_1}, \ t \ge t_0.$$

On the other hand, since φ is continuous on $[-\tau, 0]$, there exists a constant K such that

$$x(t) \le K \text{ for } t \ge t_0 - \tau. \tag{5}$$

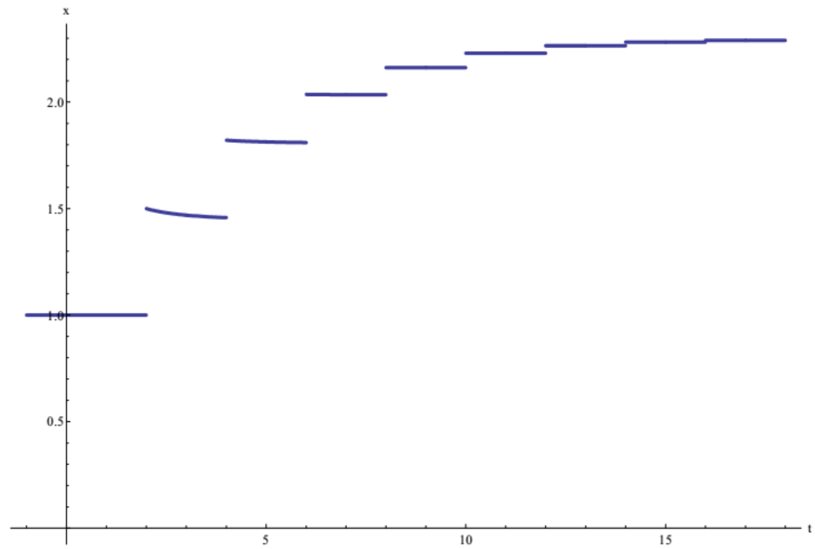

FIGURE 1. Solution of Equation 8.

Step 2. Proving the existence of $\lim_{t\to\infty} x(t)$:
For $0 \le s \le t < \infty$ we have

$$\begin{aligned} |x(t)-x(s)| \quad &\le \quad \left| \int_s^t b(u)\delta(u)x(u)du \right| + \left| \int_s^t b(u)p(u)x(u-\tau)e^{-a(u)x(u-\tau)}du \right| \\ &\quad + \left| \sum_{s<t_i<t} d_i x(t_i) \right|. \end{aligned} \tag{6}$$

Substituting (5) into (6), we get

$$|x(t)-x(s)| \le K \left(\int_s^t b(u)\delta(u)du + \int_s^t b(u)p(u)du + \sum_{s<t_i<t} |d_i| \right). \tag{7}$$

Taking (7) into account together with the conditions $(i)-(iii)$, it follows that

$$\lim_{s\to\infty} \|x(t)-x(s)\| = 0.$$

So, $\lim_{t\to\infty} x(t) \in \mathbb{R}$. □

Now, we give some examples to illustrate our results.

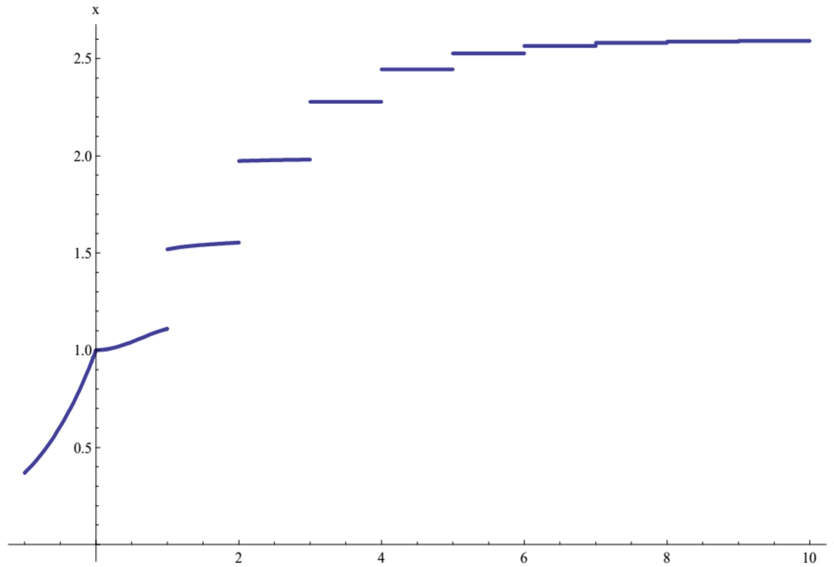

FIGURE 2. Solution of Equation 9.

Example 1. *Consider the following impulsive delay differential equation*

$$x'(t) = e^{-t}\left[-2e^{-1}x(t) + 2x(t-1)e^{-x(t-1)}\right], \qquad t > 0,\ t \neq t_i$$
$$x\left(t_i^+\right) - x\left(t_i^-\right) = \frac{1}{2^i}x(t_i), \qquad t_i = 2i,\ i \in Z^+, \tag{8}$$
$$\phi(t) = 1, \qquad -1 \leq t \leq 0.$$

Obviously, $b(t) = e^{-t}$, $\delta(t) = 2/e$, $p(t) = 2$, $\tau = 1$ *and* $d_i = 1/2^i$, $i \in Z^+$. *It is easy to see that Equation 8 satisfies the conditions of Theorem 2. So, any solution* $x(t)$ *of Equation 8 converges to a positive constant* K *as* $t \to \infty$ *(see Figure 1).*

Example 2. *Consider the following impulsive delay differential equation*

$$x'(t) = te^{-t}\left[\frac{-1}{t+1}x(t) + \left(\cos^2 t + 1\right)x(t-1)e^{-tx(t-1)}\right], \qquad t > 0,\ t \neq t_i$$
$$x\left(t_i^+\right) - x\left(t_i^-\right) = ie^{-i}x(t_i), \qquad t_i = 2i,\ i \in Z^+, \tag{9}$$
$$\phi(t) = e^t, \qquad -1 \leq t \leq 0.$$

Obviously, $b(t) = te^{-t}$, $\delta(t) = \frac{1}{t+1}$, $p(t) = \cos^2 t + 1$, $\tau = 1$ *and* $d_i = ie^{-i}$, $i \in Z^+$. *It is easy to see that Equation 9 satisfies the conditions of Theorem 2. So, any solution* $x(t)$ *of Equation 9 converges to a positive constant* K *as* $t \to \infty$ *(see Figure 2).*

3. Conclusions

This paper dealt with the asymptotic constancy of positive solutions of a non-autonomous Nicholson's blowflies model with impulses given with (1)-(2). A generalization of this model is studied in [12] without impulses with the assumption $\int_{t_0}^{\infty} b(t) = +\infty$ for all $t_0 \in R^1$. In this paper we analyze the model with impulse conditions and we assume that $\int_{t_0}^{\infty} b(t) < \infty$. We have derived sufficient conditions for asymptotic constancy of solutions of the model. Two numerical examples have been provided to illustrate the given results. The results given in this paper may also be extended for systems of equations.

There still exist interesting problems on Nicholson's model. In [19],[20] there is a detailed analysis for impulsive differential equations with piecewise constant arguments. Nicholson's blowflies model can be considered with piecewise constant arguments and the asymptotic constancy of the positive solutions may be studied.

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