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Construction of the Katětov Extension of a Hausdorff Space

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Abstract

Katětov extension κX of Hausdorff space X has been studied extensively as the largest H-closed extension of a Hausdorff space. Recall that, a Hausdorff space X is said to be an H-closed space if it is closed in every Hausdorff space in which it is embedded. Although Katětov extensions of Hausdorff spaces have been extensively studied, to date there has been very little work on either its construction or its structure (topology). In this paper, we give the detailed algorithm for constructing such a space by using filters on X. The basis generating the topology on κX contains the open sets of the form $V \cup \{\Gamma : V \in \Gamma \in \kappa X - X\}$ or $U \subset X$ where both U and V are open subsets of X and Γ is a non-convergent ultra-filter on X containing V. Moreover, using simple approach, it is proved that Katětov extension κX is a Hausdorff space, H-closed, maximal and unique extension for X.

Keywords: *Extension; Hausdorff spaces; H-closed spaces; Katětov extension; (open) filters.*
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1. Introduction

In topology, the term *extension* of a space means the process of constructing a new space which contains a given space as a dense subspace (Banaschewski 1963). One of the reasons for constructing such extensions is the possibility of transforming the problem from an old space to the one concerning an extension so that the new space is nicer than the earlier space and the problem can be easily solved (Porter and Woods 1982). The early studies of extensions of topological spaces were put forward by Alexandroff and Urysohn (1924). However, the first large body of systematic theory used for the investigation of a wide range of extension problems was presented by Stone (1937) (Banashewski 1963). Since its evolution, mathematicians were interested in the study of extensions of spaces, the basic question being how an object of a specific kind can be embedded into the other. Katětov (1940), a Czech mathematician proved the following result; Given any Hausdorff space X, it can be found an H-closed space κX in which X is densely embedded. This new space containing X as a dense subspace is what we call the Katětov extension of X. It is the largest and unique Hausdorff closure of X. Katětov also proved that if X and T are Hausdorff spaces and $h : X \to T$ is a continuous function such that $cl_T h(X) = T$, then we can find a subspace N of κX and a continuous function $H : N \to T$ from N onto T for which $X \subset N$ and $H|_X = h$. Katětov constructed such a space by adjoining to X the class of all non-convergent maximal open filters and defined the basis for a new space κX as of the form $U \cup \{\mathscr{P} \subset \tau\}$ where U is an open subset of X contained in the non-convergent maximal open filter $\mathscr{P}$ and τ is a topology on X. This basis on κX is incomplete as not every open set in X can be in $\mathscr{P}$, otherwise the convergence property of $\mathscr{P}$ would be violated. Although Katětov extensions of Hausdorff spaces have been extensively studied, to date there has been very little work on either its construction or its structure (topology). In this paper, we give the detailed procedures on the construction of the Katětov extension κX of a Hausdorff space X. We construct such a space by using the concept of filters on X. After its construction, we shall reprove the result by Katětov (1940) that such a space is the largest H-closure for X by providing a simpler proof compared to the original one. The properties of filters and H-closed spaces are the main tools in establishing our main results.

1.1. Preliminaries and definitions

The following definitions, lemmas and theorems are essential grounds to the establishment of our main results.

Definition 1.1 (Open filter)**.** *Let (X,τ) be a topological space. An open filter Γ on X is a non-empty collection of open subsets of X which satisfy the following axioms;*

(1) $X \in \Gamma$, $\emptyset \notin \Gamma$
(2) *If $U_1, U_2 \in \Gamma$, then $U_1 \cap U_2 \in \Gamma$. This is called Finite Intersection Property (FIP).*

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(3) If $U \in \Gamma$ and $H \in \tau$ such that $U \subseteq H$, then $H \in \Gamma$.

An *open ultrafilter* on the space X is a maximal open filter in the collection of open filters on X. An open filter Γ is called *free* if its *adherence*, denoted by $adh_X(\Gamma)=\bigcap_{U\in\Gamma} Cl_X U$, is empty. Otherwise, Γ is *fixed*. A point x in X is called a *limit point (or accumulation point)* of Γ if for any neighborhood U of x, there is a set A in Γ contained in U. For more details about filters, we refer the reader to Bartle (1955).

Lemma 1.2. *The following statements hold for any open ultrafilter Γ on a topological space (X,τ):*

(1) If U is an open subset of X, then $U\cap V \neq \emptyset$ for all $V \in \Gamma$ if and only if $U \in \Gamma$.
(2) If U_1, U_2 are open subets of X and $U_1 \cup U_2 \in \Gamma$, then either $U_1 \in \Gamma$ or $U_2 \in \Gamma$.
(3) If $U_1 \notin \Gamma$ and U_1 is open in X, then $U_2 = X - \overline{U_1} \in \Gamma$, where $\overline{U_1} = cl_X U_1$.
(4) If a point x in X is a limit (or an accumulation) point of Γ , then $\Gamma \longrightarrow x$.

Proof. (1) Suppose U is open subset in X and for all $V \in \Gamma$, $U\cap V \neq \emptyset$, then $\emptyset \notin \Gamma$ and hence $U \in \Gamma$.
Conversely, if both $U,V \in \Gamma$, then, $U \cap V \in \Gamma$ and since $\emptyset \notin \Gamma$, it follows that $U \cap V \neq \emptyset$.

(2) Given $U_1, U_2 \in \tau$, then $U_1 \cup U_2 \in \Gamma$ implies that for any open subset $V \in \Gamma$, we have,

$$(U_1 \cup U_2)\cap V = (U_1\cap V)\cup(U_2\cap V) \neq \emptyset.$$

This implies that

$$U_1 \cap V \in \Gamma \qquad or \qquad U_2 \cap V \in \Gamma.$$

Since $V \in \Gamma$ and both U_1, U_2 are open in X, then by (i) above, it follows that either $U_1 \in \Gamma$ or $U_2 \in \Gamma$.

(3) Given $U_1 \notin \Gamma$ and U_1 is open in X, then there exists an open subset $V \in \Gamma$ such that

$$U_1 \cap V = \emptyset.$$

Since $\overline{U_1}$ is closed in X, then $U_2 = X - \overline{U_1}$ is open in X. Now to show that $U_2 = X - \overline{U_1} \in \Gamma$, we show that for any open subset $V \in \Gamma$ with the property $U_1 \cap V = \emptyset$, $\quad (X - \overline{U_1})\cap V \neq \emptyset$.

Case 1. If $\overline{U_1}\cap V = \emptyset$, then $(X-\overline{U_1})\cap V \neq \emptyset$ because both V and $X - \overline{U_1}$ are open subsets of X and $V \subset X - \overline{U_1}$.

Case 2. If $\overline{U_1}\cap V \neq \emptyset$ and $V \not\subseteq \overline{U_1}$, then $V - (\overline{U_1}\cap V) \subset X - \overline{U_1}$. It then follows that

$$(V-(\overline{U_1}\cap V))\cap(X-\overline{U_1}) = V-(\overline{U_1}\cap V) \neq \emptyset.$$

Since $V - (\overline{U_1}\cap V) \subset V$, then $(X-\overline{U_1})\cap V \neq \emptyset$ showing that $X - \overline{U_1} \in \Gamma$.

(4) The result follows from the definition of a limit point above.

□

Lemma 1.3 (Liu 1968). *Suppose Ω is any open filter on (X,τ) and Γ is a maximal open filter such that $\Omega \subset \Gamma$, then the following statements are true:*

(1) If x is an accumulation point of Γ, then x is an accumulation point of Ω.
(2) $\Omega \longrightarrow x$ implies $\Gamma \longrightarrow x$.

Lemma 1.4. *Let X and Y be topological spaces such that X is open subset in Y and $cl_Y X = Y$. Suppose Γ is a maximal open filter on X. Then $\Gamma' = \{\, V : V$ is open in Y and $V\cap X \in \Gamma\}$ is a maximal open filter on Y. Moreover, $\Gamma \longrightarrow x$ if and only if $\Gamma' \longrightarrow x$.*

Proof. First, we show that Γ' is an open filter on Y.

(1) We show that $\emptyset \notin \Gamma'$ and $Y \in \Gamma'$. Let $V \in \Gamma'$, then $V\cap X \in \Gamma$ (from definition of Γ'). Since Γ is a maximal open filter on X, then $V\cap X \neq \emptyset$ implying that $V \neq \emptyset$ and hence $\emptyset \notin \Gamma'$. If $V = Y$, then $Y\cap X = X(\neq \emptyset) \in \Gamma$ and hence $Y \in \Gamma'$.

(2) Suppose $V_1, V_2 \in \Gamma'$, we show that $V_1\cap V_2 \in \Gamma'$. Now if $V_1, V_2 \in \Gamma'$, then $V_1\cap X, V_2\cap X \in \Gamma$ and so

$$(V_1\cap X)\cap(V_2\cap X) = (V_1\cap V_2)\cap X \in \Gamma.$$

This, in turn implies that $\qquad V_1\cap V_2 \in \Gamma'$.

(3) Next, we show that, if $V \in \Gamma'$ and U is any open subset in Y with the property that $V \subset U$, then $\quad U \in \Gamma'$. Now, $V \in \Gamma'$ implies that $V\cap X \in \Gamma$. Given $V \subset U$, then $U\cap X \supset V\cap X \in \Gamma$ and hence $U\cap X \in \Gamma$ (Γ is a maximal open filter on X) and so $U \in \Gamma'$.

So far, we have proven that Γ' is indeed an open filter on the space Y. What is left is to show that Γ' is an open ultrafilter (maximal filter) on the space Y. Suppose Γ'' is an open filter on the space Y with the property that $\Gamma' \subset \Gamma''$. We need to show that $\Gamma' = \Gamma''$. It is enough to show that if an open subset A of Y is in Γ'', then $A \in \Gamma'$, that is $\Gamma'' \subset \Gamma'$. Taking a fixed open subset $A \in \Gamma''$ of Y, we have $A\cap X$ is an open subset of X because X is an open dense subspace of Y. Since $A \in \Gamma''$, then for all $U \in \Gamma \subset \Gamma''$, we have $(A\cap X)\cap U = (A\cap U)\cap X \neq \emptyset$. Since Γ is a maximal open filter on the space X, it implies that $A\cap X \in \Gamma$, and thus $A \in \Gamma'$.

Moreover, suppose $\Gamma \longrightarrow x$ in X, then every open neighborhood U of a point x is contained in Γ. Since $\Gamma \subset \Gamma^{'}$, then by Lemma 1.3, $\Gamma^{'} \longrightarrow x$.

Conversely, if $\Gamma^{'} \longrightarrow x$, then every open subset V containing x in Y is contained in $\Gamma^{'}$. Since the space X is a dense and open subspace of the space Y, whenever $V \in \Gamma^{'}$, we have $V \cap X \in \Gamma$ concluding that $V \cap X$ is an open neighborhood of a point $x \in X$ for all open neighborhoods V of x in the topological space Y. Therefore, $\Gamma \longrightarrow x$. □

Lemma 1.5 (Liu 1968)**.** *Let X and Y be topological spaces such that X is open in Y and $cl_Y X = Y$. If $\Gamma^{'}$ is a maximal open filter on Y, then $\Gamma = \Gamma^{'} \cap X = \{U^{'} \cap X : U^{'} \in \Gamma^{'}\}$ is a maximal open filter on X. Furthermore, $\Gamma \longrightarrow x$ if and only if $\Gamma^{'} \longrightarrow x$.*

Proof. Note that Γ can be re-defined as $\Gamma = \{U \in \Gamma^{'} : U \subset X\}$ because X is a dense open subset of Y and $X \in \Gamma^{'}$. We skip the rest of the proof as it is similar to that of Lemma 1.4. □

Lemma 1.6. *If $f : X \to Y$ is a continuous function from a topological space X onto a topological space Y, then for every open filter $\mathscr{F}$ on the space Y, $f^{-1}(\mathscr{F})$ is an open filter on the space X.*

Proof. To prove that $f^{-1}(\mathscr{F})$ is an open filter on X, we show that it satisfies the following three axioms:

(1) We show that $\emptyset \notin f^{-1}(\mathscr{F})$ and $X \in f^{-1}(\mathscr{F})$. Since $\mathscr{F}$ is an open filter on Y, then $\emptyset \notin \mathscr{F}$ and $Y \in \mathscr{F}$ (from definition of open filter). By continuity of f, we have $f^{-1}(\emptyset) = \emptyset \notin f^{-1}(\mathscr{F})$ and $f^{-1}(Y) = X \in f^{-1}(\mathscr{F})$ (f is onto), as desired.

(2) Next, we show that if $U, V \in f^{-1}(\mathscr{F})$, then $U \cap V \in f^{-1}(\mathscr{F})$. Given that $U, V \in f^{-1}(\mathscr{F})$, then $f(U), f(V) \in \mathscr{F}$. Since $\mathscr{F}$ is an open filter on Y, then, $f(U) \cap f(V) = f(U \cap V) \in \mathscr{F}$. This implies that $U \cap V \in f^{-1}(\mathscr{F})$, as desired.

(3) Suppose that $U \in f^{-1}(\mathscr{F})$ and $U \subset V$, we show that $V \in f^{-1}(\mathscr{F})$. Given $U \in f^{-1}(\mathscr{F})$, then $f(U) \in \mathscr{F}$. Since $U \subset V$, we have, $f(U) \subset f(V)$ and thus $f(V) \in \mathscr{F}$ ($\mathscr{F}$ is an open filter on Y). This further implies that $V \in f^{-1}(\mathscr{F})$, as desired.

□

The following theorem will be used to characterize the H-closed spaces.

Theorem 1.7 (Porter et al. 1979 and 1982)**.** *Let (X, τ) be the Hausdorff space, then the following statements are equivalent ;*

(1) (X, τ) is H-closed.
(2) Every open ultrafilter on X is fixed.
(3) Every open filter on X is fixed.
(4) Every open cover of X has a finite open sub-cover whose union is dense in X.

2. Main Results

We are now in a position to give the structure of the Katětov extension κX of a Hausdorff topological space (X, τ).

Definition 2.1. *Let (X, τ) be a Hausdorff topological space and*

$X^{\infty} = \{\Gamma \subset \tau : \Gamma$ *is a free (or non-convergent) maximal open filter on* $X\}$.

The Katětov extension κX of a space (X, τ) is a set $\kappa X = X \cup X^{\infty}$ of disjoint union whose topology is generated by open subsets of the form as indicated in the collection β, where

$\beta = \{V \cup \{\Gamma : V \in \Gamma \in \kappa X - X\}, U : U \text{ is open in } X\}$.

From this definition, it is clear that X is open subset of κX and $cl_{\kappa X} X = \kappa X$. Moreover, κX is the largest H-closure in a set of all H-closed extensions of a space X. We shall provide the insight of these behaviors for κX in the next theorem which was given by the founder of this space, Katětov (1940) by providing a simpler proof than that provided by Katětov himself.

The following lemma is an important prerequisite in the proof of the next theorem.

Lemma 2.2. *The continuous image of an H-closed space is an H-closed space.*

Proof. Let $f : X \longrightarrow Y$ be a continuous function from an H-closed space X into a topological space Y. We show that $f(X)$ is an H-closed space. It suffices to show that every open filter on $f(X)$ converges. Let $\mathscr{F}$ be the open filter on $f(X)$. By continuity of f, $f^{-1}(\mathscr{F})$ is an open filter on X (Lemma 1.6). Since X is H-closed, there is a unique element $x \in X$ such that $f^{-1}(\mathscr{F}) \longrightarrow x$. This implies that $\mathscr{F} \longrightarrow f(x)$, $f(x) \in f(X)$. Thus, $f(X)$ is H-closed. □

Theorem 2.3 (Katětov 1940)**.** *For any Hausdorff space (X,τ), there exists a unique Hausdorff space κX, called the Katětov extension of X which is the largest H-closed extension for X.*

Proof. We define $\kappa X = X \cup X^{\infty}$ as in Definition 2.1 and let

$\beta = \{V \cup \{\Gamma : V \in \Gamma \in \kappa X - X\}, U : U \text{is open in X}\}$

be the basis generating a topology on κX.

We provide the proof for this theorem into five steps as described below;

Step 1. ***We need to show that the collection β is a basis for a topology on κX by showing that it satisfies the following two axioms:*** Notice that the nature of points in κX are of two forms; singleton points belonging to X and non-convergent open ultrafilters Γ belonging to $X^{\infty} = \kappa X - X$.

(1) We show that for each element x or Γ in κX, there is a basis element $B \in \beta$ such that $x \in B$ or $\Gamma \in B$.

Indeed, for each point $x \in X$, there exists an open neighborhood U of x in X such that $x \in U$ and by the definition of the collection β, U is an open neighborhood of x in κX.

If $\Gamma \in \kappa X - X$, then $\Gamma \in U \cup \{\Gamma\} \in \beta$ for some open subset $U \in \Gamma$ and thus the first axiom is satisfied.

(2) For the second axiom; Let $B_1, B_2 \in \beta$, we show that $B_1 \cap B_2 \in \beta$. Let $B_1 = V_1 \cup \{\Gamma\}, B_2 = V_2 \cup \{\Gamma\} \in \beta$, where $V_1, V_2 \in \Gamma$ are both open subsets of X. Then, $B_1 \cap B_2 = (V_1 \cup \{\Gamma\}) \cap (V_2 \cup \{\Gamma\}) = (V_1 \cap V_2) \cup \{\Gamma\} \in \beta$ simply because $V_1 \cap V_2 \in \Gamma$.

Therefore, we have shown that β is indeed, a basis for a topology on the Katětov extension κX. The remaining part below shows that the space κX is Hausdorff, H-closed , the largest H-closed extension of X and unique.

Step 2. ***We show that κX is a Hausdorff topological space:*** Here, three cases will be considered.

Case 1. Let x and y be any two distinct elements of X. Since X is Hausdorff, then we can find two disjoint open neighborhoods V_x, V_y of x and y, respectively. But X is a dense open subset of κX, therefore, such neighborhoods are open in κX as well and $V_x \cap V_y = \emptyset$.

Case 2. Let $x \in X$ and $\Gamma \in X^{\infty}$ be two distinct points in κX. Then, there is an open subset U of X for which $x \in U$ and $U \notin \Gamma$, for else would mean convergence of Γ to a point x should all other remaining neighborhoods of x belong to Γ. So $U \notin \Gamma$ implies that there exists some open subset $V \in \Gamma$ for which $U \cap V = \emptyset$. Then U and $V \cup \{\Gamma\}$ are disjoint open neighborhood of x and Γ in κX, respectively.

Case 3. Finally, let Γ_1, Γ_2 be two points in X^{∞} such that $\Gamma_1 \neq \Gamma_2$. Since both of them are non-convergent open ultrafilters on X, we can find two open subsets U_1, U_2 of X such that $U_1 \in \Gamma_1, U_2 \in \Gamma_2$ with the property that $U_1 \cap U_2 = \emptyset$. Then, the open sets $U_1 \cup \{\Gamma_1\}$ and $U_2 \cup \{\Gamma_2\}$ are disjoint open neighborhoods of Γ_1 and Γ_2 in κX, respectively. This completes the fact that κX is a Hausdorff space.

Step 3. ***We show that κX is an H-closed space:*** To show this, we refer Theorem 1.7. It is enough to show that every open ultrafiter (or maximal filter) on κX converges. Now, let $\mathscr{M}$ be a maximal open filter on the space κX, we show that $\mathscr{M}$ converges. Since X is a dense open subset of κX, we define an open ultrafilter on X by

$\mathscr{N} = \mathscr{M} \cap X = \{M \cap X : \quad M \in \mathscr{M}\}$

which can be re-defined as

$\mathscr{N} = \{M \in \mathscr{M} : \quad M \subset X\}$ (by Lemma 1.5).

Indeed, if $\mathscr{N}$ has a limit point x in X, then by Lemma 1.3, $\mathscr{M}$ has x as a cluster point in κX showing that $\mathscr{M} \longrightarrow x$ in κX.

However, if $\mathscr{N}$ has no limit point in X, then $\mathscr{N} \in X^{\infty}$. Let $U \cup \{\mathscr{N}\}$ be an open subset of κX containing $\mathscr{N}$ where $U \in \mathscr{N}$. Since X is a dense open subset of κX, then $U \in \mathscr{M}$. This shows that $U \cap M \neq \emptyset$ for all $M \in \mathscr{M}$. This further implies that $(U \cup \{\mathscr{N}\}) \cap M \neq \emptyset$ for all $M \in \mathscr{M}$. That is, every neighborhood $U \cup \{\mathscr{N}\}$ of $\mathscr{N}$ meets each member M of an open ultrafilter $\mathscr{M}$. This implies that $\mathscr{M} \longrightarrow \mathscr{N}$. Since $\mathscr{M}$ was arbitrarily chosen maximal open filter on κX, then every maximal open filter on κX is fixed showing that the Katětov extension κX of a Hausdorff space X is *H-closed.*

Step 4. ***We show that κX is the largest H-closed extension of X:*** Suppose Z is another largest H-closure of X, then for κX to be maximal extension on the space X, we show Z is essentially the same as κX. Since $Z \geq X$, there is a continuous inclusion $h : X \longrightarrow Z$. We need to extend the morphism h to $j : \kappa X \longrightarrow Z$ continuously. If such j exists, it will be onto. This will then mean the range $j(\kappa X)$ is an H-closed space and hence it will be closed in Z (Lemma 2.2), and has a dense subset X of Z, concluding that $j(\kappa X) = Z$.

Now, let $\Gamma \in X^{\infty} = \kappa X - X$ and $\quad \mathscr{U} = \{W : W \text{ is open in Z and } \ W \cap X \in \Gamma\}$. By Lemma 1.4, it follows that $\mathscr{U}$ is a maximal open filter on Z. Since Z is an H-closed extension, then there is a point z in Z , where by $\mathscr{U} \longrightarrow z$ and hence $\Gamma \longrightarrow z$ in Z. Depending on the nature of points on $\kappa X = X \cup X^{\infty}$, we define $j : \kappa X \longrightarrow Z$ by $j(\Gamma) = z$ and $j(x) = x$ for all $x \in X$. We now show the continuity of the function j by showing that it is continuous at every point of κX.

Case 1: Let x be a point in X. Suppose that V is an open subset of Z containing x, then since j extends h, $j^{-1}(V) \cap X$ is an open set in X and therefore $j^{-1}(V) \cap X = h^{-1}(V) = H$ is an open subset of X containing a point x. Since every open subset of X is also open in κX, then

$h^{-1}(V) = H$ is open subset of κX and $j(H) \subset V$.

Case 2: For $\Gamma \in X^{\infty}$, we define $j(\Gamma) = z$ where $\Gamma \longrightarrow z$ in Z. Let H be an open set containing z in Z, then we can find $A \in \Gamma$ such that $A \subset H$. Thus, $A \cup \{\Gamma\}$ is an open subset of κX containing Γ with the property that
$j(A \cup \{\Gamma\}) = j(A) \cup j(\{\Gamma\}) = j(A) \cup \{j(\Gamma)\} = j(A) \cup \{z\} = A \cup \{z\} \subset H$. This shows that the function j exists and is continuous and hence $\kappa X = Z$, as desired.

Step 5. ***Uniqueness of*** κX***:*** Suppose that $\kappa X^{'}$ is also the largest H-closed extension of the Hausdorff space X. For κX to be unique, we show that κX and $\kappa X^{'}$ are equal. To arrive to the desired result, we need to show that there is a homeomorphism $g : \kappa X \longrightarrow \kappa X^{'}$ such that $g(x) = x$ for all x in X and $g(\Gamma) = \Gamma^{'}$ for $\Gamma \in \kappa X - X$, and $\Gamma^{'} \in \kappa X^{'} - X$. It suffices to show that for every set U open in κX, the image $g(U)$ is open in $\kappa X^{'}$. Then, by symmetry g is the homeomorphism.

Case 1: If U does not contain $\Gamma^{'}$, then $g(U) = U$, $(U \subset X)$. Since U is open in κX and is in X, it is open in X. We know X is an open (dense) subspace of $\kappa X^{'}$, the set U is also open subset of $\kappa X^{'}$, as required.

Case 2: Suppose that U contains Γ. Then $C = \kappa X - U$ is closed in κX, and it is compact as a subspace of κX, (κX is Hausdorff). Now, since $C \subset X$, it is a compact subspace of X. Again, X is a subspace of $\kappa X^{'}$, then the space C is also a compact subspace of $\kappa X^{'}$. Since $\kappa X^{'}$ is Hausdorff, C is closed in $\kappa X^{'}$, hence $g(C) = g(\kappa X - U) = g(\kappa X) - g(U) = \kappa X^{'} - g(U)$. Since $C \subset X$, $g(C) = C$ and hence $g(U) = \kappa X^{'} - C$ is open in $\kappa X^{'}$, as desired. Therefore, $\kappa X = \kappa X^{'}$. □

3. Conclusion

We have shed a light on the construction of the Katětov extension κX of Hausdorff space X using the notion of open filters on the space X. Further, using these new tools (filters), we managed to show that, κX is indeed a unique, maximal, H-closed extension of X.

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