Determining the Velocity Distribution Profile of a Fluid in an Inclined Flat Surface Using the Finite Element Method and the Exact Differential Equation Method

Iredia Davis Erhunmwun*, Collins Oside**‡

*Department of Production Engineering, Faculty of Engineering, University of Benin, P.M.B. 1154, Benin City, Nigeria.

** Department of Mechanical Engineering, Faculty of Engineering, National Institute of Construction Technology, Uromi, Nigeria.

(iredia.erhunmwun@uniben.edu, osidecollins@gmail.com)

‡ I. D. Erhunmwun, Department of Production Engineering, University of Benin, P.M.B. 1154, Benin City, Nigeria, Tel: +234 807 072 8898,
Fax: +234 807 072 8898, iredia.erhunmwun@uniben.edu

Received: 27.07.2018 Accepted:25.04.2019

Abstract- An analysis has been carried out to determine the velocity profile of a fluid on an inclined plane using the Finite Element Method (FEM). The overall results from these finite elements were finally assembled to represent the velocity profile in the entire domain of the inclined plane. The results obtained from the finite element method shows that as the velocity distribution has a parabolic profile with the maximum velocity of 1109.8748m/s at open surface of the inclined plane. The fluid due to the no slip boundary condition has 0m/s at the walls of the inclined plane. Also, it was shown that the higher the angle of inclination and fluid viscosity, the lower the velocity and also the higher the fluid density, the higher the velocity. The result obtained from the FEM when compared with the result obtained from the exact differential equation method shows a strong agreement with a maximum percentage error of $2.3413x10^{-14}$.

Keywords: Finite Element Method, Inclined Plane, Incompressible Fluid, Interpolation Function, Weak Formulation

1. Introduction

Investigation of the properties of flow down an inclined plane is a subject of great theoretical and practical importance and has attracted the attention of many researchers [1-3]. Consideration has been given to a fluid constantly poured on the inclined plane from above. The fluid forms a steady stream moving downwards under the action of the gravity. Such an example is a river flow. This phenomenon also occurs in case of conveyor belts and in the lubrication theory.

Literature is not replete on the velocity profile of a flow down an inclined plane.

Bognár, et al. in 2018 investigated the velocity distributions on an inclined plane in the transport of non-Newtonian fluids [4]. The process was modelled by boundary layer flows. They considered the equations of continuity and motion boundary conditions on the plane and on the surface of the transported material. They finally examined the velocity distribution in case of different material properties, constant plane speed and different inclination angle.

The finite element method has been used to solve a problem on the velocity distribution in viscous incompressible fluid using the langrange interpolation function and compared their result with the exact differential equation method [5]. The plain in this case was taken to be horizontal. Also, the Finite Element Method (FEM) to determine the velocity distribution in a concentric cylindrical annulus [6]. The annulus pipe in this case was taken to be horizontal.

The aim of this study is to determine at the same time the velocity profile of a fluid in inclined plane. We will be using FEM unlike other methods that need to carry out several iterations to determine the velocities at different point.

2. Finite Element Method

Consider the flow of a Newtonian viscous fluid on an inclined flat surface, as shown in “Fig. 1”. Examples of such flow can be found in wetted-wall towers and the application of coatings to wallpaper rolls. The momentum equation, for a fully developed steady laminar flow along the z coordinate, is given by

$$-\mu_1 \frac{\partial^2 w_z}{\partial x^2} = \rho g \cos\beta \tag{1}$$

where $w =$ z component of the velocity

μ = Viscocity of the fluid

g = Acceleration due to gravity and

β = Angle between the inclined surface and the vertical

The boundary conditions associated with this problem are that the stresses is zero at $x = 0$ and the velocity is zero at $x = L$.

$$\left(\frac{dw_z}{dx}\right)\Bigg|_{x=0} = 0 \tag{2}$$

and

$$w_z(L) = 0 \tag{3}$$

The Gerlekin Finite Element Method was used to discretize the domain (inclined plane).

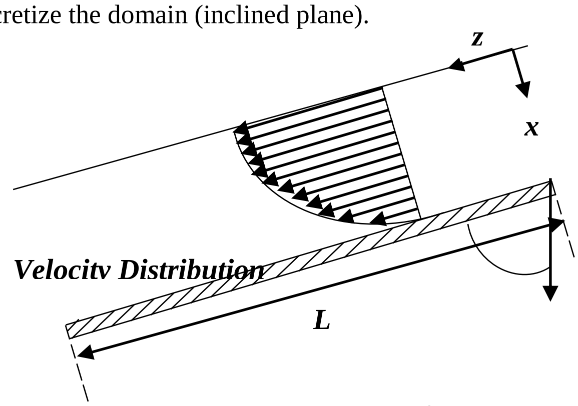

Fig. 1. Velocity Profile.

The domain of the problem consists of all points between $x = 0$ and $x = L$ i.e. $\Omega = (0, L)$. The domain was divided into a set of line elements, a typical element being of length h_e and located between two end points A and B of a typical element. The collection of such elements is called the finite element mesh of the domain. The reason for dividing the domain into finite elements was to represent the geometry of the domain and to approximate the solution over the entire domain.

2.1. Mathematical analysis

In the development of the weak form, we assumed a linear mesh and placed it over the domain. This was done by multiplying equation (1) by the weighted function (w) and integrating the final equation over the domain. This results in the mathematical expression in equation (4).

$$\int_{x_A}^{x_B} \mu \frac{\partial w}{\partial x}\frac{\partial w_z}{\partial x}dx - \int_{x_A}^{x_B} \rho g \cos\beta w dx - wQ_A - wQ_B = 0 \tag{4}$$

But $x_B = x_A + h_e$

Equation (4) is known as the weak form of the governing equation.

The weak form requires that the approximation chosen for *u* should be at least linear in *x* so that there are no terms in equation (4) that are identically zero. Since the primary variable is simply the function itself, the Lagrange family of interpolation functions is admissible. We proposed that u was the approximation over the typical finite element domain by the expression:

$$w_z = \sum_{j=1}^{n} w_j^e \psi_j^e(x) \text{ and } w = \psi_i^e(x)$$
$$i, j = 1,2,3 \tag{5}$$

where $w = \psi_i^e(x)$ is the trial function .

In Galerkin’s weighted residual method, the weighting functions are chosen to be identical to the trial functions [7].

Substitute equation (5) into equation (4), we have:

$$\mu\left[K_{ij}^e\right]\left\{w_j^e\right\} = \rho g \cos\beta\left\{F_i^e\right\} + \left\{Q_i^e\right\} \tag{6}$$

where $K_{ij}^e = \int_{x_A}^{x_A+h_e} \frac{\partial \psi_i^e(x)}{\partial x}\frac{\partial \psi_j^e(x)}{\partial x}dx \quad (7)$

$$F_i^e = \int_{x_A}^{x_A+h_e} \psi_i^e(x)dx \tag{8}$$

$$Q_i^e = \psi_i^e(x)Q_A + \psi_i^e(x)Q_B \tag{9}$$

Equation (6) is referred to as the finite element based model while equation (7) is known as the stiffness matrix and equation (8) is referred to as the flux matrix.

Hence, the one-dimensional Lagrange quadratic interpolation function becomes

$$\psi_1 = \left(1 - \frac{x}{h_e}\right)\left(1 - \frac{2x}{h_e}\right) \tag{10}$$

$$\psi_2 = \frac{4x}{h_e}\left(1 - \frac{x}{h_e}\right) \tag{11}$$

$$\psi_3 = -\frac{x}{h_e}\left(1 - \frac{2x}{h_e}\right) \tag{12}$$

where h_e = Elemental length

2.2 Evaluating the stiffness matrix $\lfloor K_{ij} \rfloor$ and flux matrix $\{F^e\}$

To evaluate the K_{ij} matrix, we substitute equations 10-12 accordingly into equations (7) and (8), respectively. Then we have;

$$K^e = \frac{1}{3h_e^3}\begin{bmatrix} 7h_e^2 - 24h_e x_A + 48x_A^2 & -8\left(h_e^2 - 3h_e x_A + 12x_A^2\right) & h_e^2 + 48x_A^2 \\ -8\left(h_e^2 - 3h_e x_A + 12x_A^2\right) & 16\left(h_e^2 + 12x_A^2\right) & -8\left(h_e^2 + 3h_e x_A + 12x_A^2\right) \\ h_e^2 + 48x_A^2 & -8\left(h_e^2 + 3h_e x_A + 12x_A^2\right) & 7h_e^2 + 24h_e x_A + 48x_A^2 \end{bmatrix} \tag{13}$$

$$F^e = \begin{Bmatrix} \dfrac{h_e}{6} - x_A + \dfrac{2x_A^2}{h_e} \\ \dfrac{2\left(h_e^2 - 6x_A^2\right)}{3h_e} \\ \dfrac{h_e}{6} + x_A + \dfrac{2x_A^2}{h_e} \end{Bmatrix} \tag{14}$$

Equation (13) represents the generalized form of the stiffness matrix for the entire domain of the fluids between stationary parallel plates and equation (14) represents the generalized form of the flux matrix for the entire domain of the fluid between stationary parallel plates.

In this work, the domain of the parallel plates was divided into four quadratic elements. Therefore,

$$\left[K^e\right] = \frac{1}{3h_e}\begin{bmatrix} 7\mu_1 & -8\mu_1 & \mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8\mu_1 & 16\mu_1 & -8\mu_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_1 & -8\mu_1 & 14\mu_1 & -8\mu_1 & \mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8\mu_1 & 16\mu_1 & -8\mu_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & -8\mu_1 & 7\mu_1 + 7\mu_2 & -8\mu_2 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8\mu_2 & 16\mu_2 & -8\mu_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_2 & -8\mu_2 & 14\mu_2 & -8\mu_2 & \mu_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8\mu_2 & 16\mu_2 & -8\mu_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu_2 & -8\mu_2 & 7\mu_2 \end{bmatrix} \tag{15}$$

$$\{F^e\} = \frac{h_e}{6}\begin{Bmatrix} 1 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 1 \end{Bmatrix} \tag{16}$$

Substitute in equation (6), equations (15) and (16) and finally, we have:

$$\begin{bmatrix} 7 & -8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -8 & 14 & -8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 16 & -8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -8 & 14 & -8 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & 16 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -8 & 14 & -8 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8 & 16 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -8 & 7 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \frac{\rho g h_e^2 \cos\beta}{2\mu}\begin{Bmatrix} 1 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_2^2 \\ Q_3^2 + Q_1^3 \\ Q_2^3 \\ Q_3^3 + Q_1^4 \\ Q_2^4 \\ Q_3^4 \end{Bmatrix} \tag{17}$$

Due to balance of internal flux, $Q_2^1,\ Q_3^1 + Q_1^2,\ Q_2^2,\ Q_3^2 + Q_1^3,\ Q_2^3,\ Q_3^3 + Q_1^4,\ Q_2^4$ are equal to zero.

Introducing the boundary conditions stated in equation (3), equation (17) reduces to

$$\begin{bmatrix} 7 & -8 & 1 & 0 & 0 & 0 & 0 & 0 \\ -8 & 16 & -8 & 0 & 0 & 0 & 0 & 0 \\ 1 & -8 & 14 & -8 & 1 & 0 & 0 & 0 \\ 0 & 0 & -8 & 16 & -8 & 0 & 0 & 0 \\ 0 & 0 & 1 & -8 & 14 & -8 & 1 & 0 \\ 0 & 0 & 0 & 0 & -8 & 16 & -8 & 0 \\ 0 & 0 & 0 & 0 & 1 & -8 & 14 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -8 & 16 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{Bmatrix} = \frac{\rho g h_e^2 \cos\beta}{2\mu}\begin{Bmatrix} 1 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 \\ Q_3^1 + Q_1^2 \\ Q_2^2 \\ Q_3^2 + Q_1^3 \\ Q_2^3 \\ Q_3^3 + Q_1^4 \\ Q_2^4 \end{Bmatrix} \tag{18}$$

3. Results and Discussion

The data used in this work are given thus:

$$\beta = 45^0,\ \ \rho = 2kgm^{-3},\ g = 9.81m/s,\ \ \mu = 0.4Pa.s,\ \ L = 8m$$

The exact differential equation solution of the problem is given in equation (19) [7].

$$w = \frac{\rho g L^2 \cos\beta}{2\mu}\left[1 - \left(\frac{x}{L}\right)^2\right] \tag{19}$$

In this paper, the problem being analysed is the one that involves the flow of a Newtonian viscous fluid on an inclined plane. Examples of such flow can be found in wetted-wall towers and the applications to wallpaper rolls. The momentum equation, for a fully developed laminar flow along the z coordinate was used to analyse the velocity distribution of an incompressible fluid flowing down an inclined plane under the influence of a pressure gradient.

The finite element method was used to discretize the entire domain. The domain was discretized into four linear elements. In other to analyse these elements, a quadratic interpolation function was used to approximate the velocity distribution in the domain.

A graph of the velocity profile of the Newtonian viscous fluid on the inclined flat surface is as shown in “Fig. 2”. The graph represents the velocity at different nodes plotted against the length of the inclined flat surface. The graph shows a parabolic relationship between the velocity and the length of the inclined flat surface. It was observed from the graph in “Fig. 2” that the velocity of the fluid at point 8m which is the boundary of the fluid and the inclined flat surface was 0m/s. This was due to the fact we applied the no slip boundary condition at the boundary (walls) of the inclined flat surface.

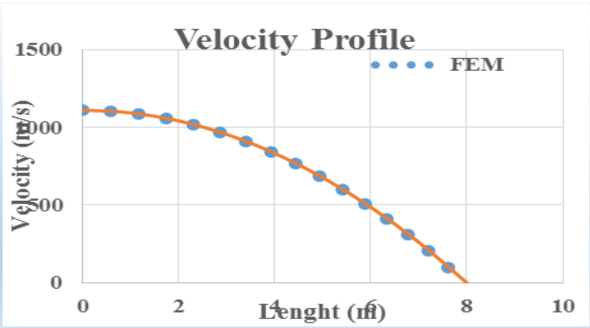

Fig. 2. A graph of velocity against length of plate.

From “Fig. 2”, between 0 and 8m represents the velocity profile. Point 0m is the open surface of the inclined flat surface while point 8m in the wall of the inclined flat surface. From this analysis the maximum velocity of 1109.8748m/s was attained at the open surface of the incline plane.

To verify the accuracy of the results obtained from the Finite Element Method, the results obtained was compared with the results obtained using the exact differential equation method. It was observed from the two methods that their results were in good agreement with one another. From the results shown in Table 1, even with just four linear elements, we were able to have a very high accuracy with a maximum percentage error of 2.3413×10^{-14}. The advantage of the finite element method over the exact differential equation method is that the FEM gives results that represent the velocities at different nodes for the whole material under consideration at the same time unlike the result from the exact differential equation method that provide discrete result at a time and needs further iteration to determine the velocity values at other points of the stationary parallel plates

Table 1. Comparison between the Exact Solution and the FEM Solution

LENGTH	FEM	EXACT	% ERROR
0	1109.8748	1109.8748	0.0000
1	1092.5330	1092.5330	0.0000
2	1040.5076	1040.5076	2.1852E-14
3	953.7987	953.7987	0.0000
4	832.4061	832.4061	1.3658E-14
5	676.3300	676.3300	1.6809E-14
6	485.5702	485.5702	2.3413E-14
7	260.1269	260.1269	2.1852E-14
8	0.0000	0.0000	0.0000

3.1. Effect of change in inclination angle on the velocity profile

In this analysis, it was observed that as the angle of inclination increases, the velocity decreases as well. It is important to note that the angle of inclination was measured between the wall of the inclined material and the vertical. This is shown in “Fig. 2”.

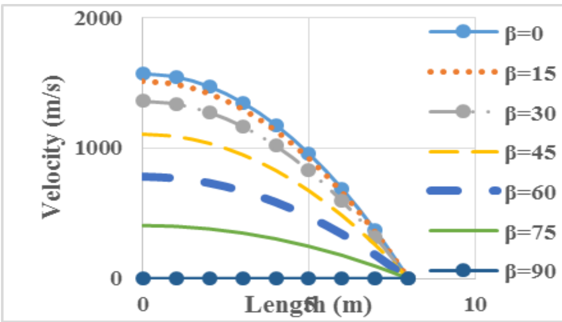

Fig. 3. A graph of velocity against length of plate at different angles of inclination

It is clear here that maximum velocity can be achieved when the angle of inclination is 0° (Zero degree). At this point, with all other parameters held constant, the maximum velocity attained was 1569.6m/s. Also, holding other parameters constant and increasing the angle of inclination, the velocity decreases until an angle of inclination of 90°. At this point, the velocity is almost zero. This can be well represented in "Fig. 4".

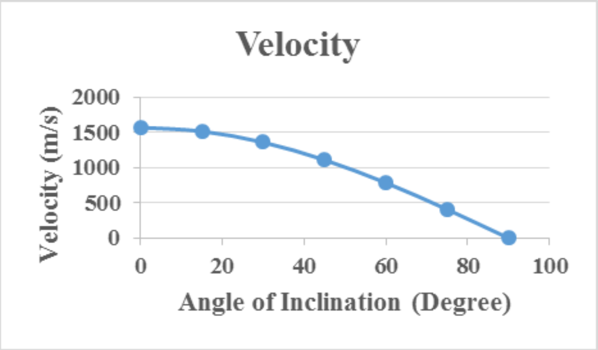

Fig. 4. A graph of velocity against angles of inclination.

In the design of a drainage system for example, the angle of inclination should be set well above 0°, else, the fluid in the drainage will not drained thereby having stagnant fluid which might have detrimental effect on the drainage.

3.2. Effect of Change in Fluid Viscosity and Density on the Velocity Profile

Examining the effect of the change in viscosity on the velocity profile, it was observed that viscosity affect the velocity profile. With an increase in the viscosity of a fluid, the velocity of the fluid was seen to decrease. This decrease in the velocity was as a result of the increase in the drag between the wall of the inclined material and the fluid. This can be shown in "Fig. 5".

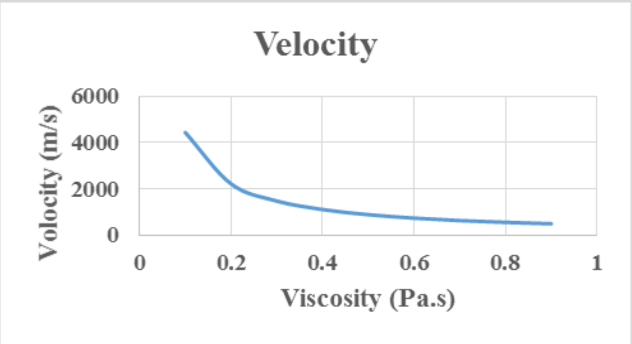

Fig. 5. A graph of velocity against change in viscosity.

Also looking at the effect of a change in the fluid density on the velocity distribution, it was observed that a change in fluid density has a linear relationship with the velocity of the fluid. This means that the higher the fluid density, the higher the velocity of the fluid. This is as shown in "Fig. 6".

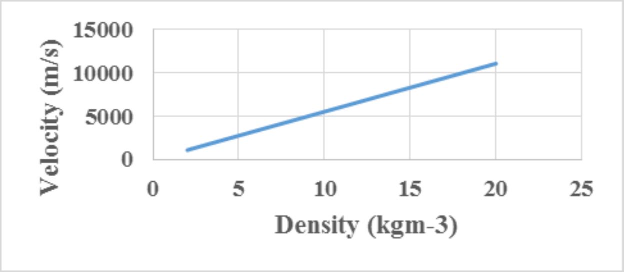

Fig. 6. A graph of velocity against change in density.

4. Conclusion

So far, the finite element method has been used to obtain the velocity profile of a fluid on an inclined plane with steady laminar flow. The results obtained from the FEM were compared with the results obtained from the exact differential equation method and it was discovered that both results agrees. It has also been shown that the higher the angle of inclination and fluid viscosity, the lower the velocity and also the higher the fluid density, the higher the velocity. The result obtained shows that the finite element method is an efficient and accurate method.

References

[1] C. Ancey, ''Flow down inclined channel as a discriminating experiment'', Research Report 2003.

[2] D. Benedetto, E. Caglioti, M. Pulvirenti, ''A kinetic equation for granular media'', RAIRO Mathematical Modeling and Numerical Analysis. Vol. 31, No. 5, pp. 615-641, 1997.

[3] D. Benedetto, M. Pulvirenti, ''On the one-domensional Boltzman equation for granular flows, Physics of Fluids, Vol. 16, No. 12, pp. 4235-4247, 2001.

[4] G. Bognár, I. Gombkötő, K. Hriczó, ''Non-Newtonian Fluid Flow down an Inclined Plane'', Recent Advances in Fluid Mechanics, Heat & Mass Transfer, pp. 129-134, 2018.

[5] J. A. Akpobi, E. D. Akpobi, ''A finite element analysis of the distribution velocity in viscous incompressible fluids using the Lagrange interpolation function'', Journal of Applied Science and Environmental Management, Vol. 11, pp. 31–38, 2007.

[6] I. D. Erhunmwun, M. H. Oladeinde, ''Analysis of flow in a concentric annulus using finite element method'', Nigeria Journal of Technology, Vol. 35, pp. 344-348, 2016.

[7] J. N. Reddy, ''Introduction to the Finite Element Method'', Second edition, McGraw-Hill series in Mechanical Engineering. 1993. Ch. 3.