Journal of
New Theory
ISSN: 2149-1402

28 (2019) 5–19
Journal of New Theory
http://www.newtheory.org
Open Access

An Approach for Multi-Criteria Decision Making Problems with Unknown Weight Information Using GRA Method Under the Picture Linguistic Fuzzy Environment

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Article History
Received: 22.12.2018
Accepted: 09.05.2019
Published: 21.05.2019
Original Article

Abstract − Picture linguistic fuzzy set is the generalize structure over existing structures of fuzzy linguistic sets to arrange uncertainty and imprecise information in decision making problems. Viewing the effectiveness of the picture linguistic fuzzy set, we developed a decision-making approach for the multi-criteria decision-making problems. We also proposed the GRA technique using Choquet integral deal- ing uncertainty in decision making problems under picture linguistic fuzzy information. Lastly, we illustrate an example to shows the effectiveness and reliability of the developed method.

Keywords − *Picture linguistic fuzzy set, Picture linguistic fuzzy Choquet integral weighted averaging (PLF-CIWA) operator, GRA method.*

1. Introduction

Fuzzy set theory concept was first time defined by Zadeh [1]. Fuzzy set are only defined membership function, but more times, it difficult to express more fuzzy information. To deal successfully with something difficult, Attanssov [2] defined the intuitionistic fuzzy set, the development of FS, which included the non-membership degree. After that Attanssov defined interval valued IFS by approaching the positive degree and negative degree to interval number [3–5], and the operational laws and comparison rules for the IvIFSs are defined. The IvIFS illustrate the fuzzy information and is more descriptive than the FS and IFS. Wang and Liu defined some geometric and averaging aggregation operators for different IFNs. Latterly, some multi-criteria decision making problems have also proposed which depend on IFS [6, 7].

Murofushi and Sugeno [8] proposed the notion of Choquet integral with respect to a fuzzy measure. It was defined by Choquet [9] in potential theory with the notion of capacity. The generalization of the classical Lebesgue integral are Choquet integral and has been tested to many other field. Choquet integral are used in many areas like as image processing, pattern recognition, information fusion and data mining [10, 11], and also utilized in economic theory [12, 13], in the context of fuzzy measure theory [14, 15]. Sugeno integral is the other important kind of fuzzy integral, and are introduced by Sugeno [16]. Sugeno integral on the fuzzy sets are generalized by Wang and Qiao [17, 18]. Yu et al. [19] proposed the Choquet integral operator to aggregate the hesitant fuzzy information for MCDM problems. Zhou [20] extended intuitionistic fuzzy Choquet integral correlation coefficient on the base

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of Shapley index. Li et al. [21] proposed the Generalized Interval Neutrosophic Choquet Aggregation Operators.

The concepts of picture fuzzy set was proposed by Coung and investigated basic operators and properties of (PFS) [22]. Picture fuzzy set are basically development of Atanassoves intuitionistic fuzzy set, which represent a membership, neutral membership and a nonmembership degree, is a very strong tool to represent vague and an uncertain information in the process of clustering analysis. When we looksome issue, which have more answers like as: yes, abstain, no, and refusal, in this case we used picture fuzzy numbers. To deal with clustering problems under the picture fuzzy environment Son [23] give the concept of generalized picture fuzzy distance measure. The decision making art are proposed by Wei [24], which is depended on the picture fuzzy weight cross-entropy. Ashraf et al. [25] proposed the series of aggregation opertors for picture fuzzy information. Zeng et al. [26] proposed the liguistic picture fuzzy TOPSIS method for picture fuzzy information. For more study, we refer to [27–32, 53–56].

Moreover, in decision position assessment are given by linguistic terms which is a linguistic values of a linguistic variable. The great deal of qualitative information arise in real decision making problem. Which is simply convert by linguistic terms, like as “very good”,“good”, “fair”,“bad” and “very bad”, etc. In some earlier application, linguistic terms were described for triangular fuzzy numbers [33, 34], trapezoidal fuzzy numbers [35, 36]. The notion of Intuitionistic linguistic set are given by Wang and Li [37], and also derived some decision making methods with ILNs. Pei et al. [38] proposed linguistic weighted aggregation operator for fuzzy risk analysis. Based on dependent operator Liu [39] derived the intuitionistic linguistic generalized dependent ordered weighted averaging operator and intuitionistic linguistic generalized dependent hybrid weighted aggregation operator. Wang et al. [40] defined the comparison rules, score function and accuracy function between two intuitionistic linguistic numbers. Wang [41] developed an ILPGWA operator and ILPGOWA operator based on power operator, and explain some individual cases of these operators with respect to the generalized criterion. Chen et al. [42] introduced the new notion of linguistic intuitionistic fuzzy number. Liu et al. [43] introduced a new linguistic term transformation tecnique in linguistic decision making. Based on Einstein T-norm and T-conorm Liu and You [44] proposed some linguistic intuitionistic fuzzy Heronian mean operators. Due to the motivation and inspiration of the above study in this article, we defined picture linguistic fuzzy set (PLFS), which is generalized form of intuitionistic linguistic fuzzy set. The application of this paper is to introduced the notion of GRA methodology for solving MADM problems under picture linguistic fuzzy information, in which the data about criteria weights are completely unknown, and the criteria values occur in the form of picture linguistic fuzzy numbers.

In preliminaries, we shortly review basic definitions and results about Choquet integral, intuitionistic linguistic fuzzy sets and picture linguistic fuzzy sets. In Section 3, we proposed the concept of picture linguistic fuzzy sets, and also introduce the GRA method for picture linguistic fuzzy MAGDM problems with incomplete weight information in Section 4. In Section 5, we illustrate our introduced algorithm with an example. In Section 6 are conclusion.

2. Preliminary

Some basic definition and notations of IFSs, ILFSs, PFS, PLFS and their operations are discussed. The concept of fuzzy measure and Choquet integral are aslo studied.

Definition 2.1. [45, 46] Let $\acute{L} = \{\pounds_p | p = 0, 1, ..., \ell - 1\}$ be the linguistic set, where as the cardinality of this set is considered as odd number, i.e. a five linguistic terms set $\acute{L}$ can be designate as;

$$\begin{aligned} \acute{L} &= (\pounds_0, \pounds_1, \pounds_2, \pounds_3, \pounds_4) \\ &= \{poor, slightly\ poor,\ fair,\ slightly\ good, good\}. \end{aligned}$$

Definition 2.2. The negation operator: neg $(\acute{L}_p) = \acute{L}_q$, where $q = \ell - 1$;
(1) Be ordered: $\pounds_p \leq \pounds_q \Longleftrightarrow p \leq q$;
(2)Maximum operator: $\max(\pounds_p, \pounds_q) = \pounds_p$ if $\pounds_p \geq \pounds_q$;
(3) Minimum operator: $\min(\pounds_p, \pounds_q) = \pounds_p$ if $\pounds_p \leq \pounds_q$.

$\pounds_{[0,\ell]} = \{\pounds_p | \pounds_0 \leq \pounds_p \leq \pounds_\ell\}$, whose elements also get all the characteristics above, and if $\pounds_p \in \acute{L}$,it is known as the actual term, otherwise, virtual term.

To make something stay the same, Herrera et al. [47] suggest that the distinct linguistic term set $\acute{L} = (\pounds_0\ , \pounds_1, ..., \pounds_{\ell-1})$ is expended to a continuous linguistic term set $\acute{L} = (\pounds_\theta | \theta \in (0, G)$, where G sufficiently large positive number which satisfies the upper characteristics. For any linguistic variables $\pounds_p, \pounds_q \in \acute{L}$, the following condition ae satisfied.

1. $\varpi \otimes \pounds_p = \pounds_{\varpi \cdot p}$
2. $\pounds_p \oplus \pounds_q = \pounds_{p+q}$
3. $\pounds_p / \pounds_q = \pounds_{p/q}$
4. $(\pounds_p)^q = \pounds_{k^q}$

Definition 2.3. [2] An IFS $E_{\breve{u}}$ on the universal set $\mathbb{R} \neq \phi$ is defined as;

$$E_{\breve{u}} = \{\langle P_{\breve{e}_{\breve{u}}}(r), I_{\breve{e}_{\breve{u}}}(r)|\ r \in \mathbb{R}\rangle\},$$

where $P_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0, 1]$ and $I_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0, 1]$ are the membership and non-membership degree of each $r \in \mathbb{R}$, respectively. Moreover $P_{\breve{e}_{\breve{u}}}(r)$ and $I_{\breve{e}_{\breve{u}}}(r)$ satisfy this condition $0 \leq P_{\breve{e}_{\breve{u}}}(r) + I_{\breve{e}_{\breve{u}}}(r) \leq 1 \forall r \in \mathbb{R}$.

Definition 2.4. [48] Let $\mathbb{R} \neq \phi$ be the universe of discourse. Then $E_{\breve{u}}$ is defined as;

$$E_{\breve{u}} = \{\langle \pounds_{\breve{e}_{\breve{u}}}(r), P_{\breve{e}_{\breve{u}}}(r), I_{\breve{e}_{\breve{u}}}(r)|\ r \in \mathbb{R}\rangle\},$$

an ILFS in a set $\mathbb{R}$ is denoted by $\pounds_{\breve{e}_{\breve{u}}}(r) \in L$ are the linguistic term, $P_{\breve{e}_{\breve{u}}}(r)$ and $I_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0, 1]$ be the membership and non-membership of each $r \in \mathbb{R}$, respectively. Moreover $P_{\breve{e}_{\breve{u}}}(r)$ and $I_{\breve{e}_{\breve{u}}}(r)$ satisfy this condition$0 \leq P_{\breve{e}_{\breve{u}}}(r) + I_{\breve{e}_{\breve{u}}}(r) \leq 1$ for all $r \in \mathbb{R}$.

Definition 2.5. [22] A PFS $E_{\breve{u}}$ on the universal set $\mathbb{R} \neq \phi$ is defined as;

$$E_{\breve{u}} = \{\langle P_{\breve{e}_{\breve{u}}}(r), I_{\breve{e}_{\breve{u}}}(r), N_{\breve{e}_{\breve{u}}}(r)|\ r \in \mathbb{R}\rangle\}.$$

where $P_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0, 1]$, $I_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0, 1]$ and $N_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0, 1]$ are the positive membership, neutral membership and negative membership of each $r \in \mathbb{R}$, respectively. Furthermore, $P_{\breve{e}_{\breve{u}}}(r)$, $I_{\breve{e}_{\breve{u}}}(r)$ and $N_{\breve{e}_{\breve{u}}}(r)$ satisfy this condition $0 \leq P_{\breve{e}_{\breve{u}}}(r) + I_{\breve{e}_{\breve{u}}}(r) + N_{\breve{e}_{\breve{u}}}(r) \leq 1 \forall r \in \mathbb{R}$.

2.1. Fuzzy measure and Choquet integral

Definition 2.6. [9] Let $\mathbb{R} = \{r_1, r_2, ..., r_n\} \neq \phi$ be the universe of discourse and $p(\mathbb{R})$ denotes the power set of $\mathbb{R}$. Then, a fuzzy measure $P_{\breve{e}_{\breve{u}}}$ on $\mathbb{R}$ is a mapping $P_{\breve{e}_{\breve{u}}}$: p($\mathbb{R}$) $\to [0, 1]$, satisfying the subsequent conditions;

1) $P_{\breve{e}_{\breve{u}}}(\phi) = 0$, $P_{\breve{e}_{\breve{u}}}(\mathbb{R}) = 1$.

2) If $E_{\breve{u}_1}, E_{\breve{u}_2} \in p(\mathbb{R})$ and $E_{\breve{u}_1} \subseteq E_{\breve{u}_2}$ then $P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) \leq P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_2})$.

Where $P_{\breve{e}_{\breve{u}}}(\{r_1, r_2, ..., r_n\})$ can be considered as the grade of subjective importance of decision criteria set $\{r_1, r_2, ..., r_n\}$. Thus, with the separate weights of criterias can also be defined. Naturally, we could say the following about any pair of criterias sets $E_{\breve{u}_1}, E_{\breve{u}_2} \in p(\mathbb{R})$, $E_{\breve{u}_1} \cap E_{\breve{u}_2} = \phi$; $E_{\breve{u}_1}$and $E_{\breve{u}_2}$ are considered to be without interaction if

$$P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1} \cup E_{\breve{u}_2}) = P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) + P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_2}) \tag{1}$$

which is known as additive measure. $E_{\breve{u}_1}$and $E_{\breve{u}_2}$ exhibit a positive synergetic interaction between them (or are complementary) if

$$P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1} \cup E_{\breve{u}_2}) > P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) + P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_2}) \tag{2}$$

which is called a superadditive measure. $E_{\breve{u}_1}$and $E_{\breve{u}_2}$ exhibit a negative synergetic interaction between them (or redundant or substitutive) if

$$P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1} \cup E_{\breve{u}_2}) < P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) + P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_2}) \tag{3}$$

known as sub-additive measure.

Since it is difficult to find the fuzzy measure according to Definition 2.6, therefore, to confirm a fuzzy measure in MAGDM problems, Sugeno [16] given below, λ-fuzzy measure:

$$P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1} \cup E_{\breve{u}_2}) = P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) + P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_2}) + \lambda P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_2}) \quad (4)$$

$\lambda \in [-1, \infty)$, $E_{\breve{u}_1} \cap E_{\breve{u}_2} = \phi$. The interaction between the criterias are determines the parameter λ . If we put $\lambda = 0$, in Equation 4, then, λ-fuzzy measure become an additive measure. And for negative and positive λ, the λ-fuzzy measure reduces to subadditive and superadditive measures, respectively. Meantime, if all the elements in $\mathbb{R}$ are independent, and we have

$$P_{\breve{e}_{\breve{u}}}(E_{\breve{u}}) = \sum_{p=1}^{n} P_{\breve{e}_{\breve{u}}}(\{r_p\}) \quad (5)$$

If we consider $\mathbb{R}$ is a finite set, then $\cup_{\mathrm{p}=1}^{n} r_{\mathrm{p}} = \mathbb{R}$, and λ-fuzzy measure $P_{\breve{e}_{\breve{u}}}$ satisfies following Equation 6

$$P_{\breve{e}_{\breve{u}}}(\mathbb{R}) = P_{\breve{e}_{\breve{u}}}\left(\cup_{p=1}^{n} r_i\right) = \begin{cases} \frac{1}{\lambda}\left(\prod_{p=1}^{n} [1 + \lambda P_{\breve{e}_{\breve{u}}}(r_p)] - 1\right) & \text{if } \lambda \neq 0 \\ \sum_{p=1}^{n} P_{\breve{e}_{\breve{u}}}(r_p) & \text{if } \lambda = 0 \end{cases} \quad (6)$$

where $r_{\mathrm{p}} \cap r_{\check{\mathrm{d}}} = \phi$ for all $p, \check{\mathrm{d}} = 1, 2, ..., n$ and $p \neq \check{\mathrm{d}}$. A fuzzy density $P_{\breve{e}_{\breve{u}}}(r_{\mathrm{p}})$ for a subset with a single element r_{p} is denoted as $P_{\breve{e}_{\breve{u}_{\mathrm{p}}}} = P_{\breve{e}_{\breve{u}}}(r_{\mathrm{p}})$.

Especially for every subset $E_{\breve{u}_1} \in p(\mathbb{R})$, we have

$$P_{\breve{e}_{\breve{u}}}(E_{\breve{u}_1}) = \begin{cases} \frac{1}{\lambda}\left(\prod_{p=1}^{n} [1 + \lambda P_{\breve{e}_{\breve{u}}}(r_p)] - 1\right) & \text{if } \lambda \neq 0 \\ \sum_{p=1}^{n} P_{\breve{e}_{\breve{u}}}(r_p) & \text{if } \lambda = 0 \end{cases} \quad (7)$$

Based on Equation 2, we determined the value of λ from $P_{\breve{e}_{\breve{u}}}(\mathbb{R}) = 1$, and is equal to solved this equation;

$$\lambda + 1 = \prod_{p=1}^{n} \left[1 + \lambda P_{\breve{e}_{\breve{u}_p}}\right] \quad (8)$$

It should be recognized that the value of λ can be uniquely determined by $P_{\breve{e}_{\breve{u}}}(\mathbb{R}) = 1$.

Definition 2.7. [16] Assume that f and $P_{\breve{e}_{\breve{u}}}$ be a positive real-valued function and fuzzy measure on $\mathbb{R}$, respectivily. The discrete Choquet integral of f with respect to $P_{\breve{e}_{\breve{u}}}$ is defined by

$$C_{\mu}(f) = \sum_{p=1}^{n} f_{\rho(p)}[P_{\breve{e}_{\breve{u}}}(A_{\rho(p)}) - P_{\breve{e}_{\breve{u}}}(A_{\rho(p\text{-}1)})] \quad (9)$$

$\rho(p)$ indicates a permutation on $\mathbb{R}$, where $f_{\rho(1)} \geq f_{\rho(2)} \geq ... \geq f_{\rho(n)}$, $A_{\rho(n)} = \{1, 2, ..., p\}$, $A_{\rho(0)} = \phi$.

3. Linguistic Picture Fuzzy Set and their Operations

We discussed in this section linguistic picture fuzzy set concept and their operationals laws.

Definition 3.1. [50] Let $\mathbb{R} \neq \phi$ be a universal set. Then, $E_{\breve{u}}$ is called a picture linguistic set, and defined as;

$$E_{\breve{u}} = \{\langle \pounds_{\breve{e}_{\breve{u}}}(r), P_{\breve{e}_{\breve{u}}}(r), I_{\breve{e}_{\breve{u}}}(r), N_{\breve{e}_{\breve{u}}}(r) |\ r \in \mathbb{R}\rangle\},$$

where $\pounds_{\breve{e}_{\breve{u}}}(r) \in L$, $P_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0, 1]$, $I_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0, 1]$ and $N_{\breve{e}_{\breve{u}}}(r) : \mathbb{R} \to [0, 1]$ are the linguistic term, the positive, neutral and negative membership degrees of each $r \in \mathbb{R}$, respectively. Furthermore $P_{\breve{e}_{\breve{u}}}(r)$, $I_{\breve{e}_{\breve{u}}}(r)$ and $N_{\breve{e}_{\breve{u}}}(r)$ satisfy that $0 \leq P_{\breve{e}_{\breve{u}}}(r) + I_{\breve{e}_{\breve{u}}}(r) + N_{\breve{e}_{\breve{u}}}(r) \leq 1\ \forall\ r \in \mathbb{R}$.

Definition 3.2. Let $E_{\breve{u}_1} = \left\langle \pounds_{\breve{e}_{\breve{u}_1}}, P_{\breve{e}_{\breve{u}_1}}, I_{\breve{e}_{\breve{u}_1}}, N_{\breve{e}_{\breve{u}_1}} \right\rangle$ and $E_{\breve{u}_2} = \left\langle \pounds_{\breve{e}_{\breve{u}_2}}, P_{\breve{e}_{\breve{u}_2}}, I_{\breve{e}_{\breve{u}_2}}, N_{\breve{e}_{\breve{u}_2}} \right\rangle$ are two PLFNs define on the universe of discourse $\mathbb{R} \neq \phi$, some operations on PLFNs are defined as follows with $\psi \geq 0$.

1. $E_{\breve{u}_1} \oplus E_{\breve{u}_2} = \left\{ \pounds_{\breve{e}_{\breve{u}_1} + \breve{e}_{\breve{u}_2}}, P_{\breve{e}_{\breve{u}_1}} + P_{\breve{e}_{\breve{u}_2}} - P_{\breve{e}_{\breve{u}_1}} \cdot P_{\breve{e}_{\breve{u}_2}},\ I_{\breve{e}_{\breve{u}_1}} \cdot I_{\breve{e}_{\breve{u}_2}},\ N_{\breve{e}_{\breve{u}_1}} \cdot N_{\breve{e}_{\breve{u}_2}} \right\}$
2. $\psi \cdot E_{\breve{u}} = \left\{ \pounds_{\psi . \breve{e}_{\breve{u}}}, 1 - (1 - P_{\breve{e}_{\breve{u}_1}})^{\psi},\ (I_{\breve{e}_{\breve{u}}})^{\psi},\ (N_{\breve{e}_{\breve{u}}})^{\psi} \right\}$
3. $E_{\breve{u}_1} \otimes E_{\breve{u}_2} = \left\{ \pounds_{\breve{e}_{\breve{u}_1} \times \breve{e}_{\breve{u}_2}}, P_{\breve{e}_{\breve{u}_1}} \cdot P_{\breve{e}_{\breve{u}_2}},\ I_{\breve{e}_{\breve{u}_1}} \cdot I_{\breve{e}_{\breve{u}_2}},\ N_{\breve{e}_{\breve{u}_1}} + N_{\breve{e}_{\breve{u}_2}} - N_{\breve{e}_{\breve{u}_1}} \cdot N_{\breve{e}_{\breve{u}_2}} \right\}$
4. $(E_{\breve{u}})^{\psi} = \left\{ \pounds_{(\breve{e}_{\breve{u}})^{\psi}}, (P_{\breve{e}_{\breve{u}}})^{\psi},\ (I_{\breve{e}_{\breve{u}}})^{\psi},\ 1 - (1 - N_{\breve{e}_{\breve{u}}})^{\psi} \right\}$

3.1. Comparison Rules for PLFNs

To rank the PLFNs, we defined some function in this section, which are the following.

Definition 3.3. Let $E_{\breve{u}} = \langle \pounds_{\breve{e}_{\breve{u}}}, P_{\breve{e}_{\breve{u}}}, I_{\breve{e}_{\breve{u}}}, N_{\breve{e}_{\breve{u}}} \rangle$ be any PLFNs. Then

1. $sc(E_{\breve{u}}) = \frac{\pounds_{\breve{e}_{\breve{u}}} \times \left(P_{\breve{e}_{\breve{u}}} - I_{\breve{e}_{\breve{u}}} - N_{\breve{e}_{\breve{u}}}\right)}{3}$ (score function).
2. $ac(E_{\breve{u}}) = \frac{\pounds_{\breve{e}_{\breve{u}}}}{2} \left(P_{\breve{e}_{\breve{u}}} + N_{\breve{e}_{\breve{u}}}\right)$ (accuracy function).
3. $cr(E_{\breve{u}}) = \frac{\pounds_{\breve{e}_{\breve{u}}}}{2} (P_{\breve{e}_{\breve{u}}})$ (certainty function).

Definition 3.4. Let $E_{\breve{u}_1} = \left\langle \pounds_{\breve{e}_{\breve{u}_1}}, P_{\breve{e}_{\breve{u}_1}}, I_{\breve{e}_{\breve{u}_1}}, N_{\breve{e}_{\breve{u}_1}} \right\rangle$ and $E_{\breve{u}_2} = \left\langle \pounds_{\breve{e}_{\breve{u}_2}}, P_{\breve{e}_{\breve{u}_2}}, I_{\breve{e}_{\breve{u}_2}}, N_{\breve{e}_{\breve{u}_2}} \right\rangle$ are two PLFNs define on the universe of discourse $\mathbb{R} \neq \phi$. With the help of Definition 3.3, we defined the following rules,

1. If $sc(E_{\breve{u}_1}) \succ sc(E_{\breve{u}_2})$,then $E_{\breve{u}_1} \succ E_{\breve{u}_2}$.
2. If $sc(E_{\breve{u}_1}) \approx sc(E_{\breve{u}_2})$,and $ac(E_{\breve{u}_1}) \succ ac(E_{\breve{u}_2})$,then $E_{\breve{u}_1} \succ E_{\breve{u}_2}$.
3. If $sc(E_{\breve{u}_1}) \approx sc(E_{\breve{u}_2})$, $ac(E_{\breve{u}_1}) \approx ac(E_{\breve{u}_2})$ and $cr(E_{\breve{u}_1}) \succ cr(E_{\breve{u}_2})$, then $E_{\breve{u}_1} \succ E_{\breve{u}_2}$.
4. If $sc(E_{\breve{u}_1}) \approx sc(E_{\breve{u}_2})$, $ac(E_{\breve{u}_1}) \approx ac(E_{\breve{u}_2})$ and $cr(E_{\breve{u}_1}) \approx cr(E_{\breve{u}_2})$, then $E_{\breve{u}_1} \approx E_{\breve{u}_2}$.

Definition 3.5. Let any collections $E_{\breve{u}_{\mathrm{p}}} = \left\langle \pounds_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, P_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, I_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, N_{\breve{e}_{\breve{u}_{\mathrm{p}}}} \right\rangle$, $p \in N$ be the PLFNs and $PLFWA : PLFN^n \rightarrow PLFN$, then $PLFWA$ describe as,

$$PLFWA\left(E_{\breve{u}_1}, E_{\breve{u}_2}, ..., E_{\breve{u}_n}\right) = \sum_{p=1}^{n} \psi_p E_{\breve{u}_p}, \tag{10}$$

such that $\psi = \{\psi_1, \psi_2, ..., \psi_n\}^T$ be the weight vector of $E_{\breve{u}_{\mathrm{p}}} = \left\langle \pounds_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, P_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, I_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, N_{\breve{e}_{\breve{u}_{\mathrm{p}}}} \right\rangle$, $p \in N$, with $\psi_{\mathrm{p}} \geq 0$ and $\sum_{\mathrm{p}=1}^{n} \psi_{\mathrm{p}} = 1$.

Theorem 3.6. Suppose that $E_{\breve{u}_{\mathrm{p}}} = \left\langle \pounds_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, P_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, I_{\breve{e}_{\breve{u}_{\mathrm{p}}}}, N_{\breve{e}_{\breve{u}_{\mathrm{p}}}} \right\rangle$, $p \in N$ be the collection of PLFNs. Then by using the Definition 3.5 and operational properties of PLFNs, we can obtained the following outcome.

$$PLFWA\left(E_{\breve{u}_1}, E_{\breve{u}_2}, ..., E_{\breve{u}_n}\right) = \left\{ \begin{array}{c} \pounds_{\sum\limits_{p=1}^{n} \psi_p \cdot \breve{e}_{\breve{u}_p}}, 1 - \Pi_{p=1}^{n}(1 - P_{\breve{e}_{\breve{u}_p}})^{\psi_p}, \\ \Pi_{p=1}^{n}(I_{\breve{e}_{\breve{u}_p}})^{\psi_p}, \\ \Pi_{p=1}^{n}(N_{\breve{e}_{\breve{u}_p}})^{\psi_p} \end{array} \right\} \tag{11}$$

Definition 3.7. Let any collections $E_{\breve{u}_p} = \left\langle \pounds_{\breve{e}_{\breve{u}_p}}, P_{\breve{e}_{\breve{u}_p}}, I_{\breve{e}_{\breve{u}_p}}, N_{\breve{e}_{\breve{u}_p}} \right\rangle$, $p \in N$ be the PLFNs and $PLFOWA : PLFN^n \to PLFN$, then $PLFOWA$ describe as,

$$PLFOWA\left(E_{\breve{u}_1}, E_{\breve{u}_2}, ..., E_{\breve{u}_n}\right) = \sum_{p=1}^{n} \psi_p E_{\breve{u}_{\rho(p)}}, \tag{12}$$

In which $\psi = \{\psi_1, \psi_2, ..., \psi_n\}$ be the weight vector of $E_{\breve{u}_p} = \left\langle \pounds_{\breve{e}_{\breve{u}_p}}, P_{\breve{e}_{\breve{u}_p}}, I_{\breve{e}_{\breve{u}_p}}, N_{\breve{e}_{\breve{u}_p}} \right\rangle$, $p \in N$, with $\psi_p \geq 0$ and $\sum_{p=1}^{n} \psi_p = 1$ and $\rho(p)$ indicates a permutation on $\mathbb{R}$.

Theorem 3.8. Suppose that $E_{\breve{u}_p} = \left\langle \pounds_{\breve{e}_{\breve{u}_p}}, P_{\breve{e}_{\breve{u}_p}}, I_{\breve{e}_{\breve{u}_p}}, N_{\breve{e}_{\breve{u}_p}} \right\rangle$, $p \in N$ be the collections of PLFNs. Then, by using the Definition 3.7 and operational properties of PLFNs, the following equation is obtained.

$$PLFOWA\left(E_{\breve{u}_1}, E_{\breve{u}_2}, ..., E_{\breve{u}_n}\right) = \left\{ \begin{array}{c} \pounds_{\sum_{p=1}^{n} \psi_p \cdot \breve{e}_{\breve{u}_{\rho(p)}}}, 1 - \Pi_{p=1}^{n}\left(1 - P_{\breve{e}_{\breve{u}_{\rho(p)}}}\right)^{\psi_p}, \\ \Pi_{p=1}^{n}\left(I_{\breve{e}_{\breve{u}_{\rho(p)}}}\right)^{\psi_p}, \\ \Pi_{p=1}^{n}\left(N_{\breve{e}_{\breve{u}_{\rho(p)}}}\right)^{\psi_p} \end{array} \right\} \tag{13}$$

Theorem 3.9. Suppose that $E_{\breve{u}_p} = \left\langle \pounds_{\breve{e}_{\breve{u}_p}}, P_{\breve{e}_{\breve{u}_p}}, I_{\breve{e}_{\breve{u}_p}}, N_{\breve{e}_{\breve{u}_p}} \right\rangle$, $p \in N$ be the collections of PLFNs and λ be a fuzzy measure on $\mathbb{R}$. Based on fuzzy measure, a Picture linguistic fuzzy Choquet integral weighted averaging ($PLFCIWA$) operator of dimension n is a mapping $PLFCIWA : PLFN^n \to PLFN$ such that

$$PLFCIWA\left(E_{\breve{u}_1}, E_{\breve{u}_2}, ..., E_{\breve{u}_n}\right) \tag{14}$$

$$= \left\{ \begin{array}{c} \pounds_{\sum_{p=1}^{n} \lambda\left(A_{\rho(p)}\right) - \lambda\left(A_{\rho(p\text{-}1)}\right) \cdot \breve{e}_{\breve{u}_{\rho(p)}}}, 1 - \Pi_{p=1}^{n}\left(1 - P_{\breve{e}_{\breve{u}_{\rho(p)}}}\right)^{\lambda\left(A_{\rho(p)}\right) - \lambda\left(A_{\rho(p\text{-}1)}\right)}, \\ \Pi_{p=1}^{n}\left(I_{\breve{e}_{\breve{u}_{\rho(p)}}}\right)^{\lambda\left(A_{\rho(p)}\right) - \lambda\left(A_{\rho(p\text{-}1)}\right)}, \\ \Pi_{p=1}^{n}\left(N_{\breve{e}_{\breve{u}_{\rho(p)}}}\right)^{\lambda\left(A_{\rho(p)}\right) - \lambda\left(A_{\rho(p\text{-}1)}\right)} \end{array} \right\}$$

where $\rho(p)$ indicates a permutation on $\mathbb{R}$ and $A_{\rho(n)} = \{1, 2, ..., p\}$, $A_{\rho(0)} = \phi$.

Definition 3.10. Let $\mathbb{R} \neq \phi$ be the universal set, and E_j, E_l be the any two picture linguistic fuzzy sets. Then, normalized Hamming distance $d_{NHD}(E_j, E_l)$ is given as for all $r \in \mathbb{R}$,

$$d_{NHD}\left(E_{\breve{u}_j}, E_{\breve{u}_l}\right) = \frac{1}{2(l-1)} \sum_{p=1}^{n} \left| \begin{array}{c} \left(P_{\breve{e}_{\breve{u}_j}}(r_p) - I_{\breve{e}_{\breve{u}_j}}(r_p) - N_{\breve{e}_{\breve{u}_j}}(r_p)\right) \pounds_{\breve{e}_{\breve{u}_j}} - \\ \left(P_{\breve{e}_{\breve{u}_l}}(r_p) - I_{\breve{e}_{\breve{u}_l}}(r_p) - N_{\breve{e}_{\breve{u}_l}}(r_p)\right) \pounds_{\breve{e}_{\breve{u}_l}} \end{array} \right| \tag{15}$$

4. Approach for Multiple criteria Decision Making with Incomplete Weight Information Using GRA Method under the Picture Linguistic Fuzzy Envioursment

Assume that $A = (a_1, ..., a_m)$ be the m alternatives and $C = \{c_1, c_2, ..., c_n\}$, denoted n criteria, and weight criteria is $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$, where $\varpi_k \geq 0$ $(k = 1, 2, ..., n)$, $\Sigma_{k=1}^{n} \varpi_k = 1$. Let assume that DM deliver information about weights of criteria may be denotes in the following form [51], for $j \neq k$,

(a) If $\{\varpi_j \geq \varpi_k\}$, then, the ranking is weak.

(b) If $\{\varpi_j - \varpi_k \geq \lambda_j (> 0)\}$, then, the ranking is strict.

(c) If $\{\varpi_j \geq \lambda_j \varpi_k\}$, $0 \leq \lambda_j \leq 1$,then, the ranking is multiple ranking.

(d) If $\{\lambda_j \leq \varpi_j \leq \lambda_j + \delta_j\}$, $0 \leq \lambda_j \leq \lambda_j + \delta_j \leq 1$, then, the ranking is an interval ranking.

For facility, Δ stand for the set of the known information about criteria weights contribute by the experts.

Let $R^k = \left[\mathrm{E}_{\breve{u}_{pq}}^{(k)}\right]_{m \times n}$ be an picture linguistic fuzzy decision matrix, provided by decision maker

$d_k (k = 1, 2, ..., l)$, as the following form:

$$R^k = \left[E^{(k)}_{\breve{u}_{pq}}\right]_{m\times n} = \begin{array}{c|cccc} & c_1 & c_2 & \cdots & c_n \\ \hline a_1 & E^{(k)}_{\breve{u}_{11}} & E^{(k)}_{\breve{u}_{12}} & \cdots & E^{(k)}_{\breve{u}_{1n}} \\ a_2 & E^{(k)}_{\breve{u}_{21}} & E^{(k)}_{\breve{u}_{22}} & \cdots & E^{(k)}_{\breve{u}_{2n}} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_m & E^{(k)}_{\breve{u}_{m1}} & E^{(k)}_{\breve{u}_{m2}} & \cdots & E^{(k)}_{\breve{u}_{mn}} \end{array} \tag{16}$$

where $E^{(k)}_{\breve{u}_{pq}} = \left(£^{(k)}_{\breve{e}_{\breve{u}_{pq}}}, P^{(k)}_{\breve{u}_{pq}}, I^{(k)}_{\breve{u}_{pq}}, N^{(k)}_{\breve{u}_{pq}}\right)$ is an PLFN representing the performance rating of the alternative $a_p \in A$ with respect to the criteria $c_p \in C$ provided by the decision makers d_k.

To extend GRA method in the process of group decision making, we first need to fuse all individual decision matrices into a collective matrix by using PLFCIWA operator.

Step 1 Suppose that for every $A = \{a_1, a_2, ..., a_m\}$, m alternative, each expert d_k $(k = 1, 2, ..., r)$ is invited to express their individual evaluation or preference according to each criterias $C_q(q = 1, 2, ..., n)$ by an picture liguistic fuzzy numbers $E^{(k)}_{\breve{u}_{pq}} = \left(£^{(k)}_{\breve{e}_{\breve{u}_{pq}}}, P^{(k)}_{\breve{u}_{pq}}, I^{(k)}_{\breve{u}_{pq}}, N^{(k)}_{\breve{u}_{pq}}\right)$ $(p = 1, 2, ..., m; q = 1, 2, ..., n, k = 1, 2, ..., r)$ expressed by the exparts d_k. In this step we construct the picture liguistic fuzzy decision making matrices, $D^s = \left[E^{(s)}_{ip}\right]_{m\times n}$ $(s = 1, 2, ..., k)$ for decision. If the criteria have two types, such as benefit criteria and cost criteria, then the picture liguistic fuzzy decision matrices, $D^s = \left[E^s_{ip}\right]_{m\times n}$ can be converted into the normalized linguistic picture fuzzy decision matrices, $R^k = \left[\mathrm{E}^{(k)}_{\breve{u}_{pq}}\right]_{m\times n}$, where $E^{(k)}_{\breve{u}_{pq}} = \begin{cases} \mathrm{E}^{(k)}_{\breve{u}_{pq}}, & \text{for benefit criteria } A_p \\ \overline{\mathrm{E}^{(k)}_{\breve{u}_{pq}}}, & \text{for cost criteria } A_p, \end{cases}$ $j = 1, 2, ..., n$, and $\overline{\mathrm{E}^{(k)}_{\breve{u}_{pq}}}$ is the complement of $E^{(k)}_{\breve{u}_{pq}}$.The normalization are not requrid, if all the criteria have the same type. Then, we obtain the decision making matrix as follow:

$$R^k = \left[E^{(k)}_{\breve{u}_{pq}}\right]_{m\times n} = \begin{array}{c|cccc} & c_1 & c_2 & \cdots & c_n \\ \hline a_1 & E^{(k)}_{\breve{u}_{11}} & E^{(k)}_{\breve{u}_{12}} & \cdots & E^{(k)}_{\breve{u}_{1n}} \\ a_2 & E^{(k)}_{\breve{u}_{21}} & E^{(k)}_{\breve{u}_{22}} & \cdots & E^{(k)}_{\breve{u}_{2n}} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_m & E^{(k)}_{\breve{u}_{m1}} & E^{(k)}_{\breve{u}_{m2}} & \cdots & E^{(k)}_{\breve{u}_{mn}} \end{array}$$

Step 2 Confirm the fuzzy density $P_{\breve{e}_{\breve{u}_p}} = P_{\breve{e}_{\breve{u}}}(a_p)$ of each expert. According to Eq.(8), parameter λ_1 of expert can be determined.

Step 3 By Definition 2.7, $E^{(k)}_{\breve{u}_{pq}}$ is reordered such that $E^{(k)}_{\breve{u}_{pq}} \geq E^{(k-1)}_{\breve{u}_{pq}}$. Utilize the picture liguistic fuzzy Choquet integral average operator;

$$PFCIWA\left(E^{(1)}_{\breve{u}_{pq}}, E^{(2)}_{\breve{u}_{pq}}, ..., E^{(r)}_{\breve{u}_{pq}}\right) \tag{17}$$
$$= \left\{ \begin{array}{c} £_{\sum\limits_{p=1}^{r} \lambda(A_{\rho(p)})-\lambda(A_{\rho(p-1)})\cdot \breve{e}_{\breve{u}_{\rho(p)}}}, 1 - \Pi^r_{p=1}(1 - P_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)})-\lambda(A_{\rho(p-1)})}, \\ \Pi^r_{p=1}(I_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)})-\lambda(A_{\rho(p-1)})}, \\ \Pi^r_{p=1}(N_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)})-\lambda(A_{\rho(p-1)})} \end{array} \right\}$$

to aggregate all the picture linguistic fuzzy decision matrices $R^k = \left[\mathrm{E}^{(k)}_{\breve{u}_{pq}}\right]_{m\times n}$ $(k = 1, 2, ..., r)$ into a collective picture linguistic fuzzy decision matrix $R = \left[\mathrm{E}^{(k)}_{\breve{u}_{pq}}\right]_{m\times n}$ where $E^{(k)}_{\breve{u}_{pq}} = \left(£^{(k)}_{\breve{e}_{\breve{u}_{pq}}}, P^{(k)}_{\breve{u}_{pq}}, I^{(k)}_{\breve{u}_{pq}}, N^{(k)}_{\breve{u}_{pq}}\right)$ $(p = 1, 2, ..., m; q = 1, 2, ..., n, k = 1, 2, ..., r)$, where $\rho(p)$ indicates a permutation on $\mathbb{R}$ and $A_{\rho(n)} = \{1, 2, ..., p\}$, $A_{\rho(0)} = \phi$ and $P_{\breve{e}_{\breve{u}}}(a_p)$ are find by Equation (9).

Step 4 The picture linguistic fuzzy positive-ideal solution (PLFPIS), stand for $P^{+} = \{P_1^{+}, P_2^{+}, ..., P_m^{+}\}$ and the picture linguistic fuzzy negative-ideal solution (PLFNIS), stand for $P^{-} = \{P_1^{-}, P_2^{-}, ..., P_m^{-}\}$ are defined as

$$P_p^{+} = \max_q sc_{pq}, \tag{18}$$

and

$$P_p^{-} = \min_q sc_{pq}, \tag{19}$$

where $P^{+} = \left(\pounds_{\breve{u}_p}^{+}, P_{\breve{u}_p}^{+}, I_{\breve{u}_p}^{+}, N_{\breve{u}_p}^{+}\right)$ and $P^{-} = \left(\pounds_{\breve{u}_p}^{-}, P_{\breve{u}_p}^{-}, I_{\breve{u}_p}^{-}, N_{\breve{u}_p}^{-}\right)$ $p = 1, 2, .., m$.

Step 5 According to linguistic picture fuzzy distance, find the distance between the alternative a_{p} and the PLFPIS P^{+} and the PLFNIS P^{-}, respectively;

$$d(E_{\breve{u}_{\mathrm{j}}}, E_{\breve{u}_{\mathrm{l}}}) = \frac{1}{2(l-1)} \sum_{p=1}^{n} \left| \begin{array}{c} \left(P_{\breve{e}_{\breve{u}_{\mathrm{j}}}}(r_p) - I_{\breve{e}_{\breve{u}_{\mathrm{j}}}}(r_p) - N_{\breve{e}_{\breve{u}_{\mathrm{j}}}}(r_p)\right) \pounds_{\breve{e}_{\breve{u}_{\mathrm{j}}}} - \\ \left(P_{\breve{e}_{\breve{u}_{\mathrm{l}}}}(r_p) - I_{\breve{e}_{\breve{u}_{\mathrm{l}}}}(r_p) - N_{\breve{e}_{\breve{u}_{\mathrm{l}}}}(r_p)\right) \pounds_{\breve{e}_{\breve{u}_{\mathrm{l}}}} \end{array} \right| \tag{20}$$

The above defined distance is called the Normalized Hamming distance [22] $d(e_j, e_k)$, and form a linguistic picture fuzzy positive-ideal separation matrix D^{+} and linguistic picture fuzzy negative-ideal separation matrix D^{-} as follows;

$$D^{+} = (D_{pq}^{+})_{m\times n} = \begin{bmatrix} d\left(E_{\breve{u}_{11}}, P_1^{+}\right) & d\left(E_{\breve{u}_{12}}, P_2^{+}\right) & ... & d\left(E_{\breve{u}_{1\mathrm{n}}}, P_n^{+}\right) \\ d\left(E_{\breve{u}_{21}}, P_1^{+}\right) & d\left(E_{\breve{u}_{22}}, P_2^{+}\right) & ... & d\left(E_{\breve{u}_{1\mathrm{n}}}, P_n^{+}\right) \\ . & . & & . \\ . & . & ... & . \\ . & . & & . \\ d\left(E_{\breve{u}_{\mathrm{m}1}}, P_1^{+}\right) & d\left(E_{\breve{u}_{\mathrm{m}2}}, P_2^{+}\right) & ... & d\left(E_{\breve{u}_{\mathrm{mn}}}, P_n^{+}\right) \end{bmatrix} \tag{21}$$

and

$$D^{-} = (D_{\mathrm{q}}^{-})_{m\times n} = \begin{bmatrix} d\left(E_{\breve{u}_{11}}, P_1^{-}\right) & d\left(E_{\breve{u}_{12}}, P_2^{-}\right) & ... & d\left(E_{\breve{u}_{1\mathrm{n}}}, P_n^{-}\right) \\ d\left(E_{\breve{u}_{21}}, P_1^{-}\right) & d\left(E_{\breve{u}_{22}}, P_2^{-}\right) & ... & d\left(E_{\breve{u}_{1\mathrm{n}}}, P_n^{-}\right) \\ . & . & & . \\ . & . & ... & . \\ . & . & & . \\ d\left(E_{\breve{u}_{\mathrm{m}1}}, P_1^{-}\right) & d\left(E_{\breve{u}_{\mathrm{m}2}}, P_2^{-}\right) & ... & d\left(E_{\breve{u}_{\mathrm{mn}}}, P_n^{-}\right) \end{bmatrix} \tag{22}$$

Step 6 Grey coefficient for every alternative is calculated from PIS and NIS by using the following equation. The grey coefficient for each alternative calculated from PIS is provided as

$$\xi_{pq}^{+} = \frac{\min_{1\le p\le m} min_{1\le q\le n} d\left(E_{\breve{u}_{pq}}, P_p^{+}\right) + \rho \max_{1\le p\le m} max_{1\le q\le n} d\left(E_{\breve{u}_{pq}}, P_p^{+}\right)}{d\left(E_{\breve{u}_{pq}}, P_p^{+}\right) + \rho \max_{1\le p\le m} max_{1\le q\le n} d\left(E_{\breve{u}_{pq}}, P_p^{+}\right)} \tag{23}$$

Where $p = 1, ..., m$ and $q = 1, ..., n$. Correspondingly, the grey coefficient of each alternative calculated from NIS is given as

$$\xi_{pq}^{-} = \frac{\min_{1\le p\le m} min_{1\le q\le n} d\left(E_{\breve{u}_{pq}}, P_k^{-}\right) + \rho \max_{1\le p\le m} max_{1\le q\le n} d\left(E_{\breve{u}_{pq}}, P_k^{-}\right)}{d\left(E_{\breve{u}_{pq}}, P_k^{-}\right) + \rho \max_{1\le p\le m} max_{1\le q\le n} d\left(E_{\breve{u}_{pq}}, P_k^{-}\right)} \tag{24}$$

Where $p = 1, ..., m$ and $q = 1, ..., n$ and the identification coefficient $\rho = 0.5$.

Step 7 Using these equation, to find the grey coefficient degree for each alternative from PIS and NIS, respectively,

$$\xi_p^{+} = \sum_{q=1}^{n} \varpi_q \xi_{pq}^{+} \tag{25}$$

$$\xi_p^{-} = \sum_{q=1}^{n} \varpi_q \xi_{pq}^{-}$$

Basic principle of the Grey method are "the chosen alternative should have the largest degree of grey relation from the PIS and the smallest degree of grey relation from the NIS". Obviously, the weights are known, the smaller ξ_{p}^{-} and the larger ξ_{p}^{+}, the finest alternative a_{p} as. But incomplete information about weights of alternatives is known. So, in this case the ξ_{p}^{-} and ξ_{p}^{+} information about weight are determined initially. So, we provide the following optimization models or multiple objective to determined the information about weight,

$$(OM1)\begin{cases} \min \xi_p^- = \sum_{q=1}^n \varpi_q \xi_{pq}^- \; p = 1, 2, ..., m \\ \max \xi_p^+ = \sum_{q=1}^n \varpi_q \xi_{pq}^+ \;\; p = 1, 2, ..., m \end{cases} \tag{26}$$

Since it given that each alternative is non-inferior, then all the alternatives have no preference relation. The above optimization models are aggregated with equal weights, into single objective optimization model,

$$(OM2)\begin{cases} \min \xi_p = \sum_{p=1}^{m}\sum_{q=1}^{n} \left(\xi_{pq}^- - \xi_{pq}^+\right) \varpi_q \end{cases} \tag{27}$$

To finding solution of OM2, we obtain optimal solution $\varpi = (\varpi_1, \varpi_2, ..., \varpi_m)^T$, which utilized as weights information alternatives. Then, we obtain ξ_p^+ $(p = 1, 2, ..., m)$ and ξ_p^- $(p = 1, 2, ..., m)$ as using the above formula, respectively.

Step 8 To find the relative closeness degree for each alternative, using the following equation;

$$\xi_p = \frac{\xi_p^+}{\xi_p^- + \xi_p^+} \tag{28}$$

Step 9 According to the ξ_p value, give ranking to the alternatives a_p and select the finest ones.

5. Discriptive Example

We shall present a numerical examples, in this section with linguistic picture fuzzy information to explain the developed approach of the paper.

Example 5.1. Let us assume that a board with four possible develop technology enterprises Z_i $(i = 1, ..., 4)$. There are four experts, and also choose four criteria to classify the four possible develop technology enterprises:

1. $(\breve{A}_1)$, the industrial development;
2. $(\breve{A}_2)$, the feasible market risk;
3. $(\breve{A}_3)$, the industrialization infrastructure, human resources and financial conditions;
4. $(\breve{A}_4)$, the job production and the development of science and technology.

Step 1 Three decision maker offering their own opinions regarding the results obtained with each emerging technology enterprise are given from the table 1-3.

Table 1. Linguistic picture fuzzy information D^1

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(\pounds_5, 0.2, 0.1, 0.6)$	$(\pounds_4, 0.5, 0.3, 0.1)$	$(\pounds_2, 0.3, 0.1, 0.5)$	$(\pounds_3, 0.4, 0.3, 0.2)$
Z_2	$(\pounds_2, 0.1, 0.4, 0.4)$	$(\pounds_3, 0.6, 0.2, 0.1)$	$(\pounds_1, 0.2, 0.2, 0.5)$	$(\pounds_5, 0.2, 0.1, 0.6)$
Z_3	$(\pounds_4, 0.2, 0.3, 0.3)$	$(\pounds_2, 0.4, 0.3, 0.2)$	$(\pounds_5, 0.3, 0.1, 0.4)$	$(\pounds_1, 0.3, 0.2, 0.4)$
Z_4	$(\pounds_1, 0.3, 0.1, 0.6)$	$(\pounds_5, 0.3, 0.2, 0.4)$	$(\pounds_3, 0.1, 0.3, 0.5)$	$(\pounds_2, 0.2, 0.3, 0.3)$

Table 2. Linguistic picture fuzzy information D^2

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(\pounds_2, 0.1, 0.3, 0.5)$	$(\pounds_5, 0.4, 0.3, 0.2)$	$(\pounds_3, 0.1, 0.1, 0.6)$	$(\pounds_4, 0.2, 0.3, 0.4)$
Z_2	$(\pounds_5, 0.2, 0.2, 0.4)$	$(\pounds_3, 0.4, 0.3, 0.2)$	$(\pounds_4, 0.3, 0.2, 0.4)$	$(\pounds_2, 0.4, 0.1, 0.4)$
Z_3	$(\pounds_3, 0.1, 0.2, 0.6)$	$(\pounds_4, 0.6, 0.1, 0.1)$	$(\pounds_2, 0.2, 0.2, 0.4)$	$(\pounds_5, 0.5, 0.2, 0.2)$
Z_4	$(\pounds_1, 0.4, 0.1, 0.5)$	$(\pounds_2, 0.5, 0.1, 0.3)$	$(\pounds_5, 0.3, 0.3, 0.3)$	$(\pounds_3, 0.6, 0.2, 0.1)$

Table 3. Linguistic picture fuzzy information D^3

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(\pounds_1, 0.3, 0.1, 0.3)$	$(\pounds_3, 0.4, 0.2, 0.1)$	$(\pounds_5, 0.2, 0.3, 0.4)$	$(\pounds_4, 0.5, 0.2, 0.1)$
Z_2	$(\pounds_4, 0.1, 0.5, 0.3)$	$(\pounds_5, 0.6, 0.1, 0.2)$	$(\pounds_1, 0.1, 0.1, 0.7)$	$(\pounds_3, 0.3, 0.1, 0.3)$
Z_3	$(\pounds_5, 0.4, 0.2, 0.3)$	$(\pounds_2, 0.4, 0.2, 0.2)$	$(\pounds_4, 0.2, 0.2, 0.5)$	$(\pounds_1, 0.6, 0.2, 0.1)$
Z_4	$(\pounds_3, 0.1, 0.2, 0.6)$	$(\pounds_4, 0.6, 0.2, 0.1)$	$(\pounds_2, 0.3, 0.1, 0.4)$	$(\pounds_2, 0.7, 0.1, 0.1)$

Since $\breve{A}_1$, $\breve{A}_2$ are cost-type criteria and $\breve{A}_3$, $\breve{A}_4$ are benefit-type criteria. First of all we normalize linguistic picture fuzzy information, which are shown in table 4,5,6.:

Table 4. Normalized linguistic picture fuzzy information R^1

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(\pounds_5, 0.6, 0.1, 0.2)$	$(\pounds_4, 0.5, 0.3, 0.1)$	$(\pounds_2, 0.5, 0.1, 0.3)$	$(\pounds_3, 0.4, 0.3, 0.2)$
Z_2	$(\pounds_2, 0.4, 0.4, 0.1)$	$(\pounds_3, 0.6, 0.2, 0.1)$	$(\pounds_1, 0.5, 0.2, 0.2)$	$(\pounds_5, 0.2, 0.1, 0.6)$
Z_3	$(\pounds_4, 0.3, 0.3, 0.2)$	$(\pounds_2, 0.4, 0.3, 0.2)$	$(\pounds_5, 0.4, 0.1, 0.3)$	$(\pounds_1, 0.3, 0.2, 0.4)$
Z_4	$(\pounds_1, 0.6, 0.1, 0.3)$	$(\pounds_5, 0.3, 0.2, 0.4)$	$(\pounds_3, 0.5, 0.3, 0.1)$	$(\pounds_2, 0.2, 0.3, 0.3)$

Table 5. Normalized linguistic picture fuzzy information R^2

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(\pounds_2, 0.5, 0.3, 0.1)$	$(\pounds_5, 0.4, 0.3, 0.2)$	$(\pounds_3, 0.6, 0.1, 0.1)$	$(\pounds_4, 0.2, 0.3, 0.4)$
Z_2	$(\pounds_5, 0.4, 0.2, 0.2)$	$(\pounds_3, 0.4, 0.3, 0.2)$	$(\pounds_4, 0.4, 0.2, 0.3)$	$(\pounds_2, 0.4, 0.1, 0.4)$
Z_3	$(\pounds_3, 0.6, 0.2, 0.1)$	$(\pounds_4, 0.6, 0.1, 0.1)$	$(\pounds_2, 0.4, 0.2, 0.2)$	$(\pounds_5, 0.5, 0.2, 0.2)$
Z_4	$(\pounds_1, 0.5, 0.1, 0.4)$	$(\pounds_2, 0.5, 0.1, 0.3)$	$(\pounds_5, 0.3, 0.3, 0.3)$	$(\pounds_3, 0.6, 0.2, 0.1)$

Table 6. Normalized linguistic picture fuzzy information R^3

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
Z_1	$(\pounds_1, 0.3, 0.1, 0.3)$	$(\pounds_3, 0.4, 0.2, 0.1)$	$(\pounds_5, 0.4, 0.3, 0.2)$	$(\pounds_4, 0.5, 0.2, 0.1)$
Z_2	$(\pounds_4, 0.3, 0.5, 0.1)$	$(\pounds_5, 0.6, 0.1, 0.2)$	$(\pounds_1, 0.7, 0.1, 0.1)$	$(\pounds_3, 0.3, 0.1, 0.3)$
Z_3	$(\pounds_5, 0.3, 0.2, 0.4)$	$(\pounds_2, 0.4, 0.2, 0.2)$	$(\pounds_4, 0.5, 0.2, 0.2)$	$(\pounds_1, 0.6, 0.2, 0.1)$
Z_4	$(\pounds_3, 0.6, 0.2, 0.1)$	$(\pounds_4, 0.6, 0.2, 0.1)$	$(\pounds_2, 0.4, 0.1, 0.3)$	$(\pounds_2, 0.7, 0.1, 0.1)$

Let us assume that the criteria weights information given by experts, are partly known;

$$\Delta = \left\{ \begin{array}{c} 0.2 \leq w_1 \leq 0.25, \\ 0.15 \leq w_2 \leq 0.2, \\ 0.28 \leq w_3 \leq 0.32, \\ 0.35 \leq w_4 \leq 0.4 \end{array} \right\}, w_p \geq 0, p = 1, 2, 3, 4, \sum_{p=1}^{4} w_p = 1$$

Then, we utilize the developed approach to get the most desirable alternative(s).

Step 2 Firstly, find fuzzy density of each decision maker, and its λ parameter. Assume that $P_{\breve{e}_{\breve{u}}}(A_1) = 0.30$, $P_{\breve{e}_{\breve{u}}}(A_2) = 0.40$, $P_{\breve{e}_{\breve{u}}}(A_3) = 0.50$. Then λ of adept can be obtained: $\lambda = -0.45$. By Eq.(6), we have $P_{\breve{e}_{\breve{u}}}(A_1, A_2) = 0.65$, $P_{\breve{e}_{\breve{u}}}(A_1, A_3) = 0.73$, $P_{\breve{e}_{\breve{u}}}(A_2, A_3) = 0.81$, $P_{\breve{e}_{\breve{u}}}(A_1, A_2, A_3) = 1$.

Step 3 According to Definition 3.4, $E_{\breve{u}_{pq}}^{(k)}$ is reordered such that $E_{\breve{u}_{pq}}^{(k)} \geq E_{\breve{u}_{pq}}^{(k-1)}$. Then, utilized the picture fuzzy Choquet integral weighted operator

$$PFCIWA(E_{\breve{u}_1}, E_{\breve{u}_2}, ..., E_{\breve{u}_n}) = \left\{ \begin{array}{c} 1 - \Pi_{p=1}^{n}(1 - P_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)}) - \lambda(A_{\rho(p\text{-}1)})}, \\ \Pi_{p=1}^{n}(I_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)}) - \lambda(A_{\rho(p\text{-}1)})}, \\ \Pi_{p=1}^{n}(N_{\breve{e}_{\breve{u}_{\rho(p)}}})^{\lambda(A_{\rho(p)}) - \lambda(A_{\rho(p\text{-}1)})} \end{array} \right\}$$

Table 7. Collective picture fuzzy information

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
$\check{Z}_1$	$\langle \pounds_{2.55}, 0.696, 0.417, 0.225 \rangle$	$\langle \pounds_{4.00}, 0.625, 0.361, 0.331 \rangle$	$\langle \pounds_{3.40}, 0.633, 0.545, 0.391 \rangle$	$\langle \pounds_{3.70}, 0.488, 0.204, 0.313 \rangle$
$\check{Z}_2$	$\langle \pounds_{3.75}, 0.739, 0.488, 0.254 \rangle$	$\langle \pounds_{3.70}, 0.488, 0.670, 0.162 \rangle$	$\langle \pounds_{2.05}, 0.739, 0.311, 0.335 \rangle$	$\langle \pounds_{3.25}, 0.613, 0.193, 0.374 \rangle$
$\check{Z}_3$	$\langle \pounds_{4.00}, 0.405, 0.654, 0.361 \rangle$	$\langle \pounds_{2.70}, 0.732, 0.274, 0.200 \rangle$	$\langle \pounds_{3.60}, 0.739, 0.260, 0.265 \rangle$	$\langle \pounds_{2.40}, 0.600, 0.278, 0.304 \rangle$
$\check{Z}_4$	$\langle \pounds_{1.70}, 0.769, 0.331, 0.418 \rangle$	$\langle \pounds_{3.60}, 0.638, 0.562, 0.311 \rangle$	$\langle \pounds_{3.35}, 0.613, 0.354, 0.311 \rangle$	$\langle \pounds_{2.35}, 0.511, 0.265, 0.358 \rangle$

Step 4 Utilize eq.(18 and eq.(19) we obtain the positive-ideal and negative-ideal solution respectively, are:

$$P^{+}=\left\{\begin{array}{c}\langle £_{2.55},0.696,0.417,0.225\rangle,\langle £_{2.70},0.732,0.274,0.200\rangle,\\ \langle £_{3.60},0.739,0.260,0.265\rangle,\langle £_{3.25},0.613,0.193,0.374\rangle\end{array}\right\}$$

$$P^{-}=\left\{\begin{array}{c}\langle £_{4.00},0.405,0.654,0.361\rangle,\langle £_{3.70},0.488,0.670,0.162\rangle,\\ \langle £_{3.40},0.633,0.545,0.391\rangle,\langle £_{2.35},0.511,0.265,0.358\rangle\end{array}\right\}$$

Step 5 Utilize equation (21) and (22) to get the positive ideal and negative ideal separation matrix, respectively as follow;

Table 8. Positive-ideal separation matrix

$D^{+}=$

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
$\check{Z}_1$	0.0000	0.0808	0.1500	0.0214
$\check{Z}_2$	0.0123	0.1645	0.0482	0.0000
$\check{Z}_3$	0.2145	0.0000	0.0000	0.0088
$\check{Z}_4$	0.0084	0.1289	0.0787	0.0344

Table 9. Negative-ideal separation matrix

$D^{-}=$

	$\breve{A}_1$	$\breve{A}_2$	$\breve{A}_3$	$\breve{A}_4$
$\check{Z}_1$	0.2145	0.0837	0.0000	0.0129
$\check{Z}_2$	0.2021	0.0000	0.1017	0.0344
$\check{Z}_3$	0.0000	0.1645	0.1500	0.0256
$\check{Z}_4$	0.2060	0.0356	0.0712	0.000

Step 6 Utilize equations (23) and (24) to get the grey relational coefficient matrices in which every alternative is obtained from PIS and NIS as follow:

$$\left[\zeta_{ij}^{+}\right]=\begin{bmatrix}0.6518 & 0.5721 & 0.8295 & 0.4711\\ 0.3667 & 1.0000 & 1.0000 & 0.3333\\ 0.5254 & 0.7829 & 0.6383 & 0.6744\\ 1.0000 & 0.6875 & 0.5562 & 1.0000\end{bmatrix}$$

$$\left[\zeta_{ij}^{-}\right]=\begin{bmatrix}0.4560 & 1.0000 & 0.5937 & 0.4039\\ 1.0000 & 0.5721 & 0.5562 & 1.0000\\ 0.5483 & 0.6689 & 0.7699 & 0.3745\\ 0.3667 & 0.5372 & 0.5184 & 0.3333\end{bmatrix}$$

Step 7 To developed the single-objective programming model, using the model (M2):

$$\min \xi(w)=-0.0709w_1+0.4283w_2-0.2594w_3-1.5440w_4$$

We gain the weight vector of criterias, to solved this model:

$$w=(0.330,0.144,0.366,0.157)$$

Now from the PIS and NIS, we get the degree of grey relational coefficient of every alternative:

$$\xi_1^{+}=0.6001,\xi_2^{+}=0.5439,\xi_3^{+}=0.6331,\xi_4^{+}=0.8823,$$

$$\xi_1^{-}=0.5355,\xi_2^{-}=0.8657,\xi_3^{-}=0.5352,\xi_4^{-}=0.4016.$$

Step 8 To find the relative relational degree of the alternative, we utilize Equation 28, and PIS and NIS:

$$\xi_1=\frac{\xi_1^{+}}{\xi_1^{-}+\xi_1^{+}}=\frac{0.6001}{0.5355+0.6001}=0.5284$$

$$\xi_2=\frac{\xi_2^{+}}{\xi_2^{-}+\xi_2^{+}}=\frac{0.5439}{0.8657+0.5439}=0.3858$$

$$\xi_3=\frac{\xi_3^{+}}{\xi_3^{-}+\xi_3^{+}}=\frac{0.6331}{0.5352+0.6331}=0.5418$$

$$\xi_4=\frac{\xi_4^{+}}{\xi_4^{-}+\xi_4^{+}}=\frac{0.8823}{0.4016+0.8823}=0.6872$$

Step 9 With the help of relative relational degree, ranking of the alternatives are the following:

$$Z_4 > Z_3 > Z_1 > Z_2,$$

and thus the most desirable alternative is Z_4.

5.1. Comparative Analysis

To justify the effectivity and efficiency of the advised procedure, we conducted a comparative analysis for comparison of our suggest approach with the GRA method for intuitionistic fuzzy set [52].

5.1.1. Comparison between intuitionistic fuzzy and picture fuzzy GRA relation Approache

In the ntuitionistic fuzzy numbers, we have only study the uncertain things from positive and negative membership degrees. They bring an effictive execution to imply the vagueness of DM. On the other hand, as stated already, in IFN the things from good and bad appearance of these two collection of fuzzy numbers, can throw away the thinking of DM perfectly. After all, dissimilar the PFNs, in some conditions the IFNs are not serviceable. The IFNs must satisfy the condition that the membership and non-membership degree sume belongs to $[0, 1]$. Thus, in some cases, there exists some problems which cannot handle by IFNs. For example, the peoples requried their opinions contain more type of answer like as: "yes", "abstain", "No" and "Refusal", in that situations picture fuzzy set are more suitable. Thus, in summary, in decision making theory, PFNs have suitable capacity to process these information.

6. Conclusion

The classical grey relational analysis method are normally applicable for tackle the MAGDM problems, in which the data occur in the form of numerical values, and still they will flop when MAGDM problems contains picture linguistic fuzzy information. In the developed approach we use the picture linguistic fuzzy Choquet integral weighted averaging (PLFCIWA) operator to marge all the individual matrices. Then, based on the traditional GRA method, an approach are given to deal with picture linguistic fuzzy MAGDM problems in which the information are incomplete. Lastly, a decision problem are developed based on the defined operators, to rank more alternatives. Thus, the proposed operations gives clear track to catch the inexact data all over the decision problem procedure.

Acknowledgement

The authors would like to thank the editor in chief, associate editor and the anonymous referees for detailed and valuable comments which helped to improve this manuscript.

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