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Research Article

Generalized autocommuting probability of a finite group relative to its subgroups

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Abstract

Let $H \subseteq K$ be two subgroups of a finite group G and $\mathrm{Aut}(K)$ the automorphism group of K. In this paper, we consider the generalized autocommuting probability of G relative to its subgroups H and K, denoted by $\mathrm{Pr}_g(H, \mathrm{Aut}(K))$, which is the probability that the autocommutator of a randomly chosen pair of elements, one from H and the other from $\mathrm{Aut}(K)$, is equal to a given element $g \in K$. We study several properties as well as obtain several computing formulae of this probability. As applications of the computing formulae, we also obtain several bounds for $\mathrm{Pr}_g(H, \mathrm{Aut}(K))$ and characterizations of some finite groups through $\mathrm{Pr}_g(H, \mathrm{Aut}(K))$.

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1. Introduction

Let G be a finite group acting on a set Ω. Let $\mathrm{Pr}(G, \Omega)$ denotes the probability that a randomly chosen element of Ω fixes a randomly chosen element of G. In 1975, Sherman [13] initiated the study of $\mathrm{Pr}(G, \Omega)$ considering G to be an abelian group and $\Omega = \mathrm{Aut}(G)$, the automorphism group of G. Note that

$$\mathrm{Pr}(G, \mathrm{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \mathrm{Aut}(G) : [x, \alpha] = 1\}|}{|G||\mathrm{Aut}(G)|}$$

where $[x, \alpha]$ is the autocommutator of x and α defined as $x^{-1}\alpha(x)$. The ratio $\mathrm{Pr}(G, \mathrm{Aut}(G))$ is called autocommuting probability of G. The case when G is non-abelian is considered in [1, 3, 12]. Few generalizations of $\mathrm{Pr}(G, \mathrm{Aut}(G))$ can also be found in [3, 4, 9, 12].

Let H and K be two subgroups of G such that $H \subseteq K$. We define

$$\mathrm{Pr}_g(H, \mathrm{Aut}(K)) = \frac{|\{(x, \alpha) \in H \times \mathrm{Aut}(K) : [x, \alpha] = g\}|}{|H||\mathrm{Aut}(K)|} \tag{1.1}$$

where $g \in K$. That is, $\mathrm{Pr}_g(H, \mathrm{Aut}(K))$ is the probability that the autocommutator of a randomly chosen pair of elements, one from H and the other from $\mathrm{Aut}(K)$, is equal to a given element $g \in K$. The ratio $\mathrm{Pr}_g(H, \mathrm{Aut}(K))$ is called generalized autocommuting probability of G relative to its subgroups H and K. Clearly, if $H = G$ and $g = 1$ then $\mathrm{Pr}_g(H, \mathrm{Aut}(K)) = \mathrm{Pr}(G, \mathrm{Aut}(G))$. Note that the cases when $H = G$ and $K = G$

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are considered in [3] and [4], respectively. If we replace $\mathrm{Aut}(K)$ by $\mathrm{Inn}(K)$, the inner automorphism group of K, in (1.1) then $\mathrm{Pr}_g(H, \mathrm{Inn}(K)) = \mathrm{Pr}_g(H, K)$ where

$$\mathrm{Pr}_g(H, K) = \frac{|\{(x, y) \in H \times K : x^{-1}y^{-1}xy = g\}|}{|H||K|}$$

which is introduced and studied in [2]. In this paper, we study several properties as well as obtain several computing formulae of $\mathrm{Pr}_g(H, \mathrm{Aut}(K))$. We also obtain some bounds for $\mathrm{Pr}_g(H, \mathrm{Aut}(K))$ and characterize some finite groups through $\mathrm{Pr}_g(H, \mathrm{Aut}(K))$ as applications.

We write $S(H, \mathrm{Aut}(K))$ to denote the set $\{[x, \alpha] : x \in H \text{ and } \alpha \in \mathrm{Aut}(K)\}$ and $[H, \mathrm{Aut}(K)] := \langle S(H, \mathrm{Aut}(K))\rangle$. We also write $L(H, \mathrm{Aut}(K)) := \{x \in H : [x, \alpha] = 1 \text{ for all } \alpha \in \mathrm{Aut}(K)\}$ and $L(G) := L(G, \mathrm{Aut}(G))$, the absolute center of G (see [6]). Note that $L(H, \mathrm{Aut}(K))$ is a normal subgroup of H contained in $H \cap Z(K)$. Further, $L(H, \mathrm{Aut}(K)) = \bigcap_{\alpha\in \mathrm{Aut}(K)} C_H(\alpha)$, where $C_H(\alpha) = \{x \in H : [x, \alpha] = 1\}$ is a subgroup of H. Let $C_{\mathrm{Aut}(K)}(x) := \{\alpha \in \mathrm{Aut}(K) : \alpha(x) = x\}$ for $x \in H$ and $C_{\mathrm{Aut}(K)}(H) = \{\alpha \in \mathrm{Aut}(K) : \alpha(x) = x \text{ for all } x \in H\}$. Then $C_{\mathrm{Aut}(K)}(x)$ is a subgroup of $\mathrm{Aut}(K)$ and $C_{\mathrm{Aut}(K)}(H) = \bigcap_{x\in H} C_{\mathrm{Aut}(K)}(x)$. We consider the action of $\mathrm{Aut}(K)$ on K given by $(\alpha, x) \mapsto \alpha(x)$ where $\alpha \in \mathrm{Aut}(K)$ and $x \in K$. Let $\mathrm{orb}_K(x) := \{\alpha(x) : \alpha \in \mathrm{Aut}(K)\}$ be the orbit of $x \in K$. Then by orbit-stabilizer theorem, we have

$$|\,\mathrm{orb}_K(x)| = \frac{|\,\mathrm{Aut}(K)|}{|C_{\mathrm{Aut}(K)}(x)|}. \tag{1.2}$$

Clearly, $\mathrm{Pr}_g(H, \mathrm{Aut}(K)) = 1$ if and only if $[H, \mathrm{Aut}(K)] = \{1\}$ and $g = 1$ if and only if $H = L(H, \mathrm{Aut}(K))$ and $g = 1$. Also, $\mathrm{Pr}_g(H, \mathrm{Aut}(K)) = 0$ if and only if $g \notin S(H, \mathrm{Aut}(K))$. Therefore, we consider $H \neq L(H, \mathrm{Aut}(K))$ and $g \in S(H, \mathrm{Aut}(K))$ throughout the paper.

2. Some properties

We begin with the following lower bounds.

Proposition 2.1. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$ and $g \in K$.*

(a) *If $g = 1$ then*

$$\mathrm{Pr}_g(H, \mathrm{Aut}(K)) \geq \tfrac{|L(H,\mathrm{Aut}(K))|}{|H|} + \tfrac{|C_{\mathrm{Aut}(K)}(H)|(|H|-|L(H,\mathrm{Aut}(K))|)}{|H||\,\mathrm{Aut}(K)|}.$$

(b) *If $g \neq 1$ then* $\mathrm{Pr}_g(H, \mathrm{Aut}(K)) \geq \frac{|L(H,\mathrm{Aut}(K))||C_{\mathrm{Aut}(K)}(H)|}{|H||\,\mathrm{Aut}(K)|}$.

Proof. Let $\mathcal{C}$ denotes the set $\{(x, \alpha) \in H \times \mathrm{Aut}(K) : [x, \alpha] = g\}$.

If $g = 1$ then we have $S := (L(H, \mathrm{Aut}(K)) \times \mathrm{Aut}(K)) \cup (H \times C_{\mathrm{Aut}(K)}(H))$ is a subset of $\mathcal{C}$. We also have

$$|S| = |L(H, \mathrm{Aut}(K))||\,\mathrm{Aut}(K)| + |C_{\mathrm{Aut}(K)}(H)||H| - |L(H, \mathrm{Aut}(K))||C_{\mathrm{Aut}(K)}(H)|.$$

Therefore,

$$\begin{aligned}\mathrm{Pr}_g(H, \mathrm{Aut}(K)) \geq & \frac{1}{|H||\,\mathrm{Aut}(K)|}\{|L(H, \mathrm{Aut}(K))||\,\mathrm{Aut}(K)|+ \\ & |C_{\mathrm{Aut}(K)}(H)||H| - |L(H, \mathrm{Aut}(K))||C_{\mathrm{Aut}(K)}(H)|\}\end{aligned}$$

and hence part (a) follows.

Now we consider the case when $g \neq 1$. Since $g \in S(H, \mathrm{Aut}(K))$ we have $\mathcal{C}$ is non-empty. Let $(y, \beta) \in \mathcal{C}$ then $(y, \beta) \notin L(H, \mathrm{Aut}(K)) \times C_{\mathrm{Aut}(K)}(H)$ otherwise $[y, \beta] = 1$. It is easy to see that the coset $(y, \beta)(L(H, \mathrm{Aut}(K)) \times C_{\mathrm{Aut}(K)}(H))$ is a subset of $\mathcal{C}$ having order $|L(H, \mathrm{Aut}(K))||C_{\mathrm{Aut}(K)}(H)|$. Therefore, $|\mathcal{C}| \geq |L(H, \mathrm{Aut}(K))||C_{\mathrm{Aut}(K)}(H)|$ and hence part (b) follows. □

Proposition 2.2. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $g \in K$ then*
$$\Pr_{g^{-1}}(H, \operatorname{Aut}(K)) = \Pr_g(H, \operatorname{Aut}(K)).$$

Proof. Let X and Y denote the sets $\{(x,\alpha) \in H \times \operatorname{Aut}(K) : [x,\alpha] = g\}$ and $\{(y,\beta) \in H \times \operatorname{Aut}(K) : [y,\beta] = g^{-1}\}$, respectively. Consider the mapping $f : X \to Y$ given by $f((x,\alpha)) = (\alpha(x), \alpha^{-1})$. Since f is bijective, we have $|X| = |Y|$. Hence, the result follows from (1.1). □

Proposition 2.3. *Let G_1 and G_2 be two finite groups. Let H_1, K_1 and H_2, K_2 be subgroups of G_1 and G_2 respectively such that $H_1 \subseteq K_1$, $H_2 \subseteq K_2$ and $\gcd(|K_1|, |K_2|) = 1$. If $(g_1, g_2) \in K_1 \times K_2$ then*
$$\Pr_{(g_1,g_2)}(H_1 \times H_2, \operatorname{Aut}(K_1 \times K_2)) = \Pr_{g_1}(H_1, \operatorname{Aut}(K_1))\Pr_{g_2}(H_2, \operatorname{Aut}(K_2)).$$

Proof. Let X denotes the set
$$\{((x,y), \alpha_{K_1\times K_2}) \in (H_1 \times H_2) \times \operatorname{Aut}(K_1 \times K_2) : [(x,y), \alpha_{K_1\times K_2}] = (g_1, g_2)\}.$$
Also, let Y and Z denote the sets $\{(x, \alpha_{K_1}) \in H_1 \times \operatorname{Aut}(K_1) : [x, \alpha_{K_1}] = g_1\}$ and $\{(y, \alpha_{K_2}) \in H_2 \times \operatorname{Aut}(K_2) : [y, \alpha_{K_2}] = g_2\}$, respectively. Since $\gcd(|K_1|, |K_2|) = 1$, by [7, Lemma 2.1], we have $\operatorname{Aut}(K_1 \times K_2) = \operatorname{Aut}(K_1) \times \operatorname{Aut}(K_2)$. Therefore, for every $\alpha_{K_1\times K_2} \in \operatorname{Aut}(K_1 \times K_2)$ there exist unique $\alpha_{K_1} \in \operatorname{Aut}(K_1)$ and $\alpha_{K_2} \in \operatorname{Aut}(K_2)$ such that $\alpha_{K_1\times K_2} = \alpha_{K_1} \times \alpha_{K_2}$, where $\alpha_{K_1} \times \alpha_{K_2}((x,y)) = (\alpha_{K_1}(x), \alpha_{K_2}(y))$ for all $(x,y) \in H_1 \times H_2$. Also, for all $(x,y) \in H_1 \times H_2$, we have $[(x,y), \alpha_{K_1\times K_2}] = (g_1, g_2)$ if and only if $[x, \alpha_{K_1}] = g_1$ and $[y, \alpha_{K_2}] = g_2$. These show that $X = Y \times Z$. Therefore
$$\frac{|X|}{|H_1 \times H_2||\operatorname{Aut}(K_1 \times K_2)|} = \frac{|Y|}{|H_1||\operatorname{Aut}(K_1)|} \cdot \frac{|Z|}{|H_2||\operatorname{Aut}(K_2)|}.$$
Hence, the result follows from (1.1). □

In the year 1940, Hall [5] introduced the concept of isoclinism between two groups. Following Hall, Moghaddam et al. [8] have defined autoisoclinism between two groups, in the year 2013. Recall that two groups G_1 and G_2 are said to be autoisoclinic if there exist isomorphisms $\psi : \frac{G_1}{L(G_1)} \to \frac{G_2}{L(G_2)}$, $\beta : [G_1, \operatorname{Aut}(G_1)] \to [G_2, \operatorname{Aut}(G_2)]$ and $\gamma : \operatorname{Aut}(G_1) \to \operatorname{Aut}(G_2)$ such that the following diagram commutes
$$\begin{array}{ccc} \frac{G_1}{L(G_1)} \times \operatorname{Aut}(G_1) & \xrightarrow{\psi\times\gamma} & \frac{G_2}{L(G_2)} \times \operatorname{Aut}(G_2) \\ \Big\downarrow a_{(G_1,\operatorname{Aut}(G_1))} & & \Big\downarrow a_{(G_2,\operatorname{Aut}(G_2))} \\ [G_1, \operatorname{Aut}(G_1)] & \xrightarrow{\beta} & [G_2, \operatorname{Aut}(G_2)] \end{array}$$
where the maps $a_{(G_i,\operatorname{Aut}(G_i))} : \frac{G_i}{L(G_i)} \times \operatorname{Aut}(G_i) \to [G_i, \operatorname{Aut}(G_i)]$, for $i = 1, 2$, are given by
$$a_{(G_i,\operatorname{Aut}(G_i))}(x_i L(G_i), \alpha_i) = [x_i, \alpha_i].$$
Such a pair $(\psi \times \gamma, \beta)$ is called an autoisoclinism between the groups G_1 and G_2. We generalize the notion of autoisoclinism in the following way:

Let H_1, K_1 and H_2, K_2 be subgroups of the groups G_1 and G_2 respectively. The pairs of subgroups (H_1, K_1) and (H_2, K_2) such that $H_1 \subseteq K_1$ and $H_2 \subseteq K_2$ are said to be autoisoclinic if there exist isomorphisms $\psi : \frac{H_1}{L(H_1, \operatorname{Aut} K_1)} \to \frac{H_2}{L(H_2, \operatorname{Aut}(K_2))}$, $\beta : [H_1, \operatorname{Aut}(K_1)] \to [H_2, \operatorname{Aut}(K_2)]$ and $\gamma : \operatorname{Aut}(K_1) \to \operatorname{Aut}(K_2)$ such that the following diagram commutes
$$\begin{array}{ccc} \frac{H_1}{L(H_1,\operatorname{Aut}(K_1))} \times \operatorname{Aut}(K_1) & \xrightarrow{\psi\times\gamma} & \frac{H_2}{L(H_2,\operatorname{Aut}(K_2))} \times \operatorname{Aut}(K_2) \\ \Big\downarrow a_{(H_1,\operatorname{Aut}(K_1))} & & \Big\downarrow a_{(H_2,\operatorname{Aut}(K_2))} \\ [H_1, \operatorname{Aut}(K_1)] & \xrightarrow{\beta} & [H_2, \operatorname{Aut}(K_2)] \end{array}$$

where the maps $a_{(H_i,\mathrm{Aut}(K_i))} : \frac{H_i}{L(H_i,\mathrm{Aut}(K_i))} \times \mathrm{Aut}(K_i) \to [H_i, \mathrm{Aut}(K_i)]$, for $i = 1, 2$, are given by

$$a_{(H_i,\mathrm{Aut}(K_i))}(x_i L(H_i, \mathrm{Aut}(K_i)), \alpha_i) = [x_i, \alpha_i].$$

Such a pair $(\psi \times \gamma, \beta)$ is said to be an autoisoclinism between the pairs of groups (H_1, K_1) and (H_2, K_2). We conclude this section with the following generalization of [3, Theorem 5.1] and [12, Lemma 2.5].

Theorem 2.4. *Let G_1 and G_2 be two finite groups with subgroups H_1, K_1 and H_2, K_2 respectively such that $H_1 \subseteq K_1$ and $H_2 \subseteq K_2$. If $(\psi \times \gamma, \beta)$ is an autoisoclinism between the pairs (H_1, K_1) and (H_2, K_2) then, for $g \in K_1$,*

$$\Pr_g(H_1, \mathrm{Aut}(K_1)) = \Pr_{\beta(g)}(H_2, \mathrm{Aut}(K_2)).$$

Proof. Let us consider the sets $\mathcal{S}_g = \{(x_1 L(H_1, \mathrm{Aut}(K_1)), \alpha_1) \in \frac{H_1}{L(H_1,\mathrm{Aut}(K_1))} \times \mathrm{Aut}(K_1) : [x_1, \alpha_1] = g\}$ and $\mathcal{T}_{\beta(g)} = \{(x_2, \alpha_2) \in \frac{H_2}{L(H_2,\mathrm{Aut}(K_2))} \times \mathrm{Aut}(K_2) : [x_2 L(H_2, \mathrm{Aut}(K_2)), \alpha_2] = \beta(g)\}$. Since (H_1, K_1) is autoisoclinic to (H_2, K_2) we have $|\mathcal{S}_g| = |\mathcal{T}_{\beta(g)}|$. Again, it is clear that

$$|\{(x_1, \alpha_1) \in H_1 \times \mathrm{Aut}(K_1) : [x_1, \alpha_1] = g\}| = |L(H_1, \mathrm{Aut}(K_1))||\mathcal{S}_g| \tag{2.1}$$

and

$$|\{(x_2, \alpha_2) \in H_2 \times \mathrm{Aut}(K_2) : [x_2, \alpha_2] = \beta(g)\}| = |L(H_2, \mathrm{Aut}(K_2))||\mathcal{T}_{\beta(g)}|. \tag{2.2}$$

Hence, the result follows from (1.1), (2.1) and (2.2). □

3. Computing formulae and applications

For any $x \in H$, let us define the set $T_{x,g}(H, K) = \{\alpha \in \mathrm{Aut}(K) : [x, \alpha] = g\}$, where g is a fixed element of K. Note that $T_{x,1}(H, K) = C_{\mathrm{Aut}(K)}(x)$. Also, $T_{x,g}(H, K)$ is non-empty if and only if $xg \in \mathrm{orb}_K(x)$. We have the following useful lemma.

Lemma 3.1. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $T_{x,g}(H, K)$ is non-empty then $T_{x,g}(H, K) = \mu C_{\mathrm{Aut}(K)}(x)$ for some $\mu \in T_{x,g}(H, K)$ and hence $|T_{x,g}(H, K)| = |C_{\mathrm{Aut}(G)}(x)|$.*

Proof. Assume that $T_{x,g}(H, K)$ is non-empty. Let μ be an element of $T_{x,g}(H, K)$. If $\nu \in \mu C_{\mathrm{Aut}(K)}(x)$ then $\nu = \mu\alpha$ for some $\alpha \in C_{\mathrm{Aut}(K)}(x)$. We have

$$[x, \nu] = [x, \mu\alpha] = x^{-1}\mu(\alpha(x)) = [x, \mu] = g$$

which implies $\nu \in T_{x,g}(H, K)$. Hence, $\mu C_{\mathrm{Aut}(K)}(x) \subseteq T_{x,g}(H, K)$.

If $\gamma \in T_{x,g}(H, K)$ then $\gamma(x) = xg$. We have $\mu^{-1}\gamma(x) = \mu^{-1}(xg) = x$ which implies $\mu^{-1}\gamma \in C_{\mathrm{Aut}(K)}(x)$. Therefore, $\gamma \in \mu C_{\mathrm{Aut}(K)}(x)$ and so $T_{x,g}(H, K) \subseteq \mu C_{\mathrm{Aut}(K)}(x)$. Hence, $T_{x,g}(H, K) = \mu C_{\mathrm{Aut}(K)}(x)$. □

The following theorem gives two computing formulae for $\Pr_g(H, \mathrm{Aut}(K))$.

Theorem 3.2. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $g \in K$ then*

$$\begin{aligned}\Pr_g(H, \mathrm{Aut}(K)) &= \frac{1}{|H||\mathrm{Aut}(K)|} \sum_{\substack{x \in H \\ xg \in \mathrm{orb}_K(x)}} |C_{\mathrm{Aut}(K)}(x)| \\ &= \frac{1}{|H|} \sum_{\substack{x \in H \\ xg \in \mathrm{orb}_K(x)}} \frac{1}{|\mathrm{orb}_K(x)|}.\end{aligned}$$

Proof. Clearly $\{(x,\alpha) \in H \times \mathrm{Aut}(K) : [x,\alpha] = g\} = \underset{x\in H}{\cup}(\{x\} \times T_{x,g}(H,K)$. Since for any two distinct elements $x, y \in H$ the sets $\{x\} \times T_{x,g}(H,K))$ and $\{y\} \times T_{y,g}(H,K)$ are disjoint, we have

$$|H||\mathrm{Aut}(K)|\mathrm{Pr}_g(H,\mathrm{Aut}(K)) = |\underset{x\in H}{\cup}(\{x\} \times T_{x,g}(H,K))| = \sum_{x\in H} |T_{x,g}(H,K)|.$$

Hence, the result follows from Lemma 3.1 and (1.2) noting that $T_{x,g}(H,K) \neq \emptyset$ if and only if $xg \in \mathrm{orb}_K(x)$. □

Considering $g = 1$ in Theorem 3.2, we get the following computing formulae for the ratio $\mathrm{Pr}(H,\mathrm{Aut}(K))$.

Corollary 3.3. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Then*

$$\mathrm{Pr}(H,\mathrm{Aut}(K)) = \frac{1}{|H||\mathrm{Aut}(K)|}\sum_{x\in H}|C_{\mathrm{Aut}(K)}(x)| = \frac{|\mathrm{orb}_K(H)|}{|H|}$$

where $\mathrm{orb}_K(H) = \{\mathrm{orb}_K(x) : x \in H\}$.

Corollary 3.4. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $C_{\mathrm{Aut}(K)}(x) = \{I\}$ for all $x \in H \setminus \{1\}$, where I is the identity element of* $\mathrm{Aut}(K)$*, then*

$$\mathrm{Pr}(H,\mathrm{Aut}(K)) = \frac{1}{|H|} + \frac{1}{|\mathrm{Aut}(K)|} - \frac{1}{|H||\mathrm{Aut}(K)|}.$$

Proof. By Corollary 3.3, we have

$$|H||\mathrm{Aut}(K)|\,\mathrm{Pr}(H,\mathrm{Aut}(K)) = \sum_{x\in H}|C_{\mathrm{Aut}(K)}(x)| = |\mathrm{Aut}(K)| + |H| - 1.$$

Hence, the result follows. □

Note that the fact $|\{(x,\alpha) \in H \times \mathrm{Aut}(K) : [x,\alpha] = 1\}| = \sum_{\alpha\in\mathrm{Aut}(K)}|C_H(\alpha)|$ also gives the following computing formula

$$\mathrm{Pr}(H,\mathrm{Aut}(K)) = \frac{1}{|H||\mathrm{Aut}(K)|}\sum_{\alpha\in\mathrm{Aut}(K)}|C_H(\alpha)|.$$

In the remaining part of this section, we shall discuss some applications of the computing formulae obtained above. More precisely, we shall obtain some bounds for $\mathrm{Pr}_g(H,\mathrm{Aut}(K))$ as well as some characterizations of finite groups in terms of $\mathrm{Pr}_g(H,\mathrm{Aut}(K))$. We begin with the following upper bound.

Proposition 3.5. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $g \in K$ then*

$$\mathrm{Pr}_g(H,\mathrm{Aut}(K)) \leq \mathrm{Pr}(H,\mathrm{Aut}(K)).$$

The equality holds if and only if $g = 1$.

Proof. Using Theorem 3.2, we have

$$\begin{aligned}\mathrm{Pr}_g(H,\mathrm{Aut}(K)) &= \frac{1}{|H||\mathrm{Aut}(K)|}\sum_{\substack{x\in H\\ xg\in\mathrm{orb}_K(x)}}|C_{\mathrm{Aut}(K)}(x)|\\ &\leq \frac{1}{|H||\mathrm{Aut}(K)|}\sum_{x\in H}|C_{\mathrm{Aut}(K)}(x)| = \mathrm{Pr}(H,\mathrm{Aut}(K)).\end{aligned}$$

The equality holds if and only if $xg \in \mathrm{orb}_K(x)$ for all $x \in H$ if and only if $g = 1$. □

Corollary 3.6. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Then*

$$\Pr_g(H, \operatorname{Aut}(K)) \leq \Pr_1(H, K).$$

Proof. By [2, Theorem 2.3], we have

$$\Pr_1(H, K) = \frac{1}{|H|}\sum_{x\in H}\frac{1}{|\operatorname{cl}_K(x)|}$$

where $\operatorname{cl}_K(x) = \{\alpha(x) : \alpha \in \operatorname{Inn}(K)\}$. Since $\operatorname{cl}_K(x) \subseteq \operatorname{orb}_K(x)$ for all $x \in H$, we have

$$\sum_{x\in H}\frac{1}{|\operatorname{cl}_K(x)|} \geq \sum_{x\in H}\frac{1}{|\operatorname{orb}_K(x)|}.$$

Therefore, by Theorem 3.2, we have $\Pr(H, \operatorname{Aut}(K)) \leq \Pr_1(H, K)$. Hence, the result follows from Proposition 3.5. □

Proposition 3.7. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Let $g \in K$ and p the smallest prime dividing $|\operatorname{Aut}(K)|$. If $g \neq 1$ then*

$$\Pr_g(H, \operatorname{Aut}(K)) \leq \frac{|H| - |L(H, \operatorname{Aut}(K))|}{p|H|} < \frac{1}{p}.$$

Proof. For $x \in L(H, \operatorname{Aut}(K))$ we have $xg \notin \operatorname{orb}_K(x)$. Therefore, by Theorem 3.2,

$$\Pr_g(H, \operatorname{Aut}(K)) = \frac{1}{|H|}\sum_{\substack{x\in H\setminus L(H,\operatorname{Aut}(K))\\ xg\in\operatorname{orb}_K(x)}}\frac{1}{|\operatorname{orb}_K(x)|}. \tag{3.1}$$

If $x \in H \setminus L(H, \operatorname{Aut}(K))$ and $xg \in \operatorname{orb}_K(x)$ then $|\operatorname{orb}_K(x)| > 1$. Also $|\operatorname{orb}_K(x)|$ divides $|\operatorname{Aut}(K)|$ and so $|\operatorname{orb}_K(x)| \geq p$. Hence, the result follows from (3.1). □

Proposition 3.8. *Let H_1, H_2 and K be subgroups of a finite group G such that $H_1 \subseteq H_2 \subseteq K$. Then*

$$\Pr_g(H_1, \operatorname{Aut}(K)) \leq |H_2 : H_1|\Pr_g(H_2, \operatorname{Aut}(K)).$$

The equality holds if and only if $xg \notin \operatorname{orb}_K(x)$ for all $x \in H_2 \setminus H_1$.

Proof. By Theorem 3.2, we have

$$\begin{aligned}|H_1||\operatorname{Aut}(K)|\Pr_g(H_1, \operatorname{Aut}(K)) &= \sum_{\substack{x\in H_1\\ xg\in\operatorname{orb}_K(x)}} |C_{\operatorname{Aut}(K)}(x)|\\ &\leq \sum_{\substack{x\in H_2\\ xg\in\operatorname{orb}_K(x)}} |C_{\operatorname{Aut}(K)}(x)|\\ &= |H_2||\operatorname{Aut}(K)|\Pr_g(H_2, \operatorname{Aut}(K)).\end{aligned}$$

Hence, the result follows. □

Proposition 3.9. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. If $g \in K$ then*

$$\Pr_g(H, \operatorname{Aut}(K)) \leq |K : H| \Pr(K, \operatorname{Aut}(K))$$

with equality if and only if $g = 1$ and $H = K$.

Proof. By Proposition 3.5, we have

$$\begin{aligned}\Pr_g(H, \operatorname{Aut}(K)) &\leq \Pr(H, \operatorname{Aut}(K))\\ &= \frac{1}{|H||\operatorname{Aut}(K)|}\sum_{x\in H}|C_{\operatorname{Aut}(K)}(x)|\\ &\leq \frac{1}{|H||\operatorname{Aut}(K)|}\sum_{x\in K}|C_{\operatorname{Aut}(K)}(x)|.\end{aligned}$$

Hence, the result follows from Corollary 3.3. □

Theorem 3.10. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$ and p the smallest prime dividing $|\operatorname{Aut}(K)|$. Then*

$$\Pr(H, \operatorname{Aut}(K)) \geq \frac{|L(H, \operatorname{Aut}(K))|}{|H|} + \frac{p(|H| - |X_H| - |L(H, \operatorname{Aut}(K))|) + |X_H|}{|H||\operatorname{Aut}(K)|}$$

and

$$\Pr(H, \operatorname{Aut}(K)) \leq \frac{(p-1)|L(H, \operatorname{Aut}(K))| + |H|}{p|H|} - \frac{|X_H|(|\operatorname{Aut}(K)| - p)}{p|H||\operatorname{Aut}(K)|},$$

where $X_H = \{x \in H : C_{\operatorname{Aut}(K)}(x) = \{I\}\}$.

Proof. Since $X_H \cap L(H, \operatorname{Aut}(K)) = \emptyset$ we have

$$\sum_{x \in H} |C_{\operatorname{Aut}(K)}(x)| = |X_H| + |\operatorname{Aut}(K)||L(H, \operatorname{Aut}(K))| + \sum_{x \in H \setminus (X_H \cup L(H, \operatorname{Aut}(K)))} |C_{\operatorname{Aut}(K)}(x)|.$$

Also $\{I\} \neq C_{\operatorname{Aut}(K)}(x) \neq \operatorname{Aut}(K)$ and so $p \leq |C_{\operatorname{Aut}(K)}(x)| \leq \frac{|\operatorname{Aut}(K)|}{p}$ for $x \in H \setminus (X_H \cup L(H, \operatorname{Aut}(K)))$. Therefore

$$\sum_{x \in H} |C_{\operatorname{Aut}(K)}(x)| \geq |X_H| + |\operatorname{Aut}(K)||L(H, \operatorname{Aut}(K))| + p(|H| - |X_H| - |L(H, \operatorname{Aut}(K))|) \quad (3.2)$$

and

$$\sum_{x \in H} |C_{\operatorname{Aut}(K)}(x)| \leq |X_H| + |\operatorname{Aut}(K)||L(H, \operatorname{Aut}(K))| + \frac{|\operatorname{Aut}(K)|(|H| - |X_H| - |L(H, \operatorname{Aut}(K))|)}{p}. \quad (3.3)$$

Hence, the result follows from Corollary 3.3, (3.2) and (3.3). □

Following two theorems give characterizations of H in terms of $\Pr(H, \operatorname{Aut}(K))$.

Theorem 3.11. *Let $H \subseteq K$ be two subgroups of a finite group G.*

(a) *If p and q are the smallest primes dividing $|\operatorname{Aut}(K)|$ and $|H|$ respectively then $\Pr(H, \operatorname{Aut}(K)) \leq \frac{p+q-1}{pq}$. In particular, if $p = q$ then $\Pr(H, \operatorname{Aut}(K)) \leq \frac{2p-1}{p^2} \leq \frac{3}{4}$.*

(b) *If $\Pr(H, \operatorname{Aut}(K)) = \frac{p+q-1}{pq}$, for some primes p and q, then pq divides $|H||\operatorname{Aut}(K)|$. Further, if p and q are the smallest primes dividing $|\operatorname{Aut}(K)|$ and $|H|$ respectively, then $\frac{H}{L(H, \operatorname{Aut}(K))} \cong \mathbb{Z}_q$. In particular, if H and $\operatorname{Aut}(K)$ are of even order and $\Pr(H, \operatorname{Aut}(K)) = \frac{3}{4}$ then $\frac{H}{L(H, \operatorname{Aut}(K))} \cong \mathbb{Z}_2$.*

Proof. (a) Since $H \neq L(H, \operatorname{Aut}(K))$ we have $|H : L(H, \operatorname{Aut}(K))| \geq q$. Therefore, by Theorem 3.10, we have

$$\Pr(H, \operatorname{Aut}(K)) \leq \frac{1}{p}\left(\frac{p-1}{|H : L(H, \operatorname{Aut}(K))|} + 1\right) \leq \frac{p+q-1}{pq}.$$

(b) Using (1.1), we have $(p+q-1)|H||\operatorname{Aut}(K)| = pq|\{(x, \alpha) \in H \times \operatorname{Aut}(K) : [x, \alpha] = 1\}|$. Since pq does not divide $(p+q-1)$, pq divides $|H||\operatorname{Aut}(K)|$.

If p and q are the smallest primes dividing $|\operatorname{Aut}(K)|$ and $|H|$ respectively then, by Theorem 3.10, we have

$$\frac{p+q-1}{pq} \leq \frac{1}{p}\left(\frac{p-1}{|H : L(H, \operatorname{Aut}(K))|} + 1\right)$$

which gives $|H : L(H, \mathrm{Aut}(K))| \leq q$. Hence, $\frac{H}{L(H,\mathrm{Aut}(K))} \cong \mathbb{Z}_q$. □

Theorem 3.12. *Let $H \subseteq K$ be two subgroups of a finite group G.*

(a) *If p, q are the smallest primes dividing $|\mathrm{Aut}(K)|$ and $|H|$ respectively and H is non-abelian then $\Pr(H, \mathrm{Aut}(K)) \leq \frac{q^2+p-1}{pq^2}$. In particular, if $p = q$ then $\Pr(H, \mathrm{Aut}(K)) \leq \frac{p^2+p-1}{p^3} \leq \frac{5}{8}$.*

(b) *If H is non-abelian and $\Pr(H, \mathrm{Aut}(K)) = \frac{q^2+p-1}{pq^2}$, for some primes p and q, then pq divides $|H||\mathrm{Aut}(K)|$. Further, if p and q are the smallest primes dividing $|\mathrm{Aut}(K)|$ and $|H|$ respectively then $\frac{H}{L(H,\mathrm{Aut}(K))} \cong \mathbb{Z}_q \times \mathbb{Z}_q$. In particular, if H and $\mathrm{Aut}(K)$ are of even order and $\Pr(H, \mathrm{Aut}(K)) = \frac{5}{8}$ then $\frac{H}{L(H,\mathrm{Aut}(K))} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. (a) Since H is non-abelian we have $|H : L(H, \mathrm{Aut}(K))| \geq q^2$. Therefore, by Theorem 3.10, we have

$$\Pr(H, \mathrm{Aut}(K)) \leq \frac{1}{p}\left(\frac{p-1}{|H : L(H, \mathrm{Aut}(K))|} + 1\right) \leq \frac{q^2+p-1}{pq^2}.$$

(b) Using (1.1), we have $(q^2+p-1)|H||\mathrm{Aut}(K)| = pq^2|\{(x, \alpha) \in H \times \mathrm{Aut}(K) : [x, \alpha] = 1\}|$. Since pq does not divide (q^2+p-1), pq divides $|H||\mathrm{Aut}(K)|$.

If p and q are the smallest primes dividing $|\mathrm{Aut}(K)|$ and $|H|$ respectively then, by Theorem 3.10, we have

$$\frac{q^2+p-1}{pq^2} \leq \frac{1}{p}\left(\frac{p-1}{|H : L(H, \mathrm{Aut}(K))|} + 1\right)$$

which gives $|H : L(H, \mathrm{Aut}(K))| \leq q^2$. Since H is non-abelian we have $|H : L(H, \mathrm{Aut}(K))| \neq 1, q$. Hence, $\frac{H}{L(H,\mathrm{Aut}(K))} \cong \mathbb{Z}_q \times \mathbb{Z}_q$. □

Proposition 3.13. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Let p, q be the smallest prime divisors of $|\mathrm{Aut}(K)|$, $|H|$ respectively and $|\mathrm{Aut}(K) : C_{\mathrm{Aut}(K)}(x)| = p$ for all $x \in H \setminus L(H, \mathrm{Aut}(K))$. Then*

$$\Pr(H, \mathrm{Aut}(K)) = \begin{cases} \frac{p+q-1}{pq} & \text{if } \frac{H}{L(H,\mathrm{Aut}(K))} \cong \mathbb{Z}_q \\ \frac{q^2+p-1}{pq^2} & \text{if } \frac{H}{L(H,\mathrm{Aut}(K))} \cong \mathbb{Z}_q \times \mathbb{Z}_q. \end{cases}$$

Proof. For all $x \in H \setminus L(H, \mathrm{Aut}(K))$ we have $|\mathrm{Aut}(K) : C_{\mathrm{Aut}(K)}(x)| = p$ and so $|C_{\mathrm{Aut}(K)}(x)| = \frac{|\mathrm{Aut}(K)|}{p}$. Therefore, by Corollary 3.3, we have

$$\begin{aligned} \Pr(H, \mathrm{Aut}(K)) &= \frac{|L(H, \mathrm{Aut}(K))|}{|H|} + \frac{1}{|H||\mathrm{Aut}(K)|} \sum_{x \in H \setminus L(H,\mathrm{Aut}(K))} |C_{\mathrm{Aut}(K)}(x)| \\ &= \frac{|L(H, \mathrm{Aut}(K))|}{|H|} + \frac{|H| - |L(H, \mathrm{Aut}(K))|}{p|H|} \\ &= \frac{1}{p}\left(\frac{p-1}{|H : L(H, \mathrm{Aut}(K))|} + 1\right). \end{aligned}$$

Hence, the result follows. □

Note that Proposition 3.13 gives partial converses of Theorems 3.11(b) and 3.12(b). We conclude this paper with the following two lower bounds analogous to the lower bounds obtained in [11, Theorem A] and [10, Theorem 1].

Theorem 3.14. *Let H and K be two subgroups of a finite group G such that $H \subseteq K$. Then*

$$\Pr(H, \mathrm{Aut}(K)) \geq \frac{1}{|S(H, \mathrm{Aut}(K))|}\left(1 + \frac{|S(H, \mathrm{Aut}(K))| - 1}{|H : L(H, \mathrm{Aut}(K))|}\right).$$

The equality holds if and only if $\text{orb}_K(x) = xS(H, \text{Aut}(K))$ *for all* $x \in H \setminus L(H, \text{Aut}(K))$.

Proof. We have $\mu(x) = x[x, \mu] \in xS(H, \text{Aut}(K))$ for all $x \in H \setminus L(H, \text{Aut}(K))$ and $\mu \in \text{Aut}(K)$. Therefore, for all $x \in H \setminus L(H, \text{Aut}(K))$ we have $\text{orb}_K(x) \subseteq xS(H, \text{Aut}(K))$ and so $|\text{orb}_K(x)| \leq |S(H, \text{Aut}(K))|$. Using Corollary 3.3, we have

$$\begin{aligned}\Pr(H, \text{Aut}(K)) &= \frac{1}{|H|}\left(\sum_{x \in L(H,\text{Aut}(K))} \frac{1}{|\text{orb}_K(x)|} + \sum_{x \in H \setminus L(H,\text{Aut}(K))} \frac{1}{|\text{orb}_K(x)|}\right) \\ &\geq \frac{|L(H, \text{Aut}(K))|}{|H|} + \frac{1}{|H|} \sum_{x \in H \setminus L(H,\text{Aut}(K))} \frac{1}{|S(H, \text{Aut}(K))|}.\end{aligned}$$

Hence, the result follows. □

Corollary 3.15. *Let* H *and* K *be two subgroups of a finite group* G *such that* $H \subseteq K$. *Then*

$$\Pr(H, \text{Aut}(K)) \geq \frac{1}{|[H, \text{Aut}(K)]|}\left(1 + \frac{|[H, \text{Aut}(K)]| - 1}{|H : L(H, \text{Aut}(K))|}\right).$$

If $H \neq L(H, \text{Aut}(K))$ *then the equality holds if and only if* $[H, \text{Aut}(K)] = S(H, \text{Aut}(K))$ *and* $\text{orb}_K(x) = x[H, \text{Aut}(K)]$ *for all* $x \in H \setminus L(H, \text{Aut}(K))$.

Proof. It is easy to see that

$$\frac{1}{n}\left(1 + \frac{n-1}{|H : L(H, \text{Aut}(K))|}\right) \geq \frac{1}{m}\left(1 + \frac{m-1}{|H : L(H, \text{Aut}(K))|}\right) \tag{3.4}$$

for any two integers $m \geq n$. If $L(H, \text{Aut}(K)) \neq H$ then equality holds in (3.4) if and only if $m = n$. Now, the result follows from Theorem 3.14 and (3.4) since $|[H, \text{Aut}(K)]| \geq |S(H, \text{Aut}(K))|$.

Note that the equality holds if and only if equality holds in Theorem 3.14 and (3.4). □

It is worth mentioning that Theorem 3.14 gives better lower bound than the lower bound given by Corollary 3.15. Also

$$\begin{aligned}\frac{1}{|[H, \text{Aut}(K)]|}\left(1 + \frac{|[H, \text{Aut}(K)]| - 1}{|H : L(H, \text{Aut}(K))|}\right) \geq &\frac{|L(H, \text{Aut}(K))|}{|H|} \\ &+ \frac{p(|H| - |L(H, \text{Aut}(K))|)}{|H||\text{Aut}(K)|}.\end{aligned}$$

Hence, Theorem 3.14 gives better lower bound than the lower bound given by Theorem 3.10.

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