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Research Article

Generalized omni-Lie algebras

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Abstract

We introduce the notion of generalized omni-Lie algebras from omni-Lie algebras constructed by Weinstein. We prove that there is a one-to-one correspondence between Dirac structures of a generalized omni-Lie algebra and Lie structures on its linear space.

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1. Introduction

The notion of omni-Lie algebras was introduced by Weinstein[7], which is the linearization of the Courant bracket. Let V be a linear space, and an omni-Lie algebra is the direct sum space $gl(V)\oplus V$ with a skew-symmetric bracket operation $[\![\cdot,\cdot]\!]$ and a non-degenerate symmetric bilinear pairing $\langle\cdot,\cdot\rangle$ given by

$$[\![A+x,B+y]\!]=[A,B]+\frac{1}{2}(Ay-Bx),$$

and

$$\langle A+x,B+y\rangle=\frac{1}{2}(Ay+Bx).$$

An omni-Lie algebra is not a Lie algebra, but its Dirac structures are Lie algebras. Actually, an omni-Lie algebra is a Lie 2-algebra since Roytenberg and Weinstein proved that every Courant algeboid gives rise to a Lie 2-algebra[5]. Recently, omni-Lie algebras were generalized to omni-Lie superalgebras, omni-Lie color algebras and omni-Lie algebroids[1,8]. In[2], they generalized omni-Lie algebras from a linear space to a linear bundle E in order to characterize all possible Lie algebroid structures on E. Dirac structures were also studied from several aspects[2,3,6].

In this paper, we introduce the notion of a generalized omni-Lie algebra, which is the (δ,α) omni-Lie algebra and discuss special situations when δ, α are fixed values. Then we study Dirac structures of the generalized omni-Lie algebra in order to characterize all Lie algebra structures on the linear space and prove that there is a one-to-one correspondence between Dirac structures of the generalized omni-Lie algebra $(\Omega,[\![\cdot,\cdot]\!],\langle\cdot,\cdot\rangle)$ and Lie algebra structures on subspaces of V if $\delta=\frac{1}{2}$. Moreover, we prove that a generalized omni-Lie algebra is a Leibniz algebra.

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2. Generalized omni-Lie algebras

Let V be a linear space over a field F. The set of all linear transformations on V is a Lie algebra denoted by $gl(V)$, given by $[A, B] = AB - BA$, for any $A, B \in gl(V)$.

Definition 2.1. A generalized omni-Lie algebra is the linear space $\Omega = gl(V) \oplus V$ with a skew-symmetric bilinear bracket operation $[\![\cdot,\cdot]\!]$ and a non-degenerate symmetric bilinear pairing $\langle\cdot,\cdot\rangle$, for any $A, B \in gl(V), x, y \in V, and\ \delta, \alpha \in \mathrm{F}$,

$$[\![A + x, B + y]\!] = [A, B] + \delta(Ay - Bx), \tag{2.1}$$

and

$$\langle A + x, B + y\rangle = \alpha(Ay + Bx). \tag{2.2}$$

We call $(\Omega, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle)$ a generalized omni-Lie algebra.

Proposition 2.2. *Let J denote the Jacobiator for the bracket $[\![\cdot,\cdot]\!]$ of Ω, then for any $e_1 = A + x, e_2 = B + y, e_3 = C + z \in \Omega$,*

$$J(e_1, e_2, e_3) = [\![[\![e_1, e_2]\!], e_3]\!] + [\![[\![e_2, e_3]\!], e_1]\!] + [\![[\![e_3, e_1]\!], e_2]\!].$$

(i) If $\delta = 0, 1$, $(\Omega, [\![\cdot,\cdot]\!])$ is a Lie algebra.
(ii) If $\alpha = \delta = \frac{1}{2}$, $(\Omega, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle)$ is an omni-Lie algebra.

Proof. By a direct calculation, we get

$$\begin{aligned}
& J(e_1, e_2, e_3) \\
= \;& [\![[\![A + x, B + y]\!], C + z]\!] + c.p. \\
= \;& [\![[A, B] + \delta(Ay - Bx), C + z]\!] + c.p. \\
= \;& [[A, B], C] + [[B, C], A] + [[C, A], B] + \delta([A, B]z - \delta C(Ay - Bx)) \\
& + \delta([B, C]x - \delta A(Bz - Cy)) + \delta([C, A]y - \delta B(Cx - Az)) \\
= \;& [[A, B], C] + [[B, C], A] + [[C, A], B] + (\delta - \delta^2)([A, B]z + [B, C]x + [C, A]y) \\
= \;& (\delta - \delta^2)([A, B]z + [B, C]x + [C, A]y).
\end{aligned}$$

If $\delta = 0, 1$, J satisfies the Jocabi identity, so $(\Omega, [\![\cdot,\cdot]\!])$ is a Lie algebra. Especially if $\delta = 1$, $(\Omega, [\![\cdot,\cdot]\!])$ is a semidirect product of $gl(V)$ and V. □

Proposition 2.3. *Let J denote the Jacobiator for the bracket $[\![\cdot,\cdot]\!]$ of Ω, for any $e_1 = A + x, e_2 = B + y, e_3 = C + z \in \Omega$, we set*

$$T(e_1, e_2, e_3) = \langle[\![e_1, e_2]\!], e_3\rangle + \langle[\![e_2, e_3]\!], e_1\rangle + \langle[\![e_3, e_1]\!], e_2\rangle,$$

$$T'(e_1, e_2, e_3) = \frac{\delta - \delta^2}{\alpha + \alpha\delta} T(e_1, e_2, e_3),$$

then we have

$$T'(e_1, e_2, e_3) = J(e_1, e_2, e_3). \tag{2.3}$$

Proof. We have proved that for any $e_1 = A + x, e_2 = B + y, e_3 = C + z \in \Omega$,

$$J(e_1, e_2, e_3) = (\delta - \delta^2)([A, B]z + [B, C]x + [C, A]y).$$

By Definition 2.1, we get

$$\begin{aligned}
& T(e_1, e_2, e_3) \\
= \;& \langle[\![e_1, e_2]\!], e_3\rangle + c.p. \\
= \;& \langle[A, B] + \delta(Ay - Bx), C + z\rangle + c.p. \\
= \;& \alpha([A, B]z + \delta C(Ay - Bx)) + \alpha([B, C]x + \delta A(Bz - Cy)) \\
& + \alpha([C, A]y + \delta B(Cx - Az)) \\
= \;& (\alpha + \alpha\delta)([A, B]z + [B, C]x + [C, A]y),
\end{aligned}$$

$$T'(e_1, e_2, e_3) = \frac{\delta - \delta^2}{\alpha + \alpha\delta} T(e_1, e_2, e_3) = J(e_1, e_2, e_3).$$

Thus, Eq. (2.3) holds. □

Let ω be a bilinear operation on V, and define the adjoint operator $\mathrm{ad}_\omega : V \to gl(V)$ by

$$\mathrm{ad}_\omega(x)(y) := \omega(x, y), \ \forall\, x, y \in V,$$

then the graph of the adjoint operator

$$\mathscr{F}_\omega = \{\mathrm{ad}_\omega(x) + x, \ \forall\, x \in V\}$$

is a subspace of Ω. $\mathscr{F}_\omega^\perp$ denote the orthogonal complement of $\mathscr{F}_\omega$ with respect to the bilinear form (2.2) of Ω.

Proposition 2.4. *If $\delta = \frac{1}{2}$, (V, ω) is a Lie algebra if and only if $\mathscr{F}_\omega$ is maximal isotropic, i.e.,*

$$\mathscr{F}_\omega = \mathscr{F}_\omega^\perp,$$

and is closed under the bracket $[\![\cdot, \cdot]\!]$.

Proof. First, for any $\mathrm{ad}_\omega(x) + x, \mathrm{ad}_\omega(y) + y \in \mathscr{F}_\omega$,

$$\begin{aligned} & \langle \mathrm{ad}_\omega(x) + x, \mathrm{ad}_\omega(y) + y \rangle \\ = \ & \alpha(\mathrm{ad}_\omega(x)(y) + \mathrm{ad}_\omega(y)(x)) \\ = \ & \alpha(\omega(x, y) + \omega(y, x)), \end{aligned}$$

which means that $\omega(\cdot, \cdot)$ is skew-symmetric if and only if its graph is isotropic and $\mathscr{F}_\omega = \mathscr{F}_\omega^\perp$. Then if $\delta = \frac{1}{2}$ and $\omega(\cdot, \cdot)$ is skew-symmetric, let us check

$$\begin{aligned} & [\![\mathrm{ad}_\omega(x) + x, \mathrm{ad}_\omega(y) + y]\!] \\ = \ & [\mathrm{ad}_\omega(x), \mathrm{ad}_\omega(y)] + \frac{1}{2}(\mathrm{ad}_\omega(x)(y) - \mathrm{ad}_\omega(y)(x)) \\ = \ & [\mathrm{ad}_\omega(x), \mathrm{ad}_\omega(y)] + \frac{1}{2}(\omega(x, y) - \omega(y, x)) \\ = \ & [\mathrm{ad}_\omega(x), \mathrm{ad}_\omega(y)] + \omega(x, y). \end{aligned}$$

Hence, the bracket is closed if and only if

$$[\mathrm{ad}_\omega(x), \mathrm{ad}_\omega(y)] = \mathrm{ad}_\omega(\omega(x, y)),$$

it follows that for any $z \in V$,

$$\begin{aligned} & [\mathrm{ad}_\omega(x), \mathrm{ad}_\omega(y)](z) - \mathrm{ad}_\omega(\omega(x, y))(z) \\ = \ & \mathrm{ad}_\omega(x)\mathrm{ad}_\omega(y)(z) - \mathrm{ad}_\omega(y)\mathrm{ad}_\omega(x)(z) - \mathrm{ad}_\omega(\omega(x, y))(z) \\ = \ & \mathrm{ad}_\omega(x)\omega(y, z) - \mathrm{ad}_\omega(y)\omega(x, z) - \omega(\omega(x, y), z) \\ = \ & \omega(x, \omega(y, z)) - \omega(y, \omega(x, z)) - \omega(\omega(x, y), z), \end{aligned}$$

it is clear that the bracket is closed if and only if the Jacobi identity of $\omega(\cdot, \cdot)$ on V is satisfied. Thus, the proof is completed. □

Definition 2.5. Let L be a maximal isotropic subspace of $\Omega = gl(V) \oplus V$ and closed under the bracket $[\![\cdot, \cdot]\!]$, then we call L a Dirac structure of the generalized omni-Lie algebra $(\Omega, [\![\cdot, \cdot]\!], \langle \cdot, \cdot \rangle)$.

Remark 2.6. By Proposition 2.3, for a Dirac structure L, we can get

$$T'(e_1, e_2, e_3) = J(e_1, e_2, e_3) = 0, \ \forall\, e_i \in L, \ i = 1, 2, 3.$$

then a Dirac structure $(L, [\![\cdot, \cdot]\!])$ is a Lie algebra.

According to the Definition 2.5, we can rewrite Proposition 2.4 that " if $\delta = \frac{1}{2}$, (V, ω) is a Lie algebra if and only if $\mathscr{F}_\omega$ is a Dirac structure of the generalized omni-Lie algebra $(\Omega, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle)$."

Then we want to know the concrete form of Dirac structures, for a maximal isotropic subspace L, let $D = L \cap gl(V)$, define D^0 to be the kernel of D,

$$D^0 := \{x \in V \mid X(x) = 0, \forall\, X \in D\} \subseteq V,$$
$$(D^0)^0 := \{X \in gl(V) \mid X(x) = 0, \forall\, x \in D^0\} \subseteq gl(V) = D.$$

Lemma 2.7. *A subspace L is maximal isotropic if and only if*

$$L = D \oplus \mathscr{F}_{\pi|D_0} = \{X + \pi(x) + x \mid \forall\, X \in D, x \in D^0\}, \tag{2.4}$$

where $\pi : V \to gl(V)$ is a skew-symmetric map.

Proof. First, suppose that L is given by (2.4), for any $X + \pi(x) + x, Y + \pi(y) + y \in L$,

$$\begin{aligned} & \langle X + \pi(x) + x, Y + \pi(y) + y\rangle \\ = \;& \alpha(X(y) + \pi(x)(y) + Y(x) + \pi(y)(x)) \\ = \;& \alpha(\pi(x)(y) + \pi(y)(x)) \\ = \;& 0. \end{aligned}$$

Thus, L is isotropic, then we prove that L is maximal isotropic. For all $B + z \in L^\perp$, we have

$$0 = \langle X, B + z\rangle = \alpha X(z), \forall\, X \in D,$$

so $z \in D^0$,

$$\begin{aligned} 0 &= \langle X + \pi(x) + x, B + z\rangle \\ &= \alpha(X(z) + \pi(x)(z) + B(x)) \\ &= \alpha(B - \pi(z))(x), \forall\, X + \pi(x) + x \in L, \end{aligned}$$

let $Z := B - \pi(z) \in D$,

$$B + z = Z + \pi(z) + z \in L^\perp = L,$$

therefore, L is maximal isotropic. The converse part is straightforward, so we omit the details. □

Lemma 2.8. *Let (D, π) be given above for a maximal isotropic subspace $L \subset \Omega$. Then L is a Dirac structure if and only if the following conditions are satisfied:*

(i) D is a subalgebra of $gl(V)$;
(ii) $\pi(\pi(x, y)) - [\pi(x), \pi(y)] \in D, \forall\, x, y \in D^0$;
(iii) $\pi(x, y) \in D^0, \forall\, x, y \in D^0$.

Such a pair (D, π) is called a characteristic pair of a Dirac structure L.

Proof. By Definition 2.5, L is a Dirac structure if and only if L is closed with respect to the bracket (2.1). First, for any $X + \pi(x) + x, Y + \pi(y) + y \in L$, by straightforward calculation, we get

$$\begin{aligned} & [\![X + \pi(x) + x, Y + \pi(y) + y]\!] \\ = \;& [X + \pi(x), Y + \pi(y)] + \delta(\pi(x)(y) - \pi(y)(x)) \\ = \;& [X, Y] + [X, \pi(y)] + [\pi(x), Y] + [\pi(x), \pi(y)] + 2\delta\pi(x, y). \end{aligned}$$

If $\delta = \frac{1}{2}$, L is closed under the bracket (2.1) if and only if $\pi(x, y) \in D^0$, $\pi(\pi(x, y)) - [\pi(x), \pi(y)] \in D$, $\forall\, x, y \in D^0$. Moreover for any $X, Y \in D, x, y, z \in D^0$, we have

$$[X, Y](z) = XY(z) - YX(z) = 0,$$
$$[X, \pi(y)](z) = X\pi(y)(z) - \pi(y)X(z) = 0,$$
$$[\pi(x), Y](z) = \pi(x)Y(z) - Y\pi(x)(z) = 0,$$

so $[X,Y],[X,\pi(y)],[\pi(x),Y]\in D$, that is to say, D is a subalgebra of $gl(V)$. $\square$

Theorem 2.9. *There is a one-to-one correspondence between Dirac structures of the generalized omni-Lie algebra* $(\Omega,[\![\cdot,\cdot]\!],\langle\cdot,\cdot\rangle)$ *and Lie algebra structures on subspaces of* V *if* $\delta=\frac{1}{2}$.

Proof. First, by Lemmas 2.7, 2.8, if L is a Dirac structure, then $L=D\oplus\mathscr{F}_{\pi|D_0}$ and satisfies three conditions in Lemma 2.8. Define operation $[\cdot,\cdot]_{D^0}$ on $D^0\subseteq V$ by

$$[x,y]_{D^0}:=\pi(x,y)\in D^0,\ \forall\, x,y\in D^0,$$

$[\cdot,\cdot]_{D^0}$ is a skew-symmetric operation because π is a skew-symmetric map. Then, we check the Jacobi identity, for any $x,y,z\in D^0$,

$$\begin{aligned}&[[x,y]_{D^0},z]_{D^0}\\ =\ &\pi([x,y]_{D^0})(z)\\ =\ &\pi(\pi(x,y))(z)\\ =\ &[\pi(x),\pi(y)](z)\\ =\ &\pi(x)\pi(y)(z)-\pi(y)\pi(x)(z)\\ =\ &[x,[y,z]_{D^0}]_{D^0}-[y,[x,z]_{D^0}]_{D^0}.\end{aligned}$$

Thus, $(D^0,[\cdot,\cdot]_{D^0})$ is a Lie algebra.

Conversely, W is a subspace of V, for any Lie algebra $(W,[\cdot,\cdot]_W)$, and define D by

$$D:=W^0=\{X\in gl(V)\mid X(x)=0,\forall x\in W\},$$
$$D^0=(W^0)^0=W.$$

Let $\mathrm{ad}:W\to gl(W)$ represents the limitation of $\pi:V\to gl(V)$ on W,

$$\mathrm{ad}_x(y)=[x,y]_W,$$

then we get a maximal isotropic subspace

$$L=D\oplus\mathscr{F}_{\pi|W}.$$

Next is to prove that L is closed under the bracket $[\![\cdot,\cdot]\!]$, if $\delta=\frac{1}{2}$, for $X+\mathrm{ad}_x+x,Y+\mathrm{ad}_y+y\in L$,

$$\begin{aligned}&[\![X+\mathrm{ad}_x+x,Y+\mathrm{ad}_y+y]\!]\\ =\ &[X+\mathrm{ad}_x,Y+\mathrm{ad}_y]+\frac{1}{2}((X+\mathrm{ad}_x)(y)-(Y+\mathrm{ad}_y)(x))\\ =\ &[X,Y]+[X,\mathrm{ad}_y]+[\mathrm{ad}_x,Y]+[\mathrm{ad}_x,\mathrm{ad}_y]+\frac{1}{2}(\mathrm{ad}_x(y)-\mathrm{ad}_y(x))\\ =\ &[X,Y]+[X,\mathrm{ad}_y]+[\mathrm{ad}_x,Y]+[\mathrm{ad}_x,\mathrm{ad}_y]+[x,y]_W.\end{aligned}$$

For any $X,Y\in D$ and $x,y\in W$,

$$[X,Y](x)=XY(x)-YX(x)=0,$$

which means $[X,Y]\in D$, D is a subalgebra of $gl(V)$.

$$[X,\mathrm{ad}_x](y)=X([x,y]_W)-[x,X(y)]=0,$$
$$[\mathrm{ad}_x,Y](y)=[x,Y(y)]-Y([x,y]_W)=0,$$

so $[X,\mathrm{ad}_y],[\mathrm{ad}_x,Y]\in D$.

Since $[\cdot,\cdot]_W$ satisfies the Jacobi identity, we obtain

$$[\mathrm{ad}_x,\mathrm{ad}_y]=\mathrm{ad}_{[x,y]_W},$$
$$[\![X+\pi(x)+x,Y+\pi(y)+y]\!]\in D\oplus\mathscr{F}_{\pi|W}.$$

Thus, L is a Dirac structure. $\square$

Let Λ *denotes the family of all Lie structures on the subspaces of* V*, and* Γ *denotes the family of all Dirac structures of the generalized omni-Lie algebra* Ω*, then according to Theorem 2.9, there exists a bijective*

$$\Psi : \Lambda \to \Gamma,$$

and an embedding

$$\varphi_W : W \to L, \forall W \in \Lambda, L \in \Gamma.$$

Definition 2.10. [4] Let L be a linear space over a field F together with a bilinear operation $\circ$: $L \times L \to L$ satisfying

$$((x \circ y) \circ z) = (x \circ (y \circ z)) - (y \circ (x \circ z)), \forall x, y, z \in L,$$

then we call $(L, \circ)$ a Leibniz algebra.

We define another bilinear operation "$\circ$" on $\Omega = gl(V) \oplus V$ by

$$(A + x) \circ (B + y) = [A, B] + \delta Ay, \forall A + x, B + y \in \Omega, \delta \in \mathrm{F}.$$

Proposition 2.11. $(\Omega, \circ)$ *is a Leibniz algebra.*

Proof. We check if the Leibniz identity is satisfied, for any $e_1 = A + x, e_2 = B + y, e_3 = C + z \in \Omega$,

$$\begin{aligned} & (e_1 \circ e_2) \circ e_3 - e_1 \circ (e_2 \circ e_3) + e_2 \circ (e_1 \circ e_3) \\ = & ([A, B] + \delta Ay) \circ (C + z) - (A + x) \circ ([B, C] + \delta Bz) + (B + y) \circ ([A, C] + \delta Az) \\ = & [[A, B], C] - [A, [B, C]] + [B, [A, C]] + \delta(ABz - BAz) - \delta ABz + \delta BAz \\ = & 0. \end{aligned}$$

By Definition 2.10, it holds. □

Proposition 2.12. *Let* V *be a Lie algebra.* D *is a derivation of* V *that satisfies*

$$D[x, y] = [Dx, y] + [x, Dy], \forall x, y \in V$$

if and only if $\mathscr{F}_\omega$ *is an invariant subspace of* D *under the operation "*$\circ$*" if* $\delta = 1$*, i.e.,*

$$D \circ \mathscr{F}_\omega \subseteq \mathscr{F}_\omega.$$

Proof. If $\delta = 1$, for $\mathrm{ad}_\omega(x) + x \in \mathscr{F}_\omega, y \in V$,

$$D \circ (\mathrm{ad}_\omega(x) + x) = [D, \mathrm{ad}_\omega(x)] + Dx.$$

The right side belongs to $\mathscr{F}_\omega$ if and only if

$$[D, \mathrm{ad}_\omega(x)] = \mathrm{ad}_\omega(Dx),$$

for convenience, we denote $\omega(x, y) := [x, y]$,

$$\begin{aligned} & [D, \mathrm{ad}_\omega(x)](y) - \mathrm{ad}_\omega(Dx)(y) \\ = & D\mathrm{ad}_\omega(x)(y) - \mathrm{ad}_\omega(x)D(y) - \mathrm{ad}_\omega(Dx)(y) \\ = & D[x, y] - [x, Dy] - [Dx, y]. \end{aligned}$$

Thus, D is a derivation of V if and only if $D \circ \mathscr{F}_\omega \subseteq \mathscr{F}_\omega$. □

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