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Research Article

Convolutions of the bi-periodic Fibonacci numbers

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Abstract

Let q_n be the bi-periodic Fibonacci numbers, defined by $q_n = c(n)q_{n-1} + q_{n-2}$ $(n \geq 2)$ with $q_0 = 0$ and $q_1 = 1$, where $c(n) = a$ if n is even, $c(n) = b$ if n is odd, where a and b are nonzero real numbers. When $c(n) = a = b = 1$, $q_n = F_n$ are Fibonacci numbers. In this paper, the convolution identities of order 2, 3 and 4 for the bi-periodic Fibonacci numbers q_n are given with binomial (or multinomial) coefficients, by using the symmetric formulas.

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1. Introduction

Convolution identities for various types of numbers (or polynomials) have been studied, with or without binomial (or multinomial) coefficients, including Bernoulli, Euler, Genocchi, Cauchy, Stirling and balancing numbers (cf. [1–3,6,9,10,15,16,19]). A typical formula is due to Euler, given by

$$\sum_{k=0}^{n}\binom{n}{k}\mathcal{B}_k\mathcal{B}_{n-k} = -n\mathcal{B}_{n-1} - (n-1)\mathcal{B}_n \quad (n \geq 0)\,,$$

where $\mathcal{B}_n$ are Bernoulli numbers, defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty}\mathcal{B}_n\frac{x^n}{n!} \quad (|x| < 2\pi)\,.$$

On the other hand, many kinds of generalizations of Fibonacci numbers have been presented in the literature. A typical one is a generalized Fibonacci sequence $\{W_n\}_{n=0}^{\infty}$, defined by $W_n = pW_{n-1} + qW_{n-2}$ $(n \geq 2)$ with $W_0 = a$ and $W_1 = b$. In [5] some new identities involving differences of products of generalized Fibonacci numbers are shown. One of different types is the bi-periodic Fibonacci sequence [7]. For any two nonzero real numbers a and b, the bi-periodic Fibonacci sequence, say $\{q_n\}_{n=0}^{\infty}$, is determined by:

$$q_0 = 0, \quad q_1 = 1, \quad q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \equiv 0 \pmod 2; \\ bq_{n-1} + q_{n-2}, & \text{if } n \equiv 1 \pmod 2; \end{cases} \quad n \geqslant 2. \tag{1.1}$$

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When $a = b = 1$, $q_n = F_n$ are Fibonacci numbers. The explicit expression of the bi-periodic Fibonacci numbers can be expressed explicitly as

$$q_{2n} = a\sum_{k=0}^{n-1}\binom{2n-k-1}{k}(ab)^{n-k-1},$$
$$q_{2n+1} = \sum_{k=0}^{n}\binom{2n-k}{k}(ab)^{n-k}.$$

Moreover, the ordinary generating function of the bi-periodic Fibonacci numbers is given by

$$F(x) := \sum_{n=0}^{\infty} q_n x^n = \frac{x(1+ax-x^2)}{1-(ab+2)x^2+x^4}.$$

For more properties about this sequence see for example [4,7,8,18,20].

Recently, in [13], the convolution identities of two Fibonacci numbers F_n are explicitly given:

$$\sum_{k=0}^{n} F_k F_{n-k} = \sum_{m=0}^{n-1} m F_m \cos\frac{(n-m-1)\pi}{2}$$

as special cases of higher-order identities. In [14], this result is generalized by using a more general form:

$$\frac{F_r + (-1)^r F_{k-r}x}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} F_{kn+r}x^n,$$

with $k > r \geq 0$, where L_n are Lucas numbers. In [11,12,17], convolution identities for Fibonacci numbers are generalized as Tribonacci numbers and Tetranacci numbers. In particular, in [12,17], symmetric formulas are used to yield the results.

In this paper, motivated by the previous results, the convolution identities for the bi-periodic Fibonacci numbers q_n are given with binomial (or multinomial) coefficients. In [15] the so-called exponential generating functions of generalized Fibonacci-type numbers u_n and Lucas-type numbers v_n are considered:

$$\frac{e^{\alpha x} - e^{\beta x}}{\sqrt{a^2+4b}} = \sum_{n=0}^{\infty} u_n \frac{x^n}{n!} \quad \text{and} \quad e^{\alpha x} + e^{\beta x} = \sum_{n=0}^{\infty} v_n \frac{x^n}{n!}.$$

Here, $u_n = au_{n-1} + bu_{n-2}$ $(n \geq 2)$ with $u_0 = 0$ and $u_1 = 1$, and $v_n = av_{n-1} + bv_{n-2}$ $(n \geq 2)$ with $v_0 = 2$ and $v_1 = a$. α and β are the roots of the quadratic equation $x^2 - ax - b = 0$, given by

$$\alpha = \frac{a+\sqrt{a^2+4b}}{2} \quad \text{and} \quad \beta = \frac{a-\sqrt{a^2+4b}}{2}.$$

Then the higher-order convolution identities with multinomial coefficients

$$\sum_{\substack{k_1+\cdots+k_r=n\\ k_1,\dots,k_r\geq 0}} \binom{n}{k_1,\dots,k_r} u_{k_1}\cdots u_{k_r} \quad \text{and} \quad \sum_{\substack{k_1+\cdots+k_r=n\\ k_1,\dots,k_r\geq 0}} \binom{n}{k_1,\dots,k_r} v_{k_1}\cdots v_{k_r}$$

are given in the linear combinations of u_n and v_n. We consider this kind of convolution identities for bi-periodic Fibonacci numbers.

This paper is organized as follows. In Section 2, convolution identities for two bi-periodic Fibonacci numbers with binomial coefficients are shown. In Section 3, convolution identities for three and four bi-periodic Fibonacci numbers with multinomial coefficients are shown. The main tools are symmetric formulas which are often used in [12,17].

2. Convolution identities with binomial coefficients

Consider the exponential generating function

$$f(x) := \sum_{n=0}^{\infty} q_n \frac{x^n}{n!} .$$

We introduce two supplementary functions

$$f_1(x) = \sum_{n=0}^{\infty} q_{2n} \frac{x^{2n}}{(2n)!} \quad \text{and} \quad f_2(x) = \sum_{n=0}^{\infty} q_{2n+1} \frac{x^{2n+1}}{(2n+1)!}$$

so that $f(x) = f_1(x) + f_2(x)$. By using the recurrence relations (1.1), we have the system of the differential equations:

$$f_1''(x) - af_2'(x) - f_1(x) = 0 , \tag{2.1}$$

$$f_2''(x) - bf_1'(x) - f_2(x) = 0 . \tag{2.2}$$

Therefore, we get two 4-th order differential equations:

$$f_1^{(4)}(x) - (ab+2)f_1''(x) + f_1(x) = 0 , \tag{2.3}$$

$$f_2^{(4)}(x) - (ab+2)f_2''(x) + f_2(x)) = 0 . \tag{2.4}$$

Since the roots of $x^4 - (ab+2)x^2 + 1 = 0$ are given by

$$\pm\alpha = \pm\sqrt{\frac{ab+2+\sqrt{ab(ab+4)}}{2}} \quad \text{and} \quad \pm\,\beta = \pm\sqrt{\frac{ab+2-\sqrt{ab(ab+4)}}{2}} ,$$

the generating function $f_1(x)$ can be expressed as

$$f_1(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x} + c_3 e^{\beta x} + c_4 e^{-\beta x} ,$$

where

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= q_0 = 0 , \\ c_1\alpha - c_2\alpha + c_3\beta - c_4\beta &= 0 , \\ c_1\alpha^2 + c_2\alpha^2 + c_3\beta^2 + c_4\beta^2 &= q_2 = a , \\ c_1\alpha^3 - c_2\alpha^3 + c_3\beta^3 - c_4\beta^3 &= 0 . \end{aligned}$$

Solving this system, we get

$$c_1 = c_2 = \frac{a}{2\sqrt{ab(ab+4)}} \quad \text{and} \quad c_3 = c_4 = -\frac{a}{2\sqrt{ab(ab+4)}} .$$

Note that $\alpha\beta = 1$, $\alpha^2 + \beta^2 = ab + 2$ and $\alpha^2 - \beta^2 = \sqrt{ab(ab+4)}$.

Similarly, the generating function $f_2(x)$ can be expressed as

$$f_2(x) = d_1 e^{\alpha x} + d_2 e^{-\alpha x} + d_3 e^{\beta x} + d_4 e^{-\beta x} ,$$

where

$$\begin{aligned} d_1 + d_2 + d_3 + d_4 &= 0 , \\ d_1\alpha - d_2\alpha + d_3\beta - d_4\beta &= q_1 = 1 , \\ d_1\alpha^2 + d_2\alpha^2 + d_3\beta^2 + d_4\beta^2 &= 0 , \\ d_1\alpha^3 - d_2\alpha^3 + d_3\beta^3 - d_4\beta^3 &= q_3 = ab + 1 . \end{aligned}$$

Solving this system, we get

$$d_1=-d_2=\frac{ab+4+\sqrt{ab(ab+4)}}{2\sqrt{2}(ab+4)\sqrt{ab+2+\sqrt{ab(ab+4)}}}\,,$$
$$d_3=-d_4=\frac{ab+4-\sqrt{ab(ab+4)}}{2\sqrt{2}(ab+4)\sqrt{ab+2-\sqrt{ab(ab+4)}}}\,.$$

Therefore, we obtain that

$$f(x)=r_1e^{\alpha x}+r_2e^{-\alpha x}+r_3e^{\beta x}+r_4e^{-\beta x}\,,$$

where

$$r_1=c_1+d_1=\frac{a}{2\sqrt{ab(ab+4)}}+\frac{ab+4+\sqrt{ab(ab+4)}}{2\sqrt{2}(ab+4)\sqrt{ab+2+\sqrt{ab(ab+4)}}}\,,$$
$$r_2=c_2+d_2=\frac{a}{2\sqrt{ab(ab+4)}}-\frac{ab+4+\sqrt{ab(ab+4)}}{2\sqrt{2}(ab+4)\sqrt{ab+2+\sqrt{ab(ab+4)}}}\,,$$
$$r_3=c_3+d_3=-\frac{a}{2\sqrt{ab(ab+4)}}+\frac{ab+4-\sqrt{ab(ab+4)}}{2\sqrt{2}(ab+4)\sqrt{ab+2-\sqrt{ab(ab+4)}}}\,,$$
$$r_4=c_4+d_4=-\frac{a}{2\sqrt{ab(ab+4)}}-\frac{ab+4-\sqrt{ab(ab+4)}}{2\sqrt{2}(ab+4)\sqrt{ab+2-\sqrt{ab(ab+4)}}}\,.$$

Now, we shall consider the sum of the product of two bi-periodic Fibonacci numbers. We need three Lemmas to get the main result.

Lemma 2.1.

$$r_1^2e^{2\alpha x}+r_2^2e^{-2\alpha x}+r_3^2e^{2\beta x}+r_4^2e^{-2\beta x}=\frac{1}{b(ab+4)}\sum_{n=0}^\infty Q_n\frac{x^n}{n!}\,,$$

where Q_n are numbers, satisfying for $n\ge 1$

$$Q_{2n}=(a+b)Q_{2n-1}+4Q_{2n-2}\,,$$
$$Q_{2n+1}=\frac{4ab}{a+b}Q_{2n}+4Q_{2n-1},$$

with $Q_0=a+b$ and $Q_1=2ab$.

Remark 2.2. We have explicit expressions: for $n\ge 0$

$$Q_{2n}=2^{2n-1}(a+b)\sum_{k=0}^n\frac{2n}{2n-k}\binom{2n-k}{k}(ab)^{n-k}\,,$$
$$Q_{2n+1}=2^{2n+1}ab\sum_{k=0}^n\frac{2n+1}{2n-k+1}\binom{2n-k+1}{k}(ab)^{n-k}\,.$$

Proof of Lemma 2.1. Assume that the exponential generating function

$$r_1^2e^{2\alpha x}+r_2^2e^{-2\alpha x}+r_3^2e^{2\beta x}+r_4^2e^{-2\beta x}$$

determines the sequence $\{\widehat{Q}_n\}_{n=0}^\infty$. By $\alpha\beta=1$ and $\alpha^2+\beta^2=ab+2$, we have the characteristic equation

$$\begin{aligned}&(x+2\alpha)(x-2\alpha)(x+2\beta)(x-2\beta)\\&=x^4-4(\alpha^2+\beta^2)x^2+16\alpha^2\beta^2\\&=x^4-4(ab+2)x^2+16\,.\end{aligned}$$

Thus, $\widehat{Q}_n$ satisfies the recurrence relation

$$\widehat{Q}_n = 4(ab+2)\widehat{Q}_{n-2} - 16\widehat{Q}_{n-4} \quad (n \geq 4)\,. \tag{2.5}$$

Now, we get that

$$\begin{aligned}
\widehat{Q}_0 &= r_1^2 + r_2^2 + r_3^2 + r_4^2 = \frac{a+b}{b(ab+4)}\,,\\
\widehat{Q}_1 &= 2r_1^2\alpha - 2r_2^2\alpha + 2r_3^2\beta - 2r_4^2\beta = \frac{2a}{ab+4}\,,\\
\widehat{Q}_2 &= 4r_1^2\alpha^2 + 4r_2^2\alpha^2 + 4r_3^2\beta^2 + 4r_4^2\beta^2 = \frac{2(a+b)(ab+2)}{b(ab+4)}\,,\\
\widehat{Q}_3 &= 8r_1^2\alpha^3 - 8r_2^2\alpha^3 + 8r_3^2\beta^3 - 8r_4^2\beta^3 = \frac{8a(ab+3)}{ab+4}\,.
\end{aligned}$$

Using the recurrence relation (2.5), by induction, we have the recurrence relations: for $n \geq 1$

$$\begin{aligned}
\widehat{Q}_{2n} &= (a+b)\widehat{Q}_{2n-1} + 4\widehat{Q}_{2n-2}\,,\\
\widehat{Q}_{2n+1} &= \frac{4ab}{a+b}\widehat{Q}_{2n} + 4\widehat{Q}_{2n-1}\,.
\end{aligned}$$

Putting $Q_n = b(ab+4)\widehat{Q}_n$, we get the desired result. □

Lemma 2.3.

$$r_1r_3e^{(\alpha+\beta)x} + r_2r_4e^{-(\alpha+\beta)x} + r_1r_4e^{(\alpha-\beta)x} + r_2r_3e^{-(\alpha-\beta)x} = -\frac{1}{b(ab+4)}\sum_{n=0}^{\infty} QQ_n\frac{x^n}{n!}\,,$$

where for $n \geq 0$

$$\begin{aligned}
QQ_{2n} &= \frac{a-b}{2}\big((ab+4)^n - (ab)^n\big) + a^{n+1}b^n\,,\\
QQ_{2n+1} &= (ab)^{n+1}\,.
\end{aligned}$$

Proof. Assume that the exponential generating function

$$r_1r_3e^{(\alpha+\beta)x} + r_2r_4e^{-(\alpha+\beta)x} + r_1r_4e^{(\alpha-\beta)x} + r_2r_3e^{-(\alpha-\beta)x}$$

determines the sequence $\{\widehat{QQ}_n\}_{n=0}^{\infty}$. By $\alpha^2+\beta^2 = ab+2$ and $\alpha^2-\beta^2 = \sqrt{ab(ab+4)}$, we have the characteristic equation

$$\begin{aligned}
&\big(x+(\alpha+\beta)\big)\big(x-(\alpha+\beta)\big)\big(x+(\alpha-\beta)\big)\big(x-(\alpha-\beta)\big)\\
&= x^4 - 2(\alpha^2+\beta^2)x^2 + (\alpha^2-\beta^2)^2\\
&= x^4 - 2(ab+2)x^2 + (ab)(ab+4)\,.
\end{aligned}$$

Thus, $\widehat{QQ}_n$ satisfies the recurrence relation

$$\widehat{QQ}_n = 2(ab+2)\widehat{QQ}_{n-2} - (ab)(ab+4)\widehat{QQ}_{n-4} \quad (n \geq 4)\,. \tag{2.6}$$

Now, we get that

$$\widehat{QQ}_0 = r_1r_3 + r_2r_4 + r_1r_4 + r_2r_3 = -\frac{a}{b(ab+4)}\,,$$

$$\widehat{QQ}_1 = r_1r_3(\alpha+\beta) - r_2r_4(\alpha+\beta) + r_1r_4(\alpha-\beta) - r_2r_3(\alpha-\beta) = -\frac{a}{ab+4}\,,$$

$$\widehat{QQ}_2 = r_1r_3(\alpha+\beta)^2 + r_2r_4(\alpha+\beta)^2 + r_1r_4(\alpha-\beta)^2 + r_2r_3(\alpha-\beta)^2 = -\frac{2(a-b)+a^2b}{b(ab+4)}\,,$$

$$\widehat{QQ}_3 = r_1r_3(\alpha+\beta)^3 - r_2r_4(\alpha+\beta)^3 + r_1r_4(\alpha-\beta)^3 - r_2r_3(\alpha-\beta)^3 = -\frac{a^2b}{ab+4}\,.$$

Using the recurrence relation (2.6), by induction, we have for $n \ge 0$

$$\widehat{QQ}_{2n} = \frac{(b-a)(ab+4)^n - (a+b)(ab)^n}{2b(ab+4)}\,,$$

$$\widehat{QQ}_{2n+1} = -\frac{a^{n+1}b^n}{ab+4}\,.$$

Putting $QQ_n = -b(ab+4)\widehat{QQ}_n$, we get the desired result. □

Lemma 2.4.

$$r_1r_2 + r_3r_4 = \frac{a-b}{2b(ab+4)}\,.$$

Proof. Since

$$r_1r_2 = r_3r_4 = \frac{a}{4b(ab+4)} - \frac{1}{4(ab+4)} = \frac{a-b}{4b(ab+4)}\,,$$

we get the result. □

Theorem 2.5. *For $n \ge 1$, we have*

$$\sum_{k=0}^{n}\binom{n}{k} q_k q_{n-k} = \frac{Q_n - 2QQ_n}{b(ab+4)}\,. \tag{2.7}$$

Proof. Since $(\hat{a}+\hat{b}+\hat{c}+\hat{d})^2 = (\hat{a}^2+\hat{b}^2+\hat{c}^2+\hat{d}^2) + 2(\hat{a}\hat{b}+\hat{a}\hat{c}+\hat{a}\hat{d}+\hat{b}\hat{c}+\hat{b}\hat{d}+\hat{c}\hat{d})$, by Lemmas 2.1, 2.3 and 2.4 we have

$$\begin{aligned}
&(r_1e^{\alpha x} + r_2e^{-\alpha x} + r_3e^{\beta x} + r_4e^{-\beta x})^2 \\
&= (r_1^2e^{2\alpha x} + r_2^2e^{-2\alpha x} + r_3^2e^{2\beta x} + r_4^2e^{-2\beta x}) \\
&\quad + 2(r_1r_3e^{(\alpha+\beta)x} + r_2r_4e^{-(\alpha+\beta)x} + r_1r_4e^{(\alpha-\beta)x} + r_2r_3e^{-(\alpha-\beta)x}) \\
&\quad + 2(r_1r_2 + r_3r_4) \\
&= \frac{1}{b(ab+4)}\sum_{n=0}^{\infty} Q_n\frac{x^n}{n!} - \frac{2}{b(ab+4)}\sum_{n=0}^{\infty} QQ_n\frac{x^n}{n!} + \frac{a-b}{b(ab+4)} \\
&= \frac{1}{b(ab+4)}\left(\sum_{n=0}^{\infty}(Q_n - 2QQ_n)\frac{x^n}{n!} + (a-b)\right).
\end{aligned}$$

Since

$$f(x)^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} q_k q_{n-k}\frac{x^n}{n!}\,,$$

by comparing the coefficients on both sides, we get the desired result. □

Examples. For $n = 1, 2, 3, 4$ in Theorem 2.5, both sides of (2.7) equal to 0, 2, $6a$, $6a^2 + 8ab + 8$, respectively.

3. Higher-order identities

Next, we shall consider the convolution identities for three bi-periodic Fibonacci numbers

$$\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} q_{k_1}q_{k_2}q_{k_3}\,.$$

We use the following symmetric formula.

Lemma 3.1 ([17]). *The following equality holds:*

$$\begin{aligned}&(\hat a+\hat b+\hat c+\hat d)^3\\ &= A(\hat a^3+\hat b^3+\hat c^3+\hat d^3)+B(\hat a\hat b\hat c+\hat a\hat b\hat d+\hat a\hat c\hat d+\hat b\hat c\hat d)\\ &\quad + C(\hat a^2+\hat b^2+\hat c^2+\hat d^2)(\hat a+\hat b+\hat c+\hat d)\\ &\quad + D(\hat a\hat b+\hat a\hat c+\hat a\hat d+\hat b\hat c+\hat b\hat d+\hat c\hat d)(\hat a+\hat b+\hat c+\hat d),\end{aligned}$$

where $A=D-2$, $B=-3D+6$ and $C=-D+3$.

Lemma 3.2.

$$r_1^3e^{3\alpha x}+r_2^3e^{-3\alpha x}+r_3^3e^{3\beta x}+r_4^3e^{-3\beta x}=\frac{1}{4b(ab+4)}\sum_{n=0}^{\infty}P_n\frac{x^n}{n!}\,,$$

where the numbers P_n satisfy for $n\ge 1$

$$\begin{aligned}P_{2n}&=\frac{3a(a+3b)}{3a+b}P_{2n-1}+9P_{2n-2}\,,\\ P_{2n+1}&=\frac{3b(3a+b)}{a+3b}P_{2n}+9P_{2n-1}\end{aligned}$$

with $P_0=0$ and $P_1=3(3a+b)$.

Remark 3.3. We have explicit expressions: for $n\ge 0$

$$\begin{aligned}P_{2n}&=3^{2n}a(a+3b)\sum_{k=0}^{n-1}\binom{2n-k-1}{k}(ab)^{n-k-1}=3^{2n}(a+3b)q_{2n}\,,\\ P_{2n+1}&=3^{2n+1}(3a+b)\sum_{k=0}^{n}\binom{2n-k}{k}(ab)^{n-k}=3^{2n+1}(3a+b)q_{2n+1}\,.\end{aligned}$$

Proof of Lemma 3.2. Assume that the exponential generating function

$$r_1^3e^{3\alpha x}+r_2^3e^{-3\alpha x}+r_3^3e^{3\beta x}+r_4^3e^{-3\beta x}$$

determines the sequence $\{\widehat{P}_n\}_{n=0}^{\infty}$. By $\alpha\beta=1$ and $\alpha^2+\beta^2=ab+2$, we have the characteristic equation

$$\begin{aligned}&(x+3\alpha)(x-3\alpha)(x+3\beta)(x-3\beta)\\ &=x^4-9(\alpha^2+\beta^2)x^2+81\alpha^2\beta^2\\ &=x^4-9(ab+2)x^2+81\,.\end{aligned}$$

Thus, $\widehat{P}_n$ satisfies the recurrence relation

$$\widehat{P}_n=9(ab+2)\widehat{P}_{n-2}-81\widehat{P}_{n-4}\quad (n\ge 4)\,. \tag{3.1}$$

Now, we get that

$$\begin{aligned}
\widehat{P}_0 &= r_1^3 + r_2^3 + r_3^3 + r_4^3 = 0\,,\\
\widehat{P}_1 &= 3r_1^3\alpha - 3r_2^3\alpha + 3r_3^3\beta - 3r_4^3\beta = \frac{3(3a+b)}{4b(ab+4)}\,,\\
\widehat{P}_2 &= 9r_1^3\alpha^2 + 9r_2^3\alpha^2 + 9r_3^3\beta^2 + 9r_4^3\beta^2 = \frac{9a(a+3b)}{4b(ab+4)}\,,\\
\widehat{P}_3 &= 27r_1^3\alpha^3 - 27r_2^3\alpha^3 + 27r_3^3\beta^3 - 27r_4^3\beta^3 = \frac{27(3a+b)(ab+1)}{4b(ab+4)}\,.
\end{aligned}$$

Using the recurrence relation (3.1), by induction, we have the recurrence relations: for $n \ge 1$

$$\begin{aligned}
\widehat{P}_{2n} &= \frac{3a(a+3b)}{3a+b}\widehat{P}_{2n-1} + 9\widehat{P}_{2n-2}\,,\\
\widehat{P}_{2n+1} &= \frac{3b(3a+b)}{a+3b}\widehat{P}_{2n} + 9\widehat{P}_{2n-1}\,.
\end{aligned}$$

Putting $P_n = 4b(ab+4)\widehat{P}_n$, we get the desired result. □

Lemma 3.4.

$$r_1r_3r_4e^{\alpha x} + r_2r_3r_4e^{-\alpha x} + r_1r_2r_3e^{\beta x} + r_1r_2r_4e^{-\beta x} = \frac{a-b}{4b(ab+4)}\sum_{n=0}^{\infty} q_n\frac{x^n}{n!}\,,$$

Proof. Since

$$r_1r_2 = r_3r_4 = \frac{a-b}{4b(ab+4)}\,,$$

we get the result. □

Theorem 3.5. *For $n \ge 1$, we have*

$$\begin{aligned}
&\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} q_{k_1}q_{k_2}q_{k_3}\\
&= \frac{1}{4b(ab+4)}\Bigg(A\cdot P_n + B(a-b)q_n + 4C\sum_{k=0}^{n}\binom{n}{k}Q_kq_{n-k}\\
&\qquad -4D\sum_{k=0}^{n}\binom{n}{k}QQ_kq_{n-k} + 2(a-b)Dq_n\Bigg)\,, \qquad (3.2)
\end{aligned}$$

where the numbers A, B, C and D satisfy the condition in Lemma 3.1.

Remark 3.6. It is clear that this value is 0 when $n = 0, 1, 2$. Assume that $n \ge 3$. When $D = 2$, by $A = B = 0$ and $C = 1$, we have a simpler form:

$$\begin{aligned}
&\sum_{\substack{k_1+k_2+k_3=n\\ k_1,k_2,k_3\ge 0}} \binom{n}{k_1,k_2,k_3} q_{k_1}q_{k_2}q_{k_3}\\
&= \frac{1}{b(ab+4)}\left(\sum_{k=0}^{n}\binom{n}{k}(Q_k - 2QQ_k)q_{n-k} + (a-b)q_n\right)\\
&= \frac{1}{b(ab+4)}\sum_{k=2}^{n-1}\binom{n}{k}(Q_k - 2QQ_k)q_{n-k}\,.
\end{aligned}$$

Notice that $Q_0 - 2QQ_0 = -(a-b)$, $Q_1 - 2QQ_1 = 0$ and $q_0 = 0$.

Proof of Theorem 3.5. By Lemma 3.1 together with Lemmas 2.1, 2.3, 2.4, 3.2, 3.4, we have

$$\begin{aligned}
&(r_1e^{\alpha x}+r_2e^{-\alpha x}+r_3e^{\beta x}+r_4e^{-\beta x})^3\\
&=A(r_1^3e^{3\alpha x}+r_2^3e^{-3\alpha x}+r_3^3e^{3\beta x}+r_4^3e^{-3\beta x})\\
&\quad+B(r_1r_2r_3e^{\beta x}+r_1r_2r_4e^{-\beta x}+r_1r_3r_4e^{\alpha x}+r_2r_3r_4e^{-\alpha x})\\
&\quad+C(r_1^2e^{2\alpha x}+r_2^2e^{-2\alpha x}+r_3^2e^{2\beta x}+r_4^2e^{-2\beta x})(r_1e^{\alpha x}+r_2e^{-\alpha x}+r_3e^{\beta x}+r_4e^{-\beta x})\\
&\quad+D(r_1r_2+r_1r_3e^{(\alpha+\beta)x}+r_1r_4e^{(\alpha-\beta)x}+r_2r_3e^{-(\alpha-\beta)x}+r_2r_4e^{-(\alpha+\beta)x}+r_3r_4)\\
&\qquad\times(r_1e^{\alpha x}+r_2e^{-\alpha x}+r_3e^{\beta x}+r_4e^{-\beta x})\\
&=A\frac{1}{4b(ab+4)}\sum_{n=0}^{\infty}P_n\frac{x^n}{n!}+B\frac{a-b}{4b(ab+4)}\sum_{n=0}^{\infty}q_n\frac{x^n}{n!}\\
&\quad+C\left(\frac{1}{b(ab+4)}\sum_{n=0}^{\infty}Q_n\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}q_n\frac{x^n}{n!}\right)\\
&\quad+D\left(-\frac{1}{b(ab+4)}\sum_{n=0}^{\infty}QQ_n\frac{x^n}{n!}+\frac{a-b}{2b(ab+4)}\right)\left(\sum_{n=0}^{\infty}q_n\frac{x^n}{n!}\right)\\
&=\frac{1}{4b(ab+4)}\sum_{n=0}^{\infty}\left(A\cdot P_n+B(a-b)q_n+4C\sum_{k=0}^{n}\binom{n}{k}Q_kq_{n-k}\right.\\
&\qquad\qquad\left.-4D\sum_{k=0}^{n}\binom{n}{k}QQ_kq_{n-k}+2(a-b)Dq_n\right)\frac{x^n}{n!}\,.
\end{aligned}$$

On the other hand,

$$\left(\sum_{n=0}^{\infty}q_n\frac{x^n}{n!}\right)^3=\sum_{n=0}^{\infty}\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\ge 0}}\binom{n}{k_1,k_2,k_3}q_{k_1}q_{k_2}q_{k_3}\frac{x^n}{n!}\,.$$

Comparing the coefficients on both sides, we get the desired result. □

Next, we shall consider the convolution identities for four bi-periodic Fibonacci numbers

$$\sum_{\substack{k_1+k_2+k_3+k_4=n\\k_1,k_2,k_3,k_4\ge 0}}\binom{n}{k_1,k_2,k_3,k_4}q_{k_1}q_{k_2}q_{k_3}q_{k_4}\,.$$

We need the following symmetric formula.

Lemma 3.7 ([17])**.** *The following equality holds:*

$$\begin{aligned}
&(\hat a+\hat b+\hat c+\hat d)^4\\
&=A(\hat a^4+\hat b^4+\hat c^4+\hat d^4)+B\hat a\hat b\hat c\hat d+C(\hat a^3+\hat b^3+\hat c^3+\hat d^3)(\hat a+\hat b+\hat c+\hat d)\\
&\quad+D(\hat a^2+\hat b^2+\hat c^2+\hat d^2)^2+E(\hat a^2+\hat b^2+\hat c^2+\hat d^2)(\hat a\hat b+\hat a\hat c+\hat a\hat d+\hat b\hat c+\hat b\hat d+\hat c\hat d)\\
&\quad+F(\hat a\hat b+\hat a\hat c+\hat a\hat d+\hat b\hat c+\hat b\hat d+\hat c\hat d)^2+G(\hat a^2+\hat b^2+\hat c^2+\hat d^2)(\hat a+\hat b+\hat c+\hat d)^2\\
&\quad+H(\hat a\hat b+\hat a\hat c+\hat a\hat d+\hat b\hat c+\hat b\hat d+\hat c\hat d)(\hat a+\hat b+\hat c+\hat d)^2\\
&\quad+I(\hat a\hat b\hat c(\hat a+\hat b+\hat c)+\hat a\hat b\hat d(\hat a+\hat b+\hat d)+\hat b\hat c\hat d(\hat b+\hat c+\hat d)+\hat a\hat c\hat d(\hat a+\hat c+\hat d))\\
&\quad+J(\hat a\hat b\hat c+\hat a\hat b\hat d+\hat b\hat c\hat d+\hat a\hat c\hat d)(\hat a+\hat b+\hat c+\hat d)\,,
\end{aligned}$$

where $A=-D+E+G+H-3$, $B=12D+12G-4J-12$, $C=-E-2G-H+4$, $F=-2D-2G-2H+6$ *and* $I=4D-E+2G-H-J$.

Lemma 3.8.

$$r_1^4e^{4\alpha x}+r_2^4e^{-4\alpha x}+r_3^4e^{4\beta x}+r_4^4e^{-4\beta x}=\frac{1}{4b^2(ab+4)^2}\sum_{n=0}^\infty R_n\frac{x^n}{n!}\,,$$

where the numbers R_n *satisfy for* $n\ge 1$

$$R_{2n}=\frac{a^2+6ab+b^2}{a+b}R_{2n-1}+2^4R_{2n-2}\,,$$
$$R_{2n+1}=\frac{2^4ab(a+b)}{a^2+6ab+b^2}R_{2n}+2^4R_{2n-1}$$

with $R_0=a^2+6ab+b^2$ *and* $R_1=8ab(a+b)$.

Remark 3.9. We have explicit expressions: for $n\ge 0$

$$R_{2n}=2^{4n-1}(a^2+6ab+b^2)\sum_{k=0}^n\frac{2n}{2n-k}\binom{2n-k}{k}(ab)^{n-k}\,,$$
$$P_{2n+1}=2^{4n+3}ab(a+b)\sum_{k=0}^n\frac{2n+1}{2n-k+1}\binom{2n-k+1}{k}(ab)^{n-k}\,.$$

Theorem 3.10. *For* $n\ge 1$

$$\begin{aligned}
&\sum_{\substack{k_1+k_2+k_3+k_4=n\\k_1,k_2,k_3,k_4\ge 0}}\binom{n}{k_1,k_2,k_3,k_4}q_{k_1}q_{k_2}q_{k_3}q_{k_4}\\
&=\frac{A}{4b^2(ab+4)^2}R_n+\frac{C}{4b(ab+4)}\sum_{k=0}^n\binom{n}{k}P_kq_{n-k}+\frac{D}{b^2(ab+4)^2}\sum_{k=0}^n\binom{n}{k}Q_kQ_{n-k}\\
&\quad+E\left(\frac{1}{b^2(ab+4)^2}\sum_{k=0}^n\binom{n}{k}Q_kQ_{n-k}+\frac{a-b}{2b^2(ab+4)^2}Q_n\right)\\
&\quad+F\left(\frac{1}{b^2(ab+4)^2}\sum_{k=0}^n\binom{n}{k}QQ_kQQ_{n-k}+\frac{a-b}{b^2(ab+4)^2}QQ_n\right)\\
&\quad+\frac{G}{b(ab+4)}\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\ge 0}}\binom{n}{k_1,k_2,k_3}Q_{k_1}q_{k_2}q_{k_3}\\
&\quad+H\left(\frac{1}{b(ab+4)}\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\ge 0}}\binom{n}{k_1,k_2,k_3}QQ_{k_1}q_{k_2}q_{k_3}+\frac{a-b}{2b(ab+4)}\binom{n}{k}Q_kQ_{n-k}\right)\\
&\quad+I\frac{a-b}{4b^2(ab+4)^2}(Q_n-QQ_n)+J\left(\frac{1}{b(ab+4)}\sum_{k=0}^nQQ_kq_{n-k}+\frac{a-b}{2b(ab+4)}q_n\right)\,,
\end{aligned}$$

where the numbers A, C, D, E, F, G, H, I *and* J *are given in Lemma 3.7.*

Remark 3.11. The above form becomes much simpler for some specific values of the numbers A to J. For example, when $A=C=G=H=I=0$, by $B=J=0$, $D=1$ and $E=F=4$, we have

$$\begin{aligned}
&\sum_{\substack{k_1+k_2+k_3+k_4=n\\k_1,k_2,k_3,k_4\ge 0}}\binom{n}{k_1,k_2,k_3,k_4}q_{k_1}q_{k_2}q_{k_3}q_{k_4}\\
&=\frac{1}{b^2(ab+4)^2}\left(2(a-b)(Q_n+2QQ_n)+\sum_{k=0}^n\binom{n}{k}(5Q_kQ_{n-k}+4QQ_kQQ_{n-k})\right)\,.
\end{aligned}$$

Proof of Theorem 3.10. We apply Lemma 3.7 as $\hat{a} = r_1 e^{\alpha x}$, $\hat{b} = r_2 e^{-\alpha x}$, $\hat{c} = r_3 e^{\beta x}$ and $\hat{d} = r_4 e^{-\beta x}$. By Lemma 3.8, we have

$$A(\hat{a}^4 + \hat{b}^4 + \hat{c}^4 + \hat{d}^4) = A\frac{1}{4b^2(ab+4)^2}\sum_{n=0}^{\infty} R_n \frac{x^n}{n!}.$$

By $r_1 r_2 = r_3 r_4 = B(a-b)/4b(ab+4)$,

$$B\hat{a}\hat{b}\hat{c}\hat{d} = \frac{(a-b)^2}{16b^2(ab+4)^2}\,.$$

By Lemma 3.2,

$$\begin{aligned} &C(\hat{a}^3 + \hat{b}^3 + \hat{c}^3 + \hat{d}^3)(\hat{a} + \hat{b} + \hat{c} + \hat{d}) \\ &= C\left(\frac{1}{4b(ab+4)}\sum_{n=0}^{\infty} P_n \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} q_n \frac{x^n}{n!}\right) \\ &= C\frac{1}{4b(ab+4)}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} P_k q_{n-k}\frac{x^n}{n!}\,. \end{aligned}$$

By Lemma 2.1

$$\begin{aligned} &D(\hat{a}^2 + \hat{b}^2 + \hat{c}^2 + \hat{d}^2)^2 \\ &= D\left(\frac{1}{b(ab+4)}\sum_{n=0}^{\infty} Q_n \frac{x^n}{n!}\right)^2 \\ &= D\frac{1}{b^2(ab+4)^2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} Q_k Q_{n-k}\frac{x^n}{n!}\,. \end{aligned}$$

By Lemma 2.3 and Lemma 2.4,

$$\begin{aligned} &E(\hat{a}^2 + \hat{b}^2 + \hat{c}^2 + \hat{d}^2)(\hat{a}\hat{b} + \hat{a}\hat{c} + \hat{a}\hat{d} + \hat{b}\hat{c} + \hat{b}\hat{d} + \hat{c}\hat{d}) \\ &= E\left(\frac{1}{b(ab+4)}\sum_{n=0}^{\infty} Q_n \frac{x^n}{n!}\right)\left(\frac{1}{b(ab+4)}\sum_{n=0}^{\infty} QQ_n \frac{x^n}{n!} + \frac{a-b}{2b(ab+4)}\right) \\ &= E\left(\frac{1}{b^2(ab+4)^2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} Q_k Q_{n-k}\frac{x^n}{n!} + \frac{a-b}{2b^2(ab+4)^2}\sum_{n=0}^{\infty} Q_n \frac{x^n}{n!}\right)\,. \end{aligned}$$

By Lemma 2.3 and Lemma 2.4,

$$\begin{aligned} &F(\hat{a}\hat{b} + \hat{a}\hat{c} + \hat{a}\hat{d} + \hat{b}\hat{c} + \hat{b}\hat{d} + \hat{c}\hat{d})^2 \\ &= F\left(\frac{1}{b(ab+4)}\sum_{n=0}^{\infty} QQ_n \frac{x^n}{n!} + \frac{a-b}{2b(ab+4)}\right)^2 \\ &= F\left(\frac{1}{b^2(ab+4)^2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} QQ_k QQ_{n-k}\frac{x^n}{n!} + \frac{a-b}{b^2(ab+4)^2}\sum_{n=0}^{\infty} QQ_n \frac{x^n}{n!} + \frac{(a-b)^2}{4b^2(ab+4)^2}\right)\,. \end{aligned}$$

By Lemma 2.1,

$$\begin{aligned} &G(\hat{a}^2 + \hat{b}^2 + \hat{c}^2 + \hat{d}^2)(\hat{a} + \hat{b} + \hat{c} + \hat{d})^2 \\ &= G\left(\frac{1}{b(ab+4)}\sum_{n=0}^{\infty} Q_n \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} q_n \frac{x^n}{n!}\right)^2 \\ &= G\frac{1}{b(ab+4)}\sum_{n=0}^{\infty}\sum_{\substack{k_1+k_2+k_3=n \\ k_1,k_2,k_3\ge 0}}\binom{n}{k_1,k_2,k_3} Q_{k_1} q_{k_2} q_{k_3}\frac{x^n}{n!}\,. \end{aligned}$$

By Lemma 2.3 and Lemma 2.4,

$$\begin{aligned}
&H(\hat a\hat b+\hat a\hat c+\hat a\hat d+\hat b\hat c+\hat b\hat d+\hat c\hat d)(\hat a+\hat b+\hat c+\hat d)^2\\
&=H\left(\frac{1}{b(ab+4)}\sum_{n=0}^\infty QQ_n\frac{x^n}{n!}+\frac{a-b}{2b(ab+4)}\right)\left(\sum_{n=0}^\infty q_n\frac{x^n}{n!}\right)^2\\
&=H\left(\frac{1}{b(ab+4)}\sum_{n=0}^\infty\sum_{\substack{k_1+k_2+k_3=n\\k_1,k_2,k_3\ge0}}\binom{n}{k_1,k_2,k_3}QQ_{k_1}q_{k_2}q_{k_3}\frac{x^n}{n!}\right.\\
&\qquad\left.+\frac{a-b}{2b(ab+4)}\sum_{n=0}^\infty\sum_{k=0}^n\binom{n}{k}Q_kQ_{n-k}\frac{x^n}{n!}\right).
\end{aligned}$$

By Lemma 2.1 and Lemma 2.3 with $r_1r_2=r_3r_4=(a-b)/4b(ab+4)$,

$$\begin{aligned}
&I(\hat a\hat b\hat c(\hat a+\hat b+\hat c)+\hat a\hat b\hat d(\hat a+\hat b+\hat d)+\hat b\hat c\hat d(\hat b+\hat c+\hat d)+\hat a\hat c\hat d(\hat a+\hat c+\hat d))\\
&=I\frac{a-b}{4b(ab+4)}(r_1r_3e^{(\alpha+\beta)x}+r_2r_4e^{-(\alpha+\beta)x}+r_1r_4e^{(\alpha-\beta)x}+r_2r_3e^{-(\alpha-\beta)x})\\
&\quad+I(r_1^2e^{2\alpha x}+r_2^2e^{-2\alpha x}+r_3^2e^{2\beta x}+r_4^2e^{-2\beta x})\\
&=-I\frac{a-b}{4b(ab+4)}\frac{1}{b(ab+4)}\sum_{n=0}^\infty QQ_n\frac{x^n}{n!}+I\frac{a-b}{4b(ab+4)}\frac{1}{b(ab+4)}\sum_{n=0}^\infty Q_n\frac{x^n}{n!}\\
&=I\frac{a-b}{4b^2(ab+4)^2}\sum_{n=0}^\infty(Q_n-QQ_n)\frac{x^n}{n!}.
\end{aligned}$$

By Lemma 2.3 and Lemma 2.4,

$$\begin{aligned}
&J(\hat a\hat b\hat c+\hat a\hat b\hat d+\hat b\hat c\hat d+\hat a\hat c\hat d)(\hat a+\hat b+\hat c+\hat d)\\
&=J\left(\frac{1}{b(ab+4)}\sum_{n=0}^\infty QQ_n\frac{x^n}{n!}+\frac{a-b}{2b(ab+4)}\right)\left(\sum_{n=0}^\infty q_n\frac{x^n}{n!}\right)\\
&=J\left(\frac{1}{b(ab+4)}\sum_{n=0}^\infty\sum_{k=0}^n QQ_kq_{n-k}\frac{x^n}{n!}+\frac{a-b}{2b(ab+4)}\sum_{n=0}^\infty q_n\frac{x^n}{n!}\right).
\end{aligned}$$

We combine all the relations to get the main result. □

4. Final remarks

One can continue to get the convolution identities of five and more bi-periodic Fibonacci numbers. The situation becomes more complicated. Even in the case of five bi-periodic Fibonacci numbers, we need the symmetric formula for $(\hat a+\hat b+\hat c+\hat d)^5$ (see [12,17]).

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