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Research Article

On centrally-extended multiplicative (generalized)-(α, β)-derivations in semiprime rings

Najat Muthana[1,2], Zakeiah Alkhamisi[*2]

[1] *Department of Mathematics, Faculty of Science, University of Jeddah, Jeddah, Saudia Arabia*
[2] *Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudia Arabia*

Abstract

Let R be a ring with center Z and α, β and d mappings of R. A mapping F of R is called a centrally-extended multiplicative (generalized)-(α, β)-derivation associated with d if $F(xy) - F(x)\alpha(y) - \beta(x)d(y) \in Z$ for all $x, y \in R$. The objective of the present paper is to study the following conditions: (i) $F(xy) \pm \beta(x)G(y) \in Z$, (ii) $F(xy) \pm g(x)\alpha(y) \in Z$ and (iii) $F(xy) \pm g(y)\alpha(x) \in Z$ for all x, y in some appropriate subsets of R, where G is a multiplicative (generalized)-(α, β)-derivation of R associated with the map g on R.

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1. Introduction

Throughout this work R will be a ring with center Z. Recall that a ring R is said to be semiprime if $aRa = 0$ then $a = 0$. For $x, y \in R$, the commutator $xy - yx$ and the anti-commutator $xy + yx$ will be written as $[x, y]$ and $(x \circ y)$ respectively. For given $x, y \in R$, put $[x, y]_0 = x$, then $[x, y]_k = [[x, y]_{k-1}, y]$ for integer $k \geq 1$. Let S be a nonempty subset of R and α a mapping of R. If $\alpha(xy) = \alpha(x)\alpha(y)$ or $\alpha(xy) = \alpha(y)\alpha(x)$ for all $x, y \in S$, then we say that α acts as homomorphism or anti-homomorphism on S, respectively. A map f: $S \to R$ is said to be α-commuting on S in case $[\alpha(x), f(x)] = 0$ satisfies for all $x \in S$. We will make some extensive use of the basic commutator identities $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = [x, z]y + x[y, z]$.

Let α and β be mappings of R. A map D on R is called an (α, β)-derivation of R if it is additive and satisfying $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$, for all $x, y \in R$. Let D be an (α, β)-derivation of R, a map F on R is called a generalized (α, β)-derivation if it is additive and satisfying $F(xy) = F(x)\alpha(y) + \beta(x)D(y)$ for all $x, y \in R$.

Recently, Bell and Daif [2] introduced the notion of centrally-extended derivations (CE-derivation) on rings. A CE-derivation D of R is a mapping of R such that $D(x+y)-D(x)-D(y) \in Z$ and $D(xy)-D(x)y-xD(y) \in Z$, for all x, y in R. Tammam et al. [5] generalized this notion to the concepts CE-(α, β)-derivation and CE-generalized (α, β)-derivation.

*Corresponding Author.
Email addresses: Nmuthana@kau.edu.sa (N. Muthana), mazasa1@yahoo.com (Z. Alkhamisi)
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A CE-(α,β)-derivation D of R is a mapping of R such that $D(x+y)-D(x)-D(y)\in Z$ and $D(xy)-D(x)\alpha(y)-\beta(x)D(y)\in Z$ hold for all $x,y\in R$. Let D be a CE-(α,β)-derivation of R, a map F on R is called a CE-generalized (α,β)-derivation if $F(x+y)-F(x)-F(y)\in Z$ and $F(xy)-F(x)\alpha(y)-\beta(x)D(y)\in Z$ are fulfilled for all $x,y\in R$.

A map F on R is said to be a multiplicative (generalized)-(α,β)-derivation (M-(generalized)-(α,β)-derivation) associated with a map d on R if $F(xy)=F(x)\alpha(y)+\beta(x)d(y)$ holds for all $x,y\in R$. According to [3], an M-(generalized)-(I,I)-derivation is simply called an M-(generalized)-derivation, where I is the identity map on R.

We begin by the following definitions.

Definition 1.1. Let R be a ring and α be a mapping of R. A map T on R is called a centrally-extended multiplicative left α-centralizer (CEM-left α-centralizer) if $T(xy)-T(x)\alpha(y)\in Z$ holds for all $x,y\in R$.

Definition 1.2. Let R be a ring and α,β be mappings of R. A map D on R is called a CEM-(α,β)-derivation if $D(xy)-D(x)\alpha(y)-\beta(x)D(y)\in Z$ holds for all $x,y\in R$.

Definition 1.3. Let R be a ring and α, β and d be mappings of R. A map F on R is called a CEM-(generalized)-(α,β)-derivation associated with d if $F(xy)-F(x)\alpha(y)-\beta(x)d(y)\in Z$ holds for all $x,y\in R$.

Hence the concept of CEM-(generalized)-(α,β)-derivation covers both the concept of CEM-(α,β)-derivation and the concept of CEM-left α-centralizer. Moreover, every CE-generalized (α,β)-derivation is a CEM-(generalized)-(α,β)-derivation and every M-(generalized)-(α,β)-derivation is a CEM-(generalized)-(α,β)-derivation. Also, every generalized (α,β)-derivation is an M-(generalized)-(α,β)-derivation.

In this paper, our aim is to investigate certain identities involving CEM-(generalized)-(α,β)-derivations on some appropriate subsets of the ring R.

2. Preliminaries

We shall require throughout this paper to the following results.

Lemma 2.1. *Let R be a semiprime ring, U a left ideal of R and α, β mappings of R such that $\beta(U)\subseteq U$. If either $[xy\alpha(z),\beta(z)]=0$ or $x[y\alpha(z),\beta(z)]=0$ holds for all $x,y,z\in U$, then $U[\alpha(z),\beta(z)]=(0)$ for all $z\in U$.*

Proof. First assume that

$$[xy\alpha(z),\beta(z)]=0 \quad \text{for all } x,y,z\in U. \tag{2.1}$$

Substituting rx for x in (2.1), where $r\in R$, and then using (2.1), we obtain

$$[r,\beta(z)]xy\alpha(z)=0 \quad \text{for all } x,y,z\in U, r\in R. \tag{2.2}$$

Replacing x by $\alpha(z)sx$, where $s\in R$, we get $[r,\beta(z)]\alpha(z)sxy\alpha(z)=0$, which implies

$$[r,\beta(z)]\alpha(z)Rxy\alpha(z)=(0) \quad \text{for all } x,y,z\in U, r\in R. \tag{2.3}$$

Interchanging x and y then subtracting one from the other, we get

$$[r,\beta(z)]\alpha(z)R[x,y]\alpha(z)=(0) \quad \text{for all } x,y,z\in U, r\in R. \tag{2.4}$$

In particular,

$$[x,\beta(z)]\alpha(z)R[x,\beta(z)]\alpha(z)=(0) \quad \text{for all } x,z\in U. \tag{2.5}$$

The semiprimeness of R yields that

$$[x,\beta(z)]\alpha(z)=0 \quad \text{for all } x,z\in U. \tag{2.6}$$

Right multiplying (2.6) by $\beta(z)$, we get

$$[x,\beta(z)]\alpha(z)\beta(z)=0 \quad \text{for all } x,z\in U. \tag{2.7}$$

Replace x by $x\beta(z)$ in (2.6) to get

$$[x,\beta(z)]\beta(z)\alpha(z)=0 \quad \text{for all } x,z\in U. \tag{2.8}$$

Now (2.7) and (2.8) together imply that

$$[x,\beta(z)][\alpha(z),\beta(z)]=0 \quad \text{for all } x,z\in U. \tag{2.9}$$

Replacing x by $\alpha(z)x$ in the last expression, we obtain

$$[\alpha(z),\beta(z)]x[\alpha(z),\beta(z)]=0 \quad \text{for all } x,z\in U, \tag{2.10}$$

that is

$$U[\alpha(z),\beta(z)]RU[\alpha(z),\beta(z)]=(0) \quad \text{for all } z\in U. \tag{2.11}$$

Hence, since R is a semiprime ring,

$$U[\alpha(z),\beta(z)]=(0) \quad \text{for all } z\in U. \tag{2.12}$$

Now suppose that

$$x[y\alpha(z),\beta(z)]=0 \quad \text{for all } x,y,z\in U. \tag{2.13}$$

Replacing y with $\alpha(z)y$ in (2.13), we get

$$x[\alpha(z)y\alpha(z),\beta(z)]=0 \quad \text{for all } x,y,z\in U. \tag{2.14}$$

Now replacing y by $y\alpha(z)u$, where $u\in U$, we have

$$x[\alpha(z)y\alpha(z)u\alpha(z),\beta(z)]=0 \quad \text{for all } x,y,z,u\in U. \tag{2.15}$$

This implies

$$x[\alpha(z)y\alpha(z),\beta(z)]u\alpha(z)+x\alpha(z)y\alpha(z)[u\alpha(z),\beta(z)]=0 \quad \text{for all } x,y,z,u\in U. \tag{2.16}$$

Then using (2.14), we obtain

$$x\alpha(z)y\alpha(z)[u\alpha(z),\beta(z)]=0 \quad \text{for all } x,y,z,u\in U, \tag{2.17}$$

and this is equivalent to

$$x\alpha(z)y([\alpha(z)u\alpha(z),\beta(z)]-[\alpha(z),\beta(z)]u\alpha(z))=0 \quad \text{for all } x,y,z,u\in U. \tag{2.18}$$

Again, using (2.14), we get

$$x\alpha(z)y[\alpha(z),\beta(z)]u\alpha(z)=0 \quad \text{for all } x,y,z,u\in U. \tag{2.19}$$

Replacing y with $\beta(z)y$ in (2.19), we obtain

$$x\alpha(z)\beta(z)y[\alpha(z),\beta(z)]u\alpha(z)=0 \quad \text{for all } x,y,z,u\in U. \tag{2.20}$$

Now replace x by $x\beta(z)$ in (2.19) to get

$$x\beta(z)\alpha(z)y[\alpha(z),\beta(z)]u\alpha(z)=0 \quad \text{for all } x,y,z,u\in U. \tag{2.21}$$

Subtracting (2.21) from (2.20), we have

$$x[\alpha(z),\beta(z)]y[\alpha(z),\beta(z)]u\alpha(z)=0 \quad \text{for all } x,y,z,u\in U. \tag{2.22}$$

Right multiplying (2.22) by $\beta(z)$, we get

$$x[\alpha(z),\beta(z)]y[\alpha(z),\beta(z)]u\alpha(z)\beta(z)=0 \quad \text{for all } x,y,z,u\in U. \tag{2.23}$$

Replacing u by $u\beta(z)$ in (2.22), we obtain

$$x[\alpha(z),\beta(z)]y[\alpha(z),\beta(z)]u\beta(z)\alpha(z)=0 \quad \text{for all } x,y,z,u\in U. \tag{2.24}$$

Now (2.23) and (2,24) together imply that

$$x[\alpha(z),\beta(z)]y[\alpha(z),\beta(z)]u[\alpha(z),\beta(z)]=0 \quad \text{for all } x,y,z,u\in U, \tag{2.25}$$

that is

$$(U[\alpha(z),\beta(z)])^3=(0) \quad \text{for all } z\in U. \tag{2.26}$$

Since a semiprime ring contains no nonzero nilpotent left ideals, it follows that

$$U[\alpha(z),\beta(z)]=(0) \quad \text{for all } z\in U. \tag{2.27}$$

□

Lemma 2.2. *Let R be a semiprime ring and U a left ideal of R. If $[y[x,z]_2, z] = 0$ for all $x, y, z \in U$, then $U[U,U] = (0)$.*

Proof. By the hypothesis, we have

$$[y[x,z]_2, z] = 0 \quad \text{for all } x, y, z \in U. \tag{2.28}$$

Substituting y by xy in (2.28) and then using (2.28), we obtain

$$[x,z]y[x,z]_2 = 0 \quad \text{for all } x, y, z \in U, \tag{2.29}$$

that is

$$U[x,z]_2 RU[x,z]_2 = (0) \quad \text{for all } x, z \in U. \tag{2.30}$$

The semiprimeness of R forces that

$$U[x,z]_2 = (0) \quad \text{for all } x, z \in U. \tag{2.31}$$

Linearizing (2.31) with respect to z, we have

$$U([[x,u],v] + [[x,v],u]) = (0) \quad \text{for all } x, u, v \in U. \tag{2.32}$$

Replacing u with uv in (2.32), then using (2.31) and (2.32) to get

$$U[u,v][x,v] = (0) \quad \text{for all } x, u, v \in U. \tag{2.33}$$

Now substituting x by xu, we obtain

$$U[u,v]x[u,v] = (0) \quad \text{for all } x, u, v \in U, \tag{2.34}$$

that is

$$U[u,v]RU[u,v] = (0) \quad \text{for all } u, v \in U, \tag{2.35}$$

Hence, the semiprimeness of R yields that $U[U,U] = (0)$. □

Lemma 2.3 ([4], Theorem 2)**.** *Let R be a semiprime ring and U a nonzero left ideal of R. For integers $n, k \geq 1$, and some $a \in R$, if $[a, x^k]_n = 0$ for all $x \in U$, then $[a, U] = (0)$.*

3. The results

Theorem 3.1. *Let R be a semiprime ring, U a left ideal of R, α, β, d and g mappings of R, F a CEM-(generalized)-(α, β)-derivation of R associated with d and G an M-(generalized)-(α, β)-derivation of R associated with g, where $\alpha(U) \subseteq U$, $\beta(U) = U$ and β acts as homomorphism on U. If $F(xy) \pm \beta(x)G(y) \in Z$ for all $x, y \in U$, then $U[(d \pm g)(x), \alpha(x)] = (0)$ for all $x \in U$. Moreover, if U is an ideal of R, $d \pm g$ is an $\alpha-$commuting map on U.*

Proof. By the hypothesis, we have

$$F(xy) \pm \beta(x)G(y) \in Z \quad \text{for all } x, y \in U. \tag{3.1}$$

Replacing y with yz in (3.1), where $z \in U$, and then we get

$$\begin{aligned} F(xyz) \pm \beta(x)G(yz) &= F(xy)\alpha(z) + \beta(xy)d(z) + a \pm \beta(x)G(y)\alpha(z) \\ &\quad \pm \beta(x)\beta(y)g(z) \\ &= (F(xy) \pm \beta(x)G(y))\alpha(z) + \beta(x)\beta(y)(d \pm g)(z) + a, \end{aligned} \tag{3.2}$$

where $a \in Z$. Applying (3.1) and (3.2) yields

$$[\beta(x)\beta(y)(d \pm g)(z), \alpha(z)] = 0 \quad \text{for all } x, y, z \in U. \tag{3.3}$$

Since $\beta(U) = U$, we get

$$[xy(d \pm g)(z), \alpha(z)] = 0 \quad \text{for all } x, y, z \in U. \tag{3.4}$$

Hence, by Lemma 2.1, we obtain

$$U[(d \pm g)(z), \alpha(z)] = (0) \quad \text{for all } z \in U. \tag{3.5}$$

Moreover, if U is an ideal of R, the semiprimeness of U yields that

$$[(d \pm g)(z), \alpha(z)] = 0 \quad \text{for all } z \in U. \tag{3.6}$$

□

Theorem 3.2. *Let R be a semiprime ring, U a left ideal of R, α, β, d and g mappings of R and F a CEM-(generalized)-(α, β)-derivation of R associated with d, where $\alpha(U) \subseteq U$, $\beta(U) = U$, α acts as homomorphism on U, and β acts as homomorphism or anti-homomorphism on U. If $F(xy) \pm g(x)\alpha(y) \in Z$ for all $x, y \in U$, then $U[d(x), \alpha(x)] = (0)$ for all $x \in U$. Moreover, if U is an ideal of R, d is an $\alpha-$commuting map on U.*

Proof. Assume that

$$F(xy) \pm g(x)\alpha(y) \in Z \quad \text{for all } x, y \in U. \tag{3.7}$$

Now we replace y with yz in (3.7), where $z \in U$, then we get

$$\begin{aligned} F(xyz) \pm g(x)\alpha(yz) &= F(xy)\alpha(z) + \beta(xy)d(z) + a \pm g(x)\alpha(y)\alpha(z) \\ &= (F(xy) \pm g(x)\alpha(y))\alpha(z) + \beta(xy)d(z) + a, \end{aligned} \tag{3.8}$$

where $a \in Z$. Applying (3.7) and (3.8) yields

$$[\beta(xy)d(z), \alpha(z)] = 0 \quad \text{for all } x, y, z \in U. \tag{3.9}$$

Since $\beta(U) = U$, we get

$$[xyd(z), \alpha(z)] = 0 \quad \text{for all } x, y, z \in U. \tag{3.10}$$

Henceforth, by Lemma 2.1, we get the required result. □

If we put $\alpha = \beta = g = I$ in Theorem 3.2, we get

Corollary 3.3 ([3], Theorem 2.9)**.** *Let R be a semiprime ring, U a nonzero left ideal of R, d a mapping of R and F an M-(generalized)-derivation of R associated with d. If $F(xy) \pm xy \in Z$ for all $x, y \in U$, then $U[d(x), x] = (0)$ for all $x \in U$.*

Theorem 3.4. *Let R be a semiprime ring, U a nonzero left ideal of R, α, d and g mappings of R, and F a CEM-(generalized)-(α, α)-derivation of R associated with d, where $\alpha(U) = U$ and α acts as anti-homomorphism on U. If $F(xy) \pm g(y)\alpha(x) \in Z$ for all $x, y \in U$, then $U[d(x), \alpha(x)] = (0)$ for all $x \in U$.*

Proof. By the hypothesis, we have

$$F(xy) \pm g(y)\alpha(x) \in Z \quad \text{for all } x, y \in U. \tag{3.11}$$

Replacing y with yz in (3.11), where $z \in U$, we get

$$F(xyz) \pm g(yz)\alpha(x) = F(xy)\alpha(z) + \alpha(xy)d(z) + a \pm g(yz)\alpha(x), \tag{3.12}$$

where $a \in Z$. Applying (3.11) and (3.12), we get

$$[\alpha(xy)d(z), \alpha(z)] + [\pm g(yz)\alpha(x) \mp g(y)\alpha(x)\alpha(z), \alpha(z)] = 0 \quad \text{for all } x, y, z \in U. \tag{3.13}$$

Now substituting zx for x in (3.13), we have for all $x, y, z \in U$

$$[\alpha(zxy)d(z), \alpha(z)] + [\pm g(yz)\alpha(x) \mp g(y)\alpha(x)\alpha(z), \alpha(z)]\alpha(z) = 0. \tag{3.14}$$

Right multiplying (3.13) by $\alpha(z)$ and then subtracting it from (3.14), we get

$$[\alpha(zxy)d(z), \alpha(z)] - [\alpha(xy)d(z), \alpha(z)]\alpha(z) = 0 \quad \text{for all } x, y, z \in U. \tag{3.15}$$

Since $\alpha(U) = U$, we obtain

$$[yx[d(z), \alpha(z)], \alpha(z)] = 0 \quad \text{for all } x, y, z \in U. \tag{3.16}$$

Replacing y with $d(z)y$, in the above relation and then using (3.16), we get

$$[d(z), \alpha(z)]yx[d(z), \alpha(z)] = 0 \quad \text{for all } x, y, z \in U, \tag{3.17}$$

that is

$$yx[d(z), \alpha(z)]Ryx[d(z), \alpha(z)] = (0) \quad \text{for all } x, y, z \in U. \tag{3.18}$$

The semiprimeness of R yields that

$$yx[d(z), \alpha(z)] = 0 \quad \text{for all } x, y, z \in U, \tag{3.19}$$

Since U is a left ideal, $[d(z), \alpha(z)]rx \in U$ for all $x, z \in U, r \in R$. In equation (3.19), replace y by x and replace x by $[d(z), \alpha(z)]rx$ to get

$$x[d(z), \alpha(z)]rx[d(z), \alpha(z)] = 0 \quad \text{for all } x, z \in U, r \in R, \tag{3.20}$$

that is

$$x[d(z), \alpha(z)]Rx[d(z), \alpha(z)] = (0) \quad \text{for all } x, z \in U. \tag{3.21}$$

Therefore we have

$$U[d(z), \alpha(z)] = (0) \quad \text{for all } z \in U. \tag{3.22}$$

□

The following theorem is an extension and generalization to [3, Theorem 2.11].

Theorem 3.5. *Let R be a semiprime ring, U a nonzero left ideal of R, α, d and g mappings of R, and F a CEM-(generalized)-(α, α)-derivation of R associated with d, where $\alpha(U) = U$ and α acts as homomorphism on U. If $F(xy) \pm g(y)\alpha(x) \in Z$ for all $x, y \in U$, then $U[d(x), \alpha(x)] = (0)$ for all $x \in U$. Moreover, if $\alpha = g$ and α is homomorphism on U, then $U \subseteq Z$, $Ud(R) \subseteq Z$ and $UF(R) \subseteq Z$.*

Proof. Suppose that

$$F(xy) \pm g(y)\alpha(x) \in Z \quad \text{for all } x, y \in U. \tag{3.23}$$

In the above relation, replacing y with yz, where $z \in U$, we get

$$F(xyz) \pm g(yz)\alpha(x) = F(xy)\alpha(z) + \alpha(xy)d(z) + a \pm g(yz)\alpha(x), \tag{3.24}$$

where $a \in Z$. Applying (3.23) and (3.24), we get

$$[\alpha(xy)d(z), \alpha(z)] + [\pm g(yz)\alpha(x) \mp g(y)\alpha(x)\alpha(z), \alpha(z)] = 0 \quad \text{for all } x, y, z \in U. \tag{3.25}$$

Now substituting xz for x in (3.25), we have for all $x, y, z \in U$

$$[\alpha(xzy)d(z), \alpha(z)] + [\pm g(yz)\alpha(x) \mp g(y)\alpha(x)\alpha(z), \alpha(z)]\alpha(z) = 0. \tag{3.26}$$

Right multiplying (3.25) by $\alpha(z)$ and then subtracting it from (3.26), we get

$$[\alpha(xzy)d(z), \alpha(z)] - [\alpha(xy)d(z), \alpha(z)]\alpha(z) = 0 \quad \text{for all } x, y, z \in U. \tag{3.27}$$

Since $\alpha(U) = U$, we have

$$[x[yd(z), \alpha(z)], \alpha(z)] = 0 \quad \text{for all } x, y, z \in U. \tag{3.28}$$

Replacing x with $yd(z)x$, in the above relation and then using (3.28), we get

$$[yd(z), \alpha(z)]x[yd(z), \alpha(z)] = 0 \quad \text{for all } x, y, z \in U, \tag{3.29}$$

that is

$$x[yd(z), \alpha(z)]Rx[yd(z), \alpha(z)] = (0) \quad \text{for all } x, y, z \in U. \tag{3.30}$$

The semiprimeness of R yields that

$$x[yd(z), \alpha(z)] = 0 \quad \text{for all } x, y, z \in U. \tag{3.31}$$

Therefore, by Lemma 2.1, we have

$$U[d(z), \alpha(z)] = (0) \quad \text{for all } z \in U. \tag{3.32}$$

Now, assume that $\alpha = g$ and α is additive on U, then (3.25) becomes

$$[\alpha(xy)d(z), \alpha(z)] + [\pm\alpha(yz)\alpha(x) \mp \alpha(y)\alpha(x)\alpha(z), \alpha(z)] = 0 \quad \text{for all } x, y, z \in U. \tag{3.33}$$

Replacing y with yz in (3.33), we have for all $x, y, z \in U$

$$[\alpha(xyz)d(z), \alpha(z)] + [\pm\alpha(yz^2)\alpha(x) \mp \alpha(yz)\alpha(x)\alpha(z), \alpha(z)] = 0. \tag{3.34}$$

Right multiplying (3.33) by $\alpha(z)$ and then subtracting it from (3.34), we get

$$[\alpha(xy)[\alpha(z), d(z)], \alpha(z)] + [\pm\alpha(yz^2)\alpha(x) \mp \alpha(yz)\alpha(x)\alpha(z), \alpha(z)] \\ -[\pm\alpha(yz)\alpha(x) \mp \alpha(y)\alpha(x)\alpha(z), \alpha(z)]\alpha(z) = 0 \quad \text{for all } x, y, z \in U. \tag{3.35}$$

By using (3.32), we have

$$[\pm\alpha(yz^2)\alpha(x) \mp \alpha(yz)\alpha(x)\alpha(z), \alpha(z)] - [\pm\alpha(yz)\alpha(x)\mp \\ \alpha(y)\alpha(x)\alpha(z), \alpha(z)]\alpha(z) = 0 \quad \text{for all } x, y, z \in U. \tag{3.36}$$

Since α is epimorphism on U, then we have

$$[\pm yz^2x \mp yzxz, z] - [\pm yzx \mp yxz, z]z = 0 \quad \text{for all } x, y, z \in U. \tag{3.37}$$

That is

$$[yz[x, z], z] - [y[x, z], z]z = 0 \quad \text{for all } x, y, z \in U, \tag{3.38}$$

which is equivalent to

$$[y[x, z]z, z] - [yz[x, z], z] = 0 \quad \text{for all } x, y, z \in U, \tag{3.39}$$

that is

$$[y[x, z]_2, z] = 0 \quad \text{for all } x, y, z \in U. \tag{3.40}$$

Thus Lemma 2.2 get us $U[U, U] = (0)$. Replacing y with $[y, z]$ in (3.23), we have

$$F(x[y, z]) \pm \alpha([y, z])\alpha(x) \in Z \quad \text{for all } x, y, z \in U, \tag{3.41}$$

that is

$$F(0) \pm [\alpha(y), \alpha(z)]\alpha(x) \in Z \quad \text{for all } x, y, z \in U. \tag{3.42}$$

Now we replace x by $[x, z]$ in (3.23) to get

$$F([x, z]y) \pm \alpha(y)\alpha([x, z]) \in Z \quad \text{for all } x, y, z \in U. \tag{3.43}$$

Thus $F([x, z]y) \in Z$ for all $x, y, z \in U$. Then $F(0) \in Z$ and from (3.42) we have $[U, U]U \subseteq Z$. Therefore we get $[x, y]_2 \in Z$ for all $x, y \in U$. In particular, $[x, y]_3 = 0$ for all $x, y \in U$. Then by Lemma 2.3, we get $U \subseteq Z$. Thus (3.23) gives $F(xy) \in Z$ for all $x, y \in U$, and so we have

$$F(x)\alpha(y) + \alpha(x)d(y) \in Z \quad \text{for all } x, y \in U. \tag{3.44}$$

Replacing x with xz in the last expression, we get

$$F(xz)\alpha(y) + \alpha(xz)d(y) \in Z \quad \text{for all } x, y, z \in U, \tag{3.45}$$

which implies that $xzd(y) \in Z$ for all $x, y, z \in U$, and then, for $r \in R$, we have $xz[d(y), r] = 0$ for all $x, y, z \in U$, that is $xRz[d(y), r] = (0)$ for all $x, y, z \in U, r \in R$. In particular, $x[d(y), r]Rx[d(y), r] = (0)$ for all $x, y \in U, r \in R$. Since R is semiprime, we get $Ud(U) \subseteq Z$. Then (3.44) gives us $F(U)U \subseteq Z$, and so $F(xr)y \in Z$ for all $x, y \in U,\ r \in R$. Then we get $b, c \in Z$ such that

$$F(x)\alpha(r)y + \alpha(x)d(r)y + by \in Z \tag{3.46}$$

and

$$F(r)\alpha(x)y + \alpha(r)d(x)y + cy \in Z \tag{3.47}$$

for all $x, y \in U,\ r \in R$. Thus, we have $xyd(r) \in Z$ and $xyF(r) \in Z$ for all $x, y \in U,\ r \in R$. Hence, since R is semiprime, we obtain $Ud(R) \subseteq Z$ and $UF(R) \subseteq Z$. □

Corollary 3.6. *Let R be a semiprime ring, U a nonzero left ideal of R, α a mapping of R, and T a CEM-left α-centralizer of R, where α is an epimorphism of U. If $T(xy) \pm \alpha(yx) \in Z$ for all $x, y \in U$, then $U \subseteq Z$ and $UT(R) \subseteq Z$.*

Note that if F is a CEM-(generalized)-(α,β)-derivation of a ring R associated with a map d on R, where α and β are mappings of R such that α acts as homomorphism on R, then $F \pm \alpha$ is a CEM-(generalized)-(α,β)-derivation of R associated with d.

Theorem 3.7. *Let R be a semiprime ring, U a nonzero left ideal of R, α and d mappings of R, and F a CEM-(generalized)-(α,α)-derivation of R associated with d, where $\alpha(U) = U$ and α acts as homomorphism on R. If one of the following conditions:*

(1) $F(xy) \pm [\alpha(x), \alpha(y)] \in Z$
(2) $F(xy) \pm (\alpha(x) \circ \alpha(y)) \in Z$

is satisfied for all $x, y \in U$, then $U[d(x), \alpha(x)] = (0)$ for all $x \in U$. Moreover, if α is homomorphism on U, then $U \subseteq Z$, $Ud(R) \subseteq Z$ and $UF(R) \subseteq Z$.

Proof. Replacing F with $F \mp \alpha$, then by Theorem 3.5 we get the desired result. □

Corollary 3.8 ([1], Theorem 2.18 and Theorem 2.19)**.** *Let R be a semiprime ring, U a nonzero left ideal of R, d a mapping on R, and F an M-(generalized)-derivation of R associated with d. If one of the following conditions:*

(1) $F(xy) \pm [x, y] \in Z$
(2) $F(xy) \pm (x \circ y) \in Z$

is satisfied for all $x, y \in U$, then $U \subseteq Z$ and $F(xy) \in Z$ for all $x, y \in U$.

The following corollary is an immediate consequence of Theorems 3.5 and 3.7.

Corollary 3.9. *Let R be a semiprime ring, α epimorphism of R, d map on R and F a CEM-(generalized)-(α,α)-derivation of R associated with d. If one of the following conditions:*

(1) $F(xy) \pm \alpha(yx) \in Z$
(2) $F(xy) \pm \alpha([x, y]) \in Z$
(3) $F(xy) \pm \alpha((x \circ y)) \in Z$

is satisfied for all $x, y \in R$, then R is commutative.

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