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Research Article

Morgan-Voyce polynomial approach for ordinary linear delay integro-differential equations with variable delays and variable bounds

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Abstract

An effective matrix method to solve the ordinary linear integro-differential equations with variable coefficients and variable delays under initial conditions is offered in this article. Our method consists of determining the approximate solution of the matrix form of Morgan-Voyce and Taylor polynomials and their derivatives in the collocation points. Then, we reconstruct the problem as a system of equations and solve this linear system. Also, some examples are given to show the validity and the residual error analysis is investigated.

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Keywords. Morgan-Voyce polynomials, integro differential equations with variable delays, matrix-collocation method, residual error analysis

1. Introduction

Many different types of Volterra-integro, delay, and fractional differential equations play an important role in such areas population dynamics, ecology, control theory, and economy. In recent times, it is seen in the literature that many articles containing such equations have been studied [5,7,19,20]. A new spectral approach for systems of pantograph type Volterra integro-differential equations has been given by Ezz-Eldien and Doha [19,20]. In the study conducted by Ayati and Biazar, the rate of convergence of the Homotopy perturbation method for nonlinear differential equations in Banach space has been discussed [4]. In addition, the nonlinear model convergence of Dickson polynomials has been developed using the residual functions in Banach space to obtain the numerical solution of the linear integro-differential-difference equations with variable coefficients given under the mixed conditions by Kürkçü and Colleagues [26–28].

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In this paper, we consider the higher-order ordinary delay integro-differential equation with variable delays as [1,6,10,11,25,30,31,33,34,40,41].

$$\sum_{k=0}^{m} P_k(t)\, y^{(k)}(t) = g(t) + \int_{t-\tau(t)}^{t-\delta(t)} K(t,s)\, y(s)\, ds \tag{1.1}$$

under the initial conditions

$$\sum_{k=0}^{m-1} a_{ik} y^{(k)}(a) = \lambda_i,\ i = 0, 1, \ldots, m-1 \tag{1.2}$$

and the integral equation with variable delays[2,11–21,23,29]

$$y(t) = g(t) + \int_{t-\tau(t)}^{t-\delta(t)} K(t,s)\, y(s)\, ds. \tag{1.3}$$

Here $P_k(t), g(t), \tau(t), \delta(t)$, and $K(t,s)$ are the functions defined on the interval $a \leq t, s \leq b$; a_{ik} and λ_i are constants; $y(x)$ is an unknown function to be determined.

The delay integro-differential and integral equations in the forms (1.1) and (1.2) have become important in the mathematical modelling of biological and physical phenomena [10,14–16] such as infectious disease, population and epidemics growth, mathematical biology, control theory, material with large memory etc. The above problems are generally hard to solve even analytically and so, numerical methods may be required to obtain their approximate solutions. The interest in the numerical solutions of (1.1) and (1.3) has increased in the recent studies. For example, these equations have been studied by using Taylor collocation method [6,23], Lagrange interpolation and Direct Quadrature methods [2,18], Runge-Kutta method [40], Fixed point method [3], and θ-methods [25,41].

In this paper, we improve a new numerical method to obtain the approximate solutions of (1.1) and (1.3) in the Morgan-Voyce series form by Sezer and co-workers [5,9,23,26–28,30,31,35,37,40].

$$y(t) \cong y_N(t) = \sum_{n=0}^{N} a_n B_n(t),\quad a \leq t \leq b \tag{1.4}$$

where $a_n,\ n = 0, 1, \ldots, N$ are unknown coefficients to be obtained and $B_n(t)$, $n = 0, 1, \ldots, N,\ N \geq m$ are the Morgan-Voyce polynomials [22].

In dealing with, A.M.Morgan-Voyce defined the set of polynomials $\{B_n(x)\}$. These polynomials are defined by the relation [22,32,38], recursively,

$$B_n(t) = (t+2)B_{n-1}(t) - B_{n-2}(t),\ n \geq 2 \tag{1.5}$$

with $B_o(t) = 1$ and $B_1(t) = t + 2$ or explicitly, for $n \geq 1$,

$$B_n(t) = \sum_{j=0}^{n} \binom{n+j+1}{n-j} t^j. \tag{1.6}$$

From (1.5) or (1.6), the first four Morgan-Voyce polynomials can be given as

$$B_o(t) = 1,\ B_1(t) = t+2,\ B_2(t) = t^2 + 4t + 3,\ B_3(t) = t^3 + 6t^2 + 10t + 4,\ \ldots$$

Also, the polynomials $B_n(t),\ n = 0, 1, 2, \ldots$ are solutions of the following differential equation

$$t(t+4)B_n''(t) + 3(t+2)B_n'(t) - n(n+2)B_n(t) = 0.$$

2. Operational matrix process

Let us take (1.1) and write the matrix forms of each term of the equation. First, we find the finite series solution (1.4) as the matrix form

$$y(t) \cong y_N(t) = B(t)A, \tag{2.1}$$

where

$$B(t) = [B_o(t),\ B_1(t),\ \dots\ ,\ B_N(t)]$$
$$A = [a_o, a_1, \dots, a_N]^T .$$

Now, we obviously rewrite the matrix form $B(t)$, by using the Morgan-Voyce polynomials $B_n(t)$ given by (1.5) or (1.6) as

$$B(t) = T(t)\boldsymbol{R}, \tag{2.2}$$

where

$$\boldsymbol{R} = \begin{bmatrix} \binom{1}{0} & \binom{2}{1} & \binom{3}{2} & \cdots & \binom{N+1}{N} \\ 0 & \binom{3}{0} & \binom{4}{1} & \cdots & \binom{N+2}{N-1} \\ 0 & 0 & \binom{5}{0} & \cdots & \binom{N+3}{N-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{2N+1}{0} \end{bmatrix}$$

and

$$T(t) = \left[1,\ t,\ t^2, \dots, t^N\right].$$

By using (2.1) and (2.2), we have

$$y_N(t) = T(t)\,\boldsymbol{R}A. \tag{2.3}$$

In addition, it is seen that the iteration [5, 26–28, 36] between the matrix $T(t)$ and its derivative $(T(t))^{(k)}$ is

$$(T(t))^{(k)} = T(t)B^k,\ k = 0, 1, \dots \tag{2.4}$$

where

$$\boldsymbol{B} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad \textit{and} \quad \boldsymbol{B}^0 = I.$$

From (2.3) and (2.4), we get the matrix form, for $k = 0, 1, \dots,$

$$y_N^{(k)}(t) = T^{(k)}(t)\,\boldsymbol{R}A = T(t)\,\boldsymbol{B}^k\boldsymbol{R}A. \tag{2.5}$$

On the other hand, the kernel function $K(t,s)$, by using the Taylor series expansion of $K(t,s)$ can be written as

$$K(t,s) = T(t)\boldsymbol{K}(T(s))^T,\quad \boldsymbol{K} = [k_{pq}]\,, \tag{2.6}$$

where

$$k_{pq} = \frac{1}{p!q!}\frac{\partial^{p+q}K(0,0)}{\partial t^p \partial s^q},\quad p, q = 0, 1, \dots, N.$$

The integral part of (1.1) can be written whereby (2.3) and (2.6) as

$$\int_{t-\tau(t)}^{t-\delta(t)} T(t)\,\boldsymbol{K}(T(s))^T T(s)\boldsymbol{R}A\,ds = T(t)\boldsymbol{K}\,\boldsymbol{Q}(t)\,\boldsymbol{R}A, \tag{2.7}$$

where

$$\boldsymbol{Q}(t)=\int_{t-\tau(t)}^{t-\delta(t)}(T(s))^{T}T(s)\,ds=[q_{mn}(t)]\,,$$

$$q_{mn}(t)=\frac{(t-\delta(t))^{m+n+1}-(t-\tau(t))^{m+n+1}}{m+n+1},\quad m,n=0,1,\ldots,N.$$

3. Matrix-collocation method

Now, we first establish the fundamental matrix equation corresponding to (1.1) by substituting the matrix relationship (2.5) and (2.7) into (1.1), as follows

$$\sum_{k=0}^{m}P_k(t)T(t)\boldsymbol{B}^k\boldsymbol{R}A=g(t)+T(t)\boldsymbol{K}\boldsymbol{Q}(t)\,\boldsymbol{R}A. \tag{3.1}$$

Then, by putting the collocation points defined by

$$t_i=a+\frac{b-a}{N}i,\quad i=0,1,\ldots,N(standart)$$

into (3.1) and facilitating, we get the following fundamental matrix equation

$$\left(\sum_{k=0}^{m}\boldsymbol{P}_k(t)\,\boldsymbol{T}\boldsymbol{B}^k-\overline{\mathbf{T}}\overline{\mathbf{K}}\overline{\mathbf{Q}}\right)\boldsymbol{R}A=G, \tag{3.2}$$

where

$$\boldsymbol{T}=\begin{bmatrix}T(t_o)\\T(t_1)\\\vdots\\T(t_N)\end{bmatrix}=\begin{bmatrix}1&t_o&\cdots&t_o^N\\1&t_1&\cdots&t_1^N\\\vdots&\vdots&\cdots&\vdots\\1&t_N&\cdots&t_N^N\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}0&1&0&\cdots&0\\0&0&2&\cdots&0\\\vdots&\vdots&\vdots&\cdots&\vdots\\0&0&0&\cdots&N\\0&0&0&\cdots&0\end{bmatrix},$$

$$\overline{\mathbf{T}}=diag\,[T(t_o),\ T(t_1),\ldots,T(t_N)]_{(N+1)\times(N+1)^2}\,,$$

$$\overline{\mathbf{K}}=diag\,[\boldsymbol{K},\ \boldsymbol{K},\ldots,\boldsymbol{K}]_{(N+1)^2\times(N+1)^2}\,,$$

$$\boldsymbol{P}_k=diag\,[P_k(t_o),\ P_k(t_1),\ldots,P_k(t_N)]_{(N+1)\times(N+1)}\,,$$

$$\overline{\mathbf{Q}}=\begin{bmatrix}\boldsymbol{Q}(t_o)\\\boldsymbol{Q}(t_1)\\\vdots\\\boldsymbol{Q}(t_N)\end{bmatrix}_{(N+1)^2\times(N+1)},and\ G=\begin{bmatrix}g(t_o)\\g(t_1)\\\vdots\\g(t_N)\end{bmatrix}_{(N+1)\times1}.$$

Nevertheless, the fundamental equation (3.2) can be expressed in the form

$$\boldsymbol{W}A=G\quad or\quad[\,\boldsymbol{W};G]\,, \tag{3.3}$$

where

$$\boldsymbol{W}=[w_{mn}]\,;\ \ m,n=0,1,\ldots,N.$$

Additionally, by (2.5), the initial conditions (1.2) can be rewritten as

$$V_iA=\lambda_i\quad or\quad[V_i;\lambda_i]\,,\ i=0,1,\ldots,m-1, \tag{3.4}$$

where

$$V_i=\sum_{k=0}^{m-1}a_{ik}T(a)\boldsymbol{B}^k\boldsymbol{R}=[u_{io},u_{i1},\ldots,u_{iN}]\,.$$

We hereby have the original matrix [5,8,24,36,39] to find the solution of (1.1) - (1.2), by changing the m rows matrices (3.4) by the any m rows of the augmented matrix (3.3)

$$\left[\tilde{\boldsymbol{W}};\tilde{G}\right] \quad or \quad \tilde{\boldsymbol{W}}A=\tilde{G}. \tag{3.5}$$

In (3.5), if $rank\ \tilde{\boldsymbol{W}} = rank\ \left[\tilde{\boldsymbol{W}};\tilde{G}\right] = N+1$, then the linear system is uniquely solved, so that the solution of the problem (1.1) - (1.2) is obtained as

$$y_N(t)=B_n(t)A \quad or \quad y_N(t)=T(t)\boldsymbol{R}A.$$

Furthermore, the fundamental matrix equation for the integral equation with variable delays (1.3) can be established as, by taking $m=0$ in (3.2),

$$\left(\boldsymbol{P}_o\boldsymbol{T}\boldsymbol{B}^o-\overline{\mathbf{T}}\overline{\mathbf{K}}\overline{\mathbf{Q}}\right)\boldsymbol{R}A=G$$

or briefly,

$$\overline{\boldsymbol{W}}A=G \quad or \quad \left[\overline{\boldsymbol{W}};G\right], \tag{3.6}$$

where

$$\overline{\boldsymbol{W}}=[\bar{w}_{ij}]=\left(\boldsymbol{P}_o\boldsymbol{T}\boldsymbol{B}^o-\overline{\mathbf{T}}\overline{\mathbf{K}}\overline{\mathbf{Q}}\right)\boldsymbol{R} \quad i,j=0,1,\ldots,N.$$

In (3.6), if $rank\ \overline{\boldsymbol{W}} = rank\ \left[\overline{\boldsymbol{W}};G\right] = N+1$, then A is uniquely determined as follows

$$A=\overline{\boldsymbol{W}}^{-1}G.$$

4. Error analysis and convergency

Now, we shall construct an error analysis using the residual function for our method. Furthermore, we shall proceed the Morgan-Voyce polynomial solutions (1.4) using the residual function of current method defined by

$$R_N(t)=L\left[y_N(t)\right]-g(t), \tag{4.1}$$

where

$$L\left[y_N(t)\right]=\sum_{k=0}^{m}P_k(t)\,y^{(k)}(t)-\int_{t-\tau(t)}^{t-\delta(t)}K(t,s)\,y(s)\,ds.$$

The $y_N(t)$ demonstrate the Morgan-Voyce polynomial solutions given by (1.4) of (1.1). Therefore, the $y_N(t)$ satisfies the problem (1.1) under the conditions (1.2). Also, the error $e_N(t)$ can be defined as a function

$$e_N(t)=y(t)-y_N(t), \tag{4.2}$$

where $y(t)$ is the exact solution of (1.1)-(1.2). From (1.1), (1.2), (3.6), and (4.1), we get the error equation

$$L\left[e_N(t)\right]=L\left[y(t)\right]-L\left[y_N(t)\right]=-R_N(t)$$

under the conditions

$$\sum_{k=0}^{m-1}a_{ik}(t)\,e_N^{(k)}(a)=0$$

or briefly, the error problem indicated by

$$\left.\begin{array}{c} L\left[e_N(t)\right]=-R_N(t) \\ \sum_{k=0}^{m-1}a_{ik}(t)\,e_N^{(k)}(a)=0 \end{array}\right\}. \tag{4.3}$$

Using the techniques of Section 3, the problem (4.3) can be solved and given as follows

$$e_{N,M}(t)=\sum_{n=0}^{M}a_n^{*}B_n(t), \quad (M>N).$$

Consequently, the corrected solution is $y_{N,M}(t) = y_N(t) + e_{N,M}(t)$. As well, we build the error function $e_N(t) = y(t) - y_N(t)$, the estimated error function $e_{N,M}(t)$ and the corrected error function $E_{N,M}(t) = e_N(t) - e_{N,M}(t) = y(t) - y_{N,M}(t)$.

Using the convergence of the Homotophy perturbation method to investigate the convergence rate in the Banach space [4] and the studies on the convergence of Dickson polynomials using the residual functions in this space [28], we will now set out the following convergence criteria for Morgan-Voyce polynomial solutions. For this aim, the residual function $R_N(t)$ given by (4.1) can be defined on the interval $[a,b]$ or (a,b) as $R_N(t) : [a,b] \longrightarrow \mathbb{R}$ or $R_N(t) : [a+\varepsilon, b-\varepsilon] \longrightarrow \mathbb{R}$ (ε is a sufficiently small value) and $R_N(t)$ can be written in the Taylor series form

$$R_N(t) = r_0 + r_1 t + r_2 t^2 + ... + r_N t^N = \sum_{n=0}^{N} r_n t^n,$$

where $\mathbb{R}$ is the set of real numbers. Now, we can use the following theorem for our investigation.

Theorem 4.1. *[28] Let B be a Banach space. The residual function sequence $\{R_N(t)\}_{N=2}^{\infty}$ is convergent in B and the following inequality is satisfied so that $0 < \mu_N < 1$. Here μ_N is constant in B:*

$$||R_{N+1}(t)|| < \mu_N ||R_N(t)|. \tag{4.4}$$

5. Illustrative examples

Now, the various numerical examples are given to prove the accuracy and the efficiency of the cuurent method. These examples were performed on the computer using a program written in MATLAB2016b.

Example 1. Firstly, we take the second-order integro-differential equation having variable coefficients and variable delays

$$y''(t) + ty'(t) + e^{t+t^2} y(t) = g(t) + \int_{t-t^2}^{t-ln(t+1)} e^t y(s)\, ds \tag{5.1}$$

with the condition $y(0) = 1, y'(0) = -1$, $0 \le t \le 1$ and the exact solution $y(t) = e^{-t}$. Here

$$g(t) = (1-t)e^{-t} + t + 1$$

and $m = 2, P_o(t) = e^{t+t^2}, P_1(t) = t, P_2(t) = 1, K(t,s) = e^t$. For $N = 3$, the collocation points are computed as $t_o = 0,\ t_1 = \frac{1}{3},\ t_2 = \frac{2}{3},\ t_3 = 1$. By using (3.2), the fundamental matrix equation is written as

$$\left(\sum_{k=0}^{2} \boldsymbol{P}_k(t)\, \boldsymbol{T} \boldsymbol{B}^k - \overline{\mathbf{T}}\overline{\mathbf{K}}\overline{\mathbf{Q}} \right) \boldsymbol{R} A = G, \tag{5.2}$$

where

$$\boldsymbol{P}_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{\frac{4}{9}} & 0 & 0 \\ 0 & 0 & e^{\frac{10}{9}} & 0 \\ 0 & 0 & 0 & e^2 \end{bmatrix}, \ \boldsymbol{P}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{P}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} \\ 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} \\ 1 & 1 & 1 & 1 \end{bmatrix}, \ \boldsymbol{R} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 10 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{K} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{6} & 0 & 0 & 0 \end{bmatrix},$$

Table 1. Solutions for Example 1.

t	Exact	$N=3, M=4$ App.	$N=3, M=4$ Corr.	$N=4, M=5$ App.	$N=4, M=5$ Corr.	$N=12, M=13$ App.	$N=12, M=13$ Corr.
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.2	0.81873	0.81429	0.81822	0.81869	0.81873	0.81873	0.81873
0.4	0.67032	0.65631	0.66887	0.67021	0.67031	0.67032	0.67032
0.6	0.54881	0.52484	0.54651	0.54865	0.54879	0.54881	0.54881
0.8	0.44933	0.41862	0.44652	0.44913	0.44931	0.44933	0.44933
1	0.36788	0.33641	0.36507	0.36769	0.36786	0.36788	0.36788

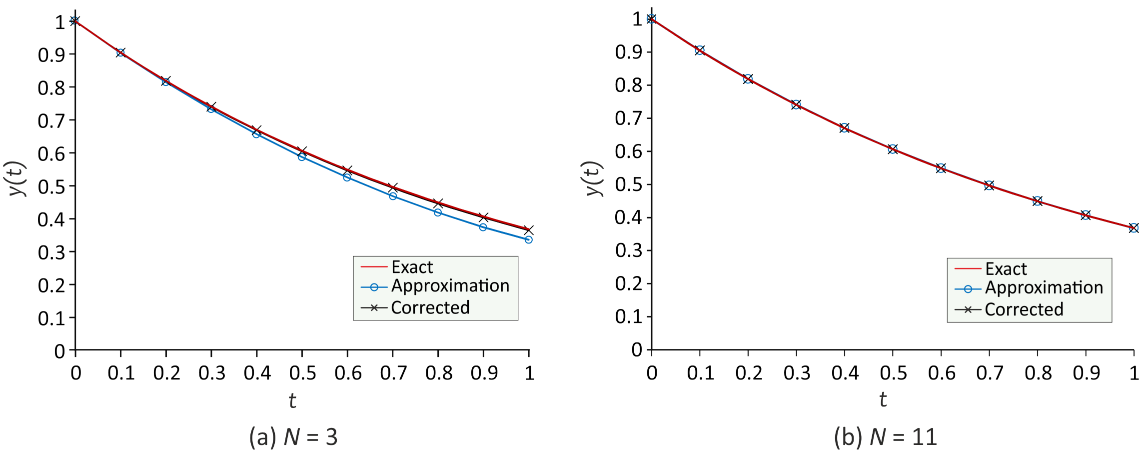

Figure 1. Solutions for Example 1.

$$Q = [q_{mn}], \quad q_{mn} = \frac{(t - ln(t+1))^{m+n+1} - (t - t^2)^{m+n+1}}{m+n+1},$$

$$\overline{\mathbf{K}} = diag\,[\boldsymbol{K}, \boldsymbol{K}, \boldsymbol{K}, \boldsymbol{K}]_{16\times 16}\,, \; \overline{\boldsymbol{Q}} = \begin{bmatrix} Q(0) \\ Q(\frac{1}{3}) \\ Q(\frac{2}{3}) \\ Q(1) \end{bmatrix}_{16\times 4}, \; G = \begin{bmatrix} 2 \\ 1.81102 \\ 1.83781 \\ 2 \end{bmatrix}.$$

The augmented matrix of (5.3) is computed as

$$[\boldsymbol{W}; G] = \left[\begin{array}{cccc|c} 1.0000 & 2.0000 & 5.0000 & 16.0000 & 2.0000 \\ 1.8060 & 4.4981 & 11.3632 & 32.6590 & 1.8110 \\ 3.1664 & 9.0489 & 24.6074 & 71.0787 & 1.8378 \\ 6.5708 & 21.4051 & 64.1298 & 193.4816 & 2.0000 \end{array}\right]. \tag{5.3}$$

As well, the conditions are written as the matrix form, respectively

$$[V_o; \lambda_o] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \end{array}\right] \text{ and } [V_1; \lambda_1] = \left[\begin{array}{cccc|c} 0 & 1 & 4 & 10 & -1 \end{array}\right].$$

Thereby, we have the solution for $N = 3$,

$$y_3(t) = \sum_{n=0}^{3} a_n B_n(t) = 1 - t + 0.3623\, t^2 - 0.0259\, t^3.$$

The approximate and corrected solutions are given in Table 1 and Figure 1. Moreover, it is seen that the error values are reduced while N and M values are increasing in Table 2 and Figure 2.

Table 2. Errors for Example 1.

t	$N=3, M=4$ Abs.	Corr.	$N=4, M=5$ Abs.	Corr.	$N=12, M=13$ Abs.	Corr.
0	4.44089e-16	1.55431e-15	1.99840e-15	1.77636e-15	6.66134e-16	6.66134e-16
0.2	4.44538e-03	5.07806e-04	4.20981e-05	5.25094e-06	5.02265e-13	9.20375e-14
0.4	1.40075e-02	1.45071e-03	1.08912e-04	1.28535e-05	1.05538e-12	1.92846e-13
0.6	2.39735e-02	2.30254e-03	1.64220e-04	1.89756e-05	1.48770e-12	2.71561e-13
0.8	3.07102e-02	2.81196e-03	1.96092e-04	2.24573e-05	1.70275e-12	3.13360e-13
1	3.14685e-02	2.80681e-03	1.93955e-04	2.18761e-05	2.99744e-12	2.99649e-13

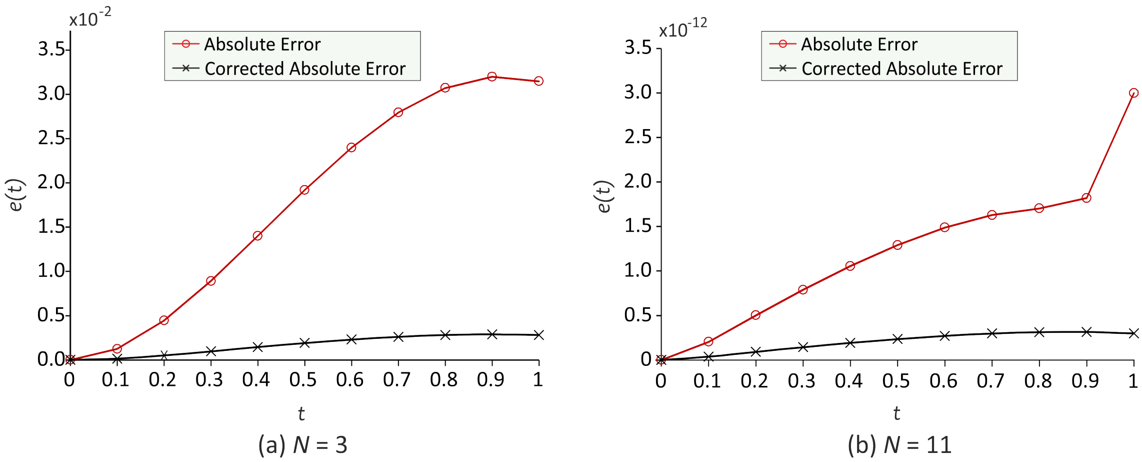

Figure 2. Errors for Example 1.

By using the Theorem 4.1, the residual functions sequence can be calculated as

$$\begin{aligned}\{|R_N(1)|\}_{N=2}^{\infty} &= \Big\{|R_2(1)|, |R_3(1)|, |R_4(1)|, |R_5(1)|, |R_6(1)|, \dots\Big\}\\ &= \{0.037785, 0.002807, 0.000264, 0.000022, 0.000001, \dots\}\end{aligned}$$

$$\mu_N = \left\{\frac{|R_3(1)|}{|R_2(1)|}, \frac{|R_4(1)|}{|R_3(1)|}, \frac{|R_5(1)|}{|R_4(1)|}, \frac{|R_6(1)|}{|R_5(1)|}, \dots\right\} = \{0.0743, 0.0942, 0.0827, 0.0586, 0.0557, \dots\}$$

so,

$$\frac{|R_{N+1}(1)|}{|R_N(1)|} < 1.$$

This shows us that the ratio is approaching zero as N increases. Thus, the residual function sequence $\{R_N(1)\}_{N=2}^{\infty}$ is convergent in B Banach space.

Example 2. Secondly, we solve the delay Volterra integral equation

$$y(t) = (t - 2t^3)e^{-t} + \int_{t-t^2}^{t+t^2} e^{s-t} y\,(s)\,ds \tag{5.4}$$

having the exact solution $y(t) = te^{-t}$.

Similarly, all results in this example are approached to exact solution while N and M are increasing in Table 3 and Figure 3. Furthermore, the errors are illustrated in Table 4 and Figure 4.

Table 3. Solutions for Example 2.

t	Exact	$N=3, M=4$ App.	$N=3, M=4$ Corr.	$N=4, M=5$ App.	$N=4, M=5$ Corr.	$N=12, M=13$ App.	$N=12, M=13$ Corr.
0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	0.16375	0.15034	0.16367	0.16375	0.16375	0.16375	0.16375
0.4	0.26813	0.28020	0.26814	0.26811	0.26813	0.26813	0.26813
0.6	0.32929	0.38746	0.32857	0.32896	0.32930	0.32929	0.32929
0.8	0.35946	0.47001	0.35761	0.35885	0.35948	0.35946	0.35946
1	0.36788	0.52573	0.36538	0.36740	0.36790	0.36788	0.36788

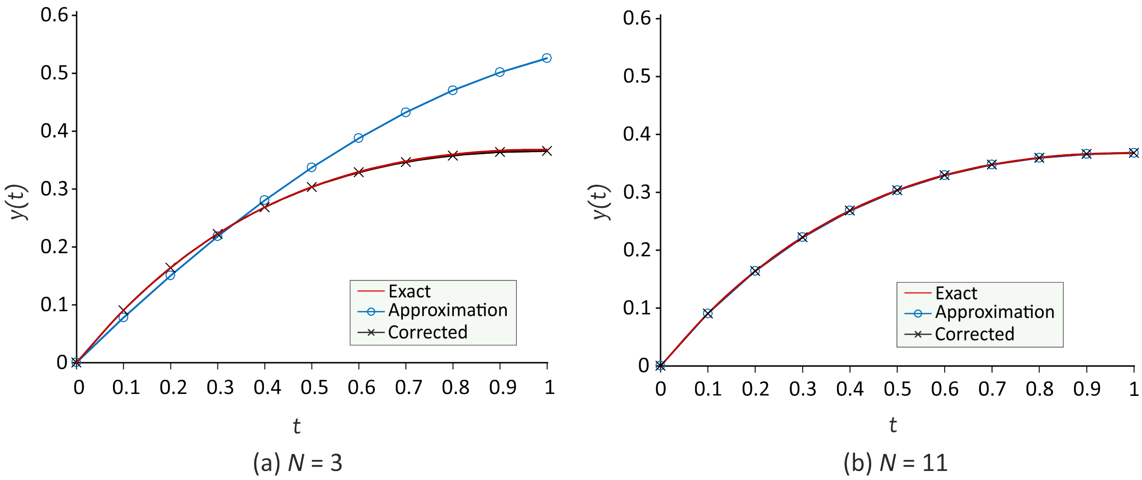

Figure 3. Solutions for Example 2.

Table 4. Errors for Example 2.

t	$N=3, M=4$ Abs.	$N=3, M=4$ Corr.	$N=4, M=5$ Abs.	$N=4, M=5$ Corr.	$N=12, M=13$ Abs.	$N=12, M=13$ Corr.
0	8.32667e-17	1.22402e-14	6.77236e-15	7.11584e-15	6.66067e-14	3.42044e-16
0.2	1.34041e-02	7.77876e-05	4.79841e-08	1.47481e-07	3.47222e-14	4.14668e-14
0.4	1.20736e-02	7.00774e-06	2.14200e-05	5.14742e-07	7.57394e-13	4.15057e-13
0.6	5.81763e-02	7.17626e-04	3.26620e-04	8.08662e-06	1.48396e-11	2.98256e-12
0.8	1.10549e-01	1.85086e-03	6.14730e-04	2.00986e-05	6.10013e-11	7.54613e-12
1	1.57852e-01	2.50300e-03	4.78612e-04	2.13810e-05	3.74857e-10	1.21689e-11

By using the Theorem 4.1, the residual functions sequence can be calculated as

$$\{|R_N(1)|\}_{N=2}^{\infty} = \Big\{|R_2(1)|, |R_3(1)|, |R_4(1)|, |R_5(1)|, |R_6(1)|, \ldots\Big\}$$
$$= \{0.044511, 0.002503, 0.001497, 0.000021, 0.000004, \ldots\}$$

$$\mu_N = \left\{\frac{|R_3(1)|}{|R_2(1)|}, \frac{|R_4(1)|}{|R_3(1)|}, \frac{|R_5(1)|}{|R_4(1)|}, \frac{|R_6(1)|}{|R_5(1)|}, \ldots\right\} = \{0.0562, 0.5979, 0.0143, 0.1710, 0.0445, \ldots\}$$

so,

$$\frac{|R_{N+1}(1)|}{|R_N(1)|} < 1.$$

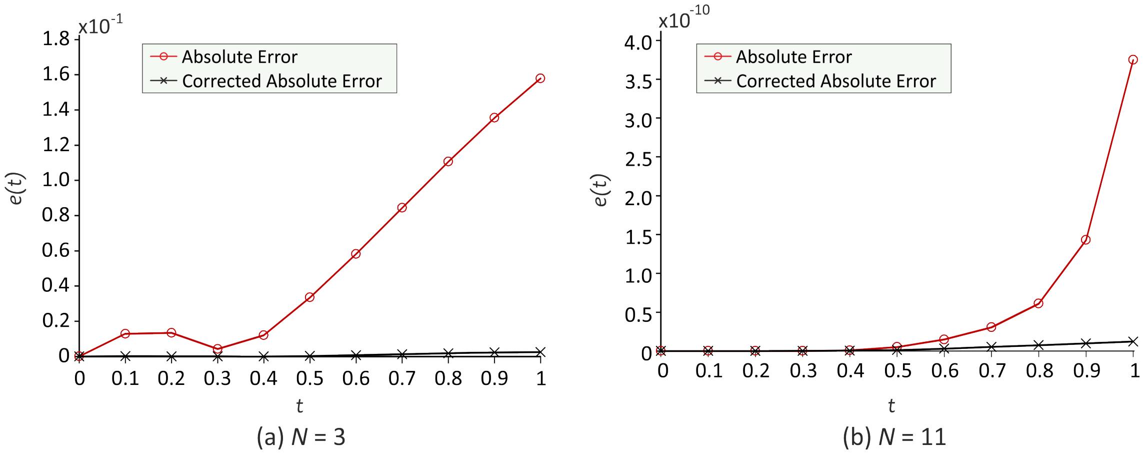

Figure 4. Errors for Example 2.

Table 5. Solutions for Example 3.

t	Exact	$N=3, M=4$ App.	$N=3, M=4$ Corr.	$N=4, M=5$ App.	$N=4, M=5$ Corr.	$N=12, M=13$ App.	$N=12, M=13$ Corr.
0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	0.16375	0.15942	0.16320	0.16423	0.16373	0.16375	0.16375
0.4	0.26813	0.23533	0.26463	0.27148	0.26806	0.26813	0.26813
0.6	0.32929	0.22423	0.31973	0.33904	0.32911	0.32929	0.32929
0.8	0.35946	0.12261	0.34092	0.37936	0.35913	0.35946	0.35946
1	0.36788	-0.07303	0.33760	0.40153	0.36734	0.36788	0.36788

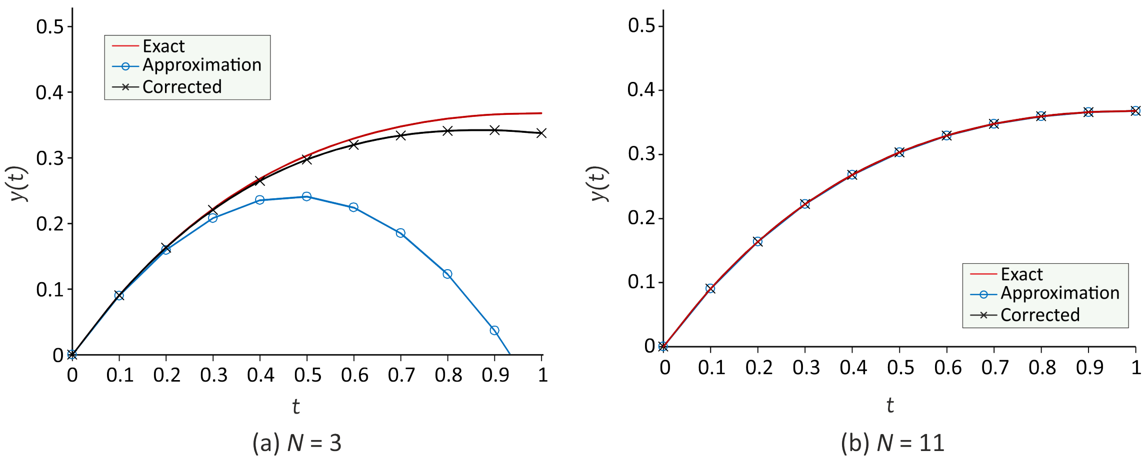

Figure 5. Solutions for Example 3.

Example 3. Now, we deal with the third order integro-differential equation with variable bounds

$$y'''(t) + 3y(t) = (3 + 2t - 2t^3)e^{-t} + \int_{t-t^2}^{t+t^2} e^{s-t} y(s)\, ds$$

subject to the initial contitions $y(0) = 0, y'(0) = 1, y''(0) = -2$, $0 \le t \le 1$ and the exact solution $y(t) = te^{-t}$.
The results and their errors are given in the following tables and figures.

Table 6. Errors for Example 3.

t	$N=3, M=4$ Abs.	Corr.	$N=4, M=5$ Abs.	Corr.	$N=12, M=13$ Abs.	Corr.
0	2.77556e-16	1.99840e-15	5.38458e-15	6.42195e-15	1.79918e-14	2.78734e-14
0.2	4.33035e-03	5.41234e-04	4.79773e-04	1.25048e-05	2.51920e-09	9.53329e-12
0.4	3.28016e-02	3.49699e-03	3.34722e-03	7.08251e-05	1.27584e-08	4.65800e-11
0.6	1.05060e-01	9.56027e-03	9.75674e-03	1.79510e-04	3.09833e-08	1.11678e-10
0.8	2.36852e-01	1.85478e-02	1.99010e-02	3.36175e-04	5.68061e-08	2.03426e-10
1	4.40904e-01	3.02838e-02	3.36511e-02	5.38484e-04	8.97459e-08	3.20448e-10

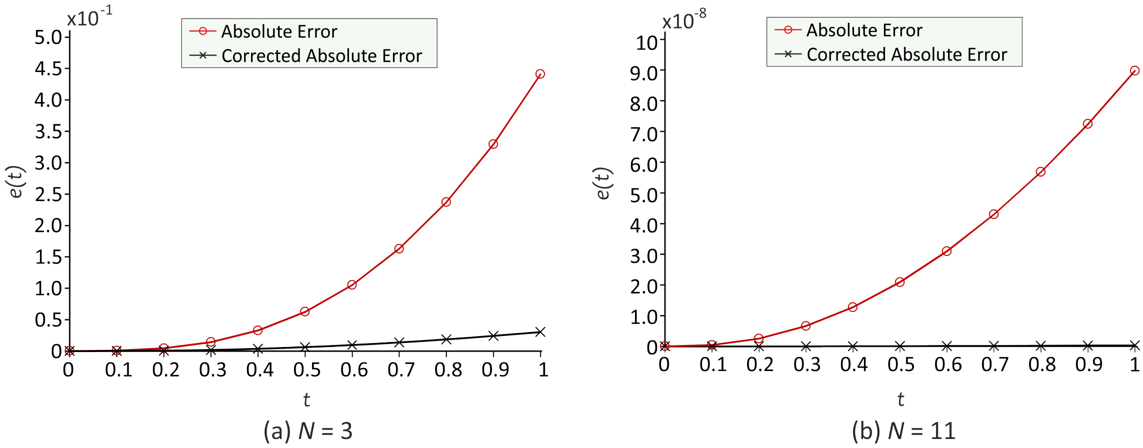

Figure 6. Errors for Example 3.

By using the Theorem 4.1, the residual functions sequence can be calculated as

$$\{|R_N(1)|\}_{N=2}^{\infty} = \Big\{|R_2(1)|, |R_3(1)|, |R_4(1)|, |R_5(1)|, |R_6(1)|, \dots\Big\}$$
$$= \{0.554399, 0.030284, 0.030520, 0.000538, 0.000149, \dots\}$$

$$\mu_N = \left\{\frac{|R_3(1)|}{|R_2(1)|}, \frac{|R_4(1)|}{|R_3(1)|}, \frac{|R_5(1)|}{|R_4(1)|}, \frac{|R_6(1)|}{|R_5(1)|}, \dots\right\} = \{0.0546, 1.0078, 0.0176, 0.2771, 0.3967, \dots\}$$

so,

$$\frac{|R_{N+1}(1)|}{|R_N(1)|} < 1.$$

In order to improve the solution space of the above examples, our program has been run for different values of N (for $N = 1, \dots, 50$). The mean square error (MSE) values are computed and shown in Figure 7. The MSE values for all problems are fluctuated between 10^{-10} and 10^{-16} for $N > 12$. The fluctuation is based on the truncation errors.

6. Conclusion

The ordinary linear delay integro-differential equations with variable delays and/or integral equations with variable bounds are solved by using Morgan-Voyce polynomials and its related matrix form. If the problem does not have the exact solution, the main purpose is to reduce the error generated by the residual functions by these techniques. Obviously, when the value N increases in tables and graphs, the accuracy of the solution is improved. In order to calculate the above solutions and errors, code was written in MATLAB2016R and all calculations were made by using these codes. As a result, it is seen that this novel method effectively and reliably provides solutions of ordinary variable-delay integro

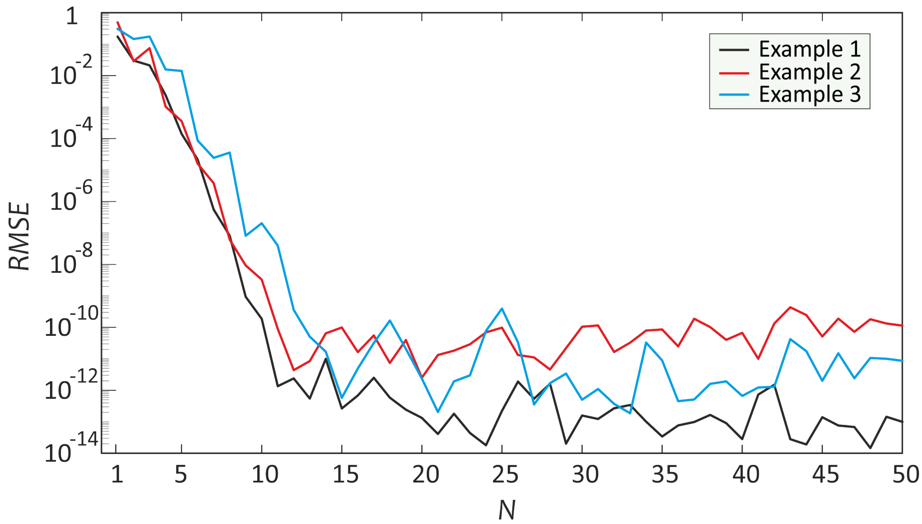

Figure 7. MSE values for N=1, 2, ..., 50 for Example 1, Example 2 and Example 3.

differential equations.

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