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On Fuhrmann's Theorem in Abstract Spaces

Nilgün SÖNMEZ

Abstract

We prove that Fuhrmann's Theorem holds on every Ptolemaic space.

Keywords: Fuhrmann's theorem, Ptolemy's theorem, Ptolemaic space , Abstract space

1. INTRODUCTION

The classical Fuhrmann's Theorem, [2], states that if P is an inscribed hexagon in the plane R^2 with oriented vertices $p_1, \ldots, p_6$ and d denotes the Euclidean distance in R^2, then

$$d(p_1,p_4).d(p_2,p_5).d(p_3,p_6) = d(p_1,p_2).d(p_3,p_4).d(p_5,p_6) + d(p_1,p_6).d(p_2,p_3).d(p_4,p_5) + d(p_1,p_2).d(p_4,p_5).d(p_3,p_6) + d(p_2,p_3).d(p_5,p_6).d(p_1,p_4) + d(p_3,p_4).d(p_2,p_5).d(p_1,p_6). \quad (1)$$

The classical Fuhrmann's Theorem follows as an elementary corollary of Ptolemy's theorem in R^2, proved by the Ancient Greek mathematician Claudius Ptolemaeus (Ptolemy) of Alexandria almost 1800 years ago.

2. PRELIMINARIES

Teorem 1 *(Ptolemy's Theorem in R^2)* Given an inscribed quadrilateral $Q = (p_1, p_2, p_3, p_4)$ then

$$d(p_1,p_3).d(p_2,p_4) = d(p_1,p_2).d(p_3,p_4) + d(p_1,p_4).d(p_2,p_3). \quad (2)$$

Now, Fuhrmann's Theorem may be proved as follows: We apply Theorem 1 to the quadrilaterals

$$Q_1 = (p_1, p_2, p_4, p_5),\ Q_2 = (p_2, p_3, p_4, p_6)$$

$$Q_3 = (p_1, p_4, p_5, p_6),\ Q_4 = (p_1, p_2, p_5, p_6),$$

to obtain the relations

$$d(p_1,p_4).d(p_2,p_5) = d(p_1,p_2).d(p_4,p_5) + d(p_2,p_4).d(p_1,p_5), \quad (3)$$

$$d(p_2,p_4).d(p_3,p_6) = d(p_2,p_3).d(p_4,p_6) + d(p_3,p_4).d(p_2,p_6), \quad (4)$$

$$d(p_1,p_5).d(p_4,p_6) = d(p_1,p_4).d(p_5,p_6) + d(p_4,p_5).d(p_1,p_6), \quad (5)$$

$$d(p_1,p_5).d(p_2,p_6) = d(p_1,p_2).d(p_5,p_6) + d(p_2,p_5).d(p_1,p_6). \quad (6)$$

We multiply (3) by $d(p_3,p_6)$ to obtain

$$d(p_1,p_4).d(p_2,p_5).d(p_3,p_6) = d(p_1,p_2).d(p_4,p_5).d(p_3,p_6) + d(p_2,p_4).d(p_1,p_5).d(p_3,p_6)$$

using (4)

$$d(p_1, p_4).\, d(p_2, p_5).\, d(p_3, p_6) = d(p_1, p_2).\, d(p_4, p_5).\, d(p_3, p_6) + d(p_1, p_5).\big(d(p_2, p_3).\, d(p_4, p_6) + d(p_3, p_4).\, d(p_2, p_6)\big)$$

$$= d(p_1, p_2).\, d(p_4, p_5).\, d(p_3, p_6) + d(p_1, p_5).\, d(p_2, p_3).\, d(p_4, p_6) + d(p_1, p_5).\, d(p_3, p_4).\, d(p_2, p_6).$$

We now use (5) and (6) for the last two terms to obtain

$$d(p_1, p_4).\, d(p_2, p_5).\, d(p_3, p_6) = d(p_1, p_2).\, d(p_4, p_5).\, d(p_3, p_6) + d(p_2, p_3).\big(d(p_1, p_4).\, d(p_5, p_6) + d(p_4, p_5).\, d(p_1, p_6)\big) + d(p_3, p_4).\big(d(p_1, p_2).\, d(p_5, p_6) + d(p_2, p_5).\, d(p_1, p_6)\big)$$

and Fuhrmann's Theorem follows.

It is therefore clear that Fuhrmann's Theorem does not rely on the Euclidean space itself but rather on its Ptolemaic property, see Definition 1 of Ptolemaic spaces below. This property is intuitive and has been generalised to more abstract spaces, for example see [1,3], [4], among a great variety of other references on Ptolemaic spaces.

Let (X, d) be a metric space and suppose there is remote point which we shall denote by ω. We consider the one point compactification of (X, d): $\tilde{X} = X \cup \{\omega\}$ and $\tilde{d}$ is defined on $\tilde{X} \mathrm{x} \tilde{X}$ by

$$\tilde{d}(x, y) = \begin{cases} d(x, y) & if\ x, y \in X, \\ +\infty & if\ x \in X, y = \omega, \\ +\infty & if\ x = \omega, y \in X, \\ 0 & if\ x = y = \omega. \end{cases}$$

To lighten the notation, we will drop the tildes and henceforth (X, d) shall denote the compactified space with the extended metric.

Definition 1 The space (X, d) is called Ptolemaic if for every p_1, p_2, p_3, p_4 pairwise distinct points of X, the following relation holds:

$$d(p_1, p_3).\, d(p_2, p_4) \leq d(p_1, p_2).\, d(p_3, p_4) + d(p_1, p_4).\, d(p_2, p_3). \qquad (7)$$

Definition 2 A Ptolemaic circle c in X is a curve homeomorphic to S^1 such that for every p_1, p_2, p_3, p_4 pairwise distinct points on c Equation (2) holds.

It turns out that a variety of nice spaces are Ptolemaic. For instance, the extended Euclidean space $\overline{R^n} = R^n \cup \{\infty\}$ with the extended Euclidean metric is Ptolemaic and its Ptolemaic circles are the usual Euclidean circles as well as the straight lines. This space of course may be identified via stereographic projection with the sphere S^n and the metric is identified to the chordal metric. The sphere S^n is in turn identified to the boundary of R^{n+1} which is the usual Euclidean space, the first among the $\mathbb{K}$-hyperbolic spaces $H^n_{\mathbb{K}}$, where $\mathbb{K}$ can be the set of the real numbers R, the set of the complex numbers C, the set of quaternions $\mathbb{H}$ and the set of octonions $\mathbb{O}$ (the latter only for $n = 2$). The boundaries of those spaces may be identified to what is called the generalised Heisenberg group $h_{\mathbb{K}}$ together with a point at infinity ∞. There is a natural metric defined on those spaces, the so-called Korányi metric d_h. Now, it is known (see for instance [4]) that all spaces $\partial H^n_{\mathbb{K}}$ endowed with the extension to infinity of the Korányi metric d_h are Ptolemaic and have Ptolemaic circles.

3. MAIN RESULT

Teorem 2 Fuhrmann's Theorem holds on every Ptolemaic circle on the boundary of $H^n_{\mathbb{K}}$.

Let $\overline{R^n}$ be the extended Euclidean space; that is

$$\overline{R^n} = R^n \cup \{\infty\}.$$

We will denote the Korányi metric d_h=d in $\overline{R^n}$, by requiring

$$d(p, \infty) = +\infty, if\ p \neq \infty, d(\infty, \infty) = 0$$

and let $p = (p_1, p_2, p_3, p_4) \in \overline{R^n}$ be arbitrary. There are six distances in $(0,+\infty]$ involved:

$$d(p_i, p_j),\ \ i, j = 1, \ldots, 4, i \neq j$$

We adopt the convention: $(+\infty):(+\infty) = 1$, and to p we associate the cross-ratio $|X^d|(p)$ defined by

$$\left|X^d\right|(p)=\frac{d(p_4,p_2)}{d(p_4,p_1)}\cdot\frac{d(p_3,p_1)}{d(p_3,p_2)}$$

[5].

For every $i,j,k,l=1,\dots,4$, such that $p_i,p_j,p_k,p_l\in\overline{R^n}$ are pairwise disjoint, the following symmetry conditions are clearly satisfied:

$$\begin{aligned}\left|X^d\right|(p)(p_i,p_j,p_k,p_l)&=\left|X^d\right|(p)(p_j,p_i,p_l,p_k)\\&=\left|X^d\right|(p)(p_k,p_l,p_i,p_j)\\&=\left|X^d\right|(p)(p_k,p_l,p_j,p_i).\end{aligned}$$

Let now $p=(p_1,p_2,p_3,p_4)\in\overline{R^n}$ and set

$$|X_1^d|(p)=\left|X^d\right|(p)(p_1,p_2,p_3,p_4),$$

$$|X_2^d|(p)=\left|X^d\right|(p)(p_1,p_3,p_2,p_4).$$

[5]. The cross-ratios of all possible permutations of points of p are functions of $|X_1^d|(p)$ and $|X_2^d|(p)$.

We apply it to the quadrilaterals

$Q_1=(p_1,p_2,p_4,p_5)$, $Q_2=(p_2,p_3,p_4,p_6)$

$Q_3=(p_1,p_4,p_5,p_6)$, $Q_4=(p_1,p_2,p_5,p_6)$.

We assume that

$$p_1=\infty, p_2=(x_2,0,u), p_3=(x_3,0,u),$$

$$p_4=(x_4,0,u),\ p_5=(x_5,0,u),$$

$$p_6=(0,0,u)$$

where $x_5>x_4>x_3>x_2>0$.

We now use (3). Then:

$$|X_1^d|(p)=\left|X^d\right|(p)(p_1,p_2,p_4,p_5)$$

$$=\frac{d(p_5,p_2)}{d(p_5,p_1)}\cdot\frac{d(p_4,p_1)}{d(p_4,p_2)}$$

$$=\frac{x_5-x_2}{x_4-x_2}$$

and

$$|X_2^d|(p)=\left|X^d\right|(p)(p_1,p_4,p_2,p_5)$$

$$=\frac{d(p_5,p_4)}{d(p_5,p_1)}\cdot\frac{d(p_2,p_1)}{d(p_2,p_4)}$$

$$=\frac{x_5-x_4}{x_4-x_2}$$

The p_2 and p_4 separate p_1 and p_5 since $|X_1^d|(p)-|X_2^d|(p)=1$. We now use (4). Then:

$$|X_1^d|(p)=\left|X^d\right|(p)(p_2,p_3,p_4,p_6)$$

$$=\frac{d(p_6,p_3)}{d(p_6,p_2)}\cdot\frac{d(p_4,p_2)}{d(p_4,p_3)}$$

$$=\frac{x_3}{x_2}\cdot\frac{x_4-x_2}{x_4-x_3}$$

and

$$|X_2^d|(p)=\left|X^d\right|(p)(p_2,p_4,p_3,p_6)$$

$$=\frac{d(p_6,p_4)}{d(p_6,p_2)}\cdot\frac{d(p_3,p_2)}{d(p_3,p_4)}$$

$$=\frac{x_4}{x_2}\cdot\frac{x_3-x_2}{x_3-x_4}$$

$$=-\frac{x_4}{x_2}\cdot\frac{x_3-x_2}{x_4-x_3}$$

The p_2 and p_6 separate p_3 and p_4 since $|X_1^d|(p)+|X_2^d|(p)=1$. We now use (5). Then:

$$|X_1^d|(p)=\left|X^d\right|(p)(p_1,p_4,p_5,p_6)$$

$$=\frac{d(p_6,p_4)}{d(p_6,p_1)}\cdot\frac{d(p_5,p_1)}{d(p_5,p_4)}$$

$$=-\frac{x_4}{x_5-x_4}$$

and

$$|X_2^d|(p)=\left|X^d\right|(p)(p_1,p_5,p_4,p_6)$$

$$=\frac{d(p_6,p_5)}{d(p_6,p_1)}\cdot\frac{d(p_4,p_1)}{d(p_4,p_5)}$$

$$=\frac{x_5}{x_5-x_4}$$

The p_4 and p_5 separate p_1 and p_6 since $|X_2^d|(p)-|X_1^d|(p)=1$. We now use (6). Then:

$$|X_1^d|(p)=\left|X^d\right|(p)(p_1,p_2,p_5,p_6)$$

$$=\frac{d(p_6,p_2)}{d(p_6,p_1)}\cdot\frac{d(p_5,p_1)}{d(p_5,p_2)}$$

$$=-\frac{x_2}{x_5-x_2}$$

and

$$|X_2^d|(p) = |X^d|(p)(p_1, p_5, p_2, p_6)$$

$$=\frac{d(p_6,p_5)}{d(p_6,p_1)}\cdot\frac{d(p_2,p_1)}{d(p_2,p_5)}$$

$$=\frac{x_5}{x_5-x_2}$$

The p_1 and p_6 separate p_2 and p_5 since $|X_1^d|(p) + |X_2^d|(p) = 1$.

4. CONCLUSION

Fuhrmann's theorem is satisfied in abstract Ptolemaic spaces.

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