Research Article GU J Sci 32(2): 637-647 (2019)

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Faber Polynomial Coefficient Estimates for Analytic Bi-Close-to-Convex Functions Defined by Subordination

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Article Info

Received: 24/12/2017
Accepted: 07/01/2019

Keywords

Bi-univalent functions
Bi-close-to-convex functions
Coefficient estimates
Faber polynomial
Subordination

Abstract

In this work, the Faber polynomial expansions and a different method were employed to estimate the $|a_n|$ coefficients of a subclass of bi-close-to-convex functions, which is introduced by subordination concept in the open unit disk. Further, we generalize some of the previous outcomes.

1. INTRODUCTION

Suppose $\mathcal{A}$ be a class of analytic functions in the open unit disk $\mathbb{U} = \{z \in C : |z| < 1\}$, as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

All univalent functions in the subclass of $\mathcal{A}$ are denoted by $\mathcal{S}$. For α, $0 \le \alpha < 1$, the important subclasses of starlike, convex and close-to-convex functions are expressed by (see for details [1,2]),

$$S^*(\alpha) = \left\{ f \in \mathcal{S} : Re\left[\frac{zf'(z)}{f(z)}\right] > \alpha,\ z \in \mathbb{U} \right\}$$

$$\mathcal{C}(\alpha) = \left\{ f \in \mathcal{S} : Re\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha,\ z \in \mathbb{U} \right\}$$

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A}\ |\ \exists \psi \in \mathcal{C}(0)\ Re\left[\frac{f'(z)}{\psi'(z)}\right] > \alpha,\ z \in \mathbb{U} \right\},$$

respectively, and by Alexander's Theorem we know, $\psi \in \mathcal{C}(0)$ if and only if $\phi = z\psi' \in \mathcal{S}^*(0)$. Hence, we can rewrite $\mathcal{K}(\alpha)$ as follows:

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$$\mathcal{K}(\alpha)=\left\{f\in\mathcal{A}\,|\,\exists\phi\in\mathcal{S}^*(0)\ Re\left[\frac{zf'(z)}{\phi(z)}\right]>\alpha,\ z\in\mathbb{U}\right\}.$$

Considering the Koebe one-quarter theorem [1], the image of $\mathbb{U}$ under $f\in\mathcal{S}$ includes a disk of radius $1/4$. Obviously, the inverse f^{-1} of $f\in\mathcal{S}$ is expressed by

$$f^{-1}(f(z))=z\quad(z\in\mathbb{U})\qquad\text{and}\qquad f(f^{-1}(w))=w\ \left(|w|<r_0(f);r_0(f)\geq\frac{1}{4}\right),$$

where

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots\tag{2}$$

If both f and f^{-1} are univalent in $\mathbb{U}$, then function $f\in\mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ and the class was denoted by Σ.

A major problem in geometric function theory is calculation of the bounds for the coefficients $|a_n|$ as they give information about the geometric properties of these functions. For example, the bound for the $|a_2|$ of functions $f\in\mathcal{S}$ gives the distortion and growth bounds followed by covering theorems, see, for example, [3-8]. The coefficient estimate issue i.e. bound of $|a_n|\,(n\in N-\{1,2\})$ for each $f\in\Sigma$ is still an open problem.

Faber [9] introduced the Faber polynomials, which is an important factor in diverse fields of mathematical sciences, especially in geometric function theory. Several authors worked on utilizing the Faber polynomial expansions to estimate coefficient for bi-univalent functions, [10-14]. By employing the Faber polynomial expansion of functions $f\in\mathcal{S}$ given in (1), the coefficients of its inverse function $g=f^{-1}$ is written as (see, for details, [15,16]):

$$g(w)=f^{-1}(w)=w+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\cdots,a_n)w^n,$$

where

$$\begin{aligned}K_{n-1}^{-n}&=\frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1}+\frac{(-n)!}{\big(2(-n+1)\big)!(n-3)!}a_2^{n-3}a_3\\&+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4+\frac{(-n)!}{\big(2(-n+2)\big)!(n-5)!}a_2^{n-5}\\&\cdot[a_5+(-n+2)a_3^2]+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}[a_6+(-2n+5)a_3a_4]+\sum_{j\geq7}a_2^{n-j},\end{aligned}$$

so that $V_j\ \ (7\leqq j\leqq n)$ is a homogeneous polynomial in the quantities $a_2,a_3,\cdots,a_n$ and expressions such as (for instance) (-n)! are to be introduced by symbols as follows:

$$(-n)!\equiv\Gamma(1-n):=(-n)(-n-1)(-n-2)\cdots\ \big(n\in N_0:=N\cup\{0\}\ \ (\mathrm{N}:=\{1,2,3,\cdots\})\big)$$

Particularly, the first three expirations of K_{n-1}^{-n} are rendered by

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3\left(2a_2^2 - a_3\right) \quad \text{and} \quad K_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).$$

In general, for any $p \in Z = \{0, \pm 1, \pm 2, \cdots\}$, an expansion of K_n^p is rendered below ([15,17,18]; see also [16, p. 349])

$$K_{n-1}^p = pa_n + \frac{p(p-1)}{2}D_{n-1}^2 + \frac{p!}{(p-3)!3!}D_{n-1}^3 + \cdots + \frac{p!}{(p-n+1)!(n-1)!}D_{n-1}^{n-1}, \ \left(p \in \mathbb{Z}\right)$$

where (see, for details, [18]) $D_n^p = D_n^p(a_2, a_3, \cdots).$ We also have

$$D_n^m(a_2, a_3, \cdots, a_{n+1}) = \sum \frac{m!(a_2)^{\mu_1} \cdots (a_{n+1})^{\mu_n}}{\mu_1! \cdots \mu_n!}, \tag{3}$$

where the all nonnegative integers of $\mu_1, \cdots, \mu_n$ are summed, meeting the following conditions:

$$\mu_1 + \mu_2 + \cdots + \mu_n = m$$
$$\mu_1 + 2\mu_2 + \cdots + n\mu_n = n.$$

Note that $D_n^n(a_2, a_3, \cdots, a_{n+1}) = a_2^n$.

The purpose of our study is using the Faber polynomial expansions to obtain estimates of coefficients $|a_n|$ for bi-close-to-convex functions, which is stated by subordinations in $\mathbb{U}$. Further, we generalize some of the previous outcomes.

2. PRELIMINARIES

First, some definitions and lemmas are mentioned in this paper.

Definition 2.1. [1] Let h and H be analytic in $\mathbb{U}$. We state that h is subordinate to H, written as $h(z) \prec H(z)$, provided there exists an analytic function ϖ, described on $\mathbb{U}$ with the conditions $\varpi(0) = 0$ and $|\varpi(z)| < 1$, satisfying $h(z) = H(\varpi(z))$. In particular, if H is univalent then $h(z) \prec H(z)$ is equivalent to $h(\mathbb{U}) \subseteq H(\mathbb{U})$ and $h(0) = H(0)$.

Different categories of starlike and convex functions were introduced by Ma and Minda [19], where each factor $zf'(z)/f(z)$ or $1+zf''(z)/f(z)$ is subordinated to the total function. To this aim, they determined an analytic function with the characteristics of a positive real part of $\mathbb{U}$, $\varphi(0) = 1, \varphi'(0) > 0$, and maps $\mathbb{U}$ onto a region starlike respecting 1 and symmetric respecting the real axis. So, we let $\varphi(z)$ is analytic function with the characteristics of a positive real part in $\mathbb{U}$ and $\varphi(\mathbb{U})$ symmetric respecting the real axis, such that

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \qquad (B_1 > 0).$$

Recently, Sivasubramanian et al. [20] introduced two subclasses $\mathcal{K}_\Sigma[\alpha]$ and $\mathcal{K}_\Sigma(\beta)$ and only obtained estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses.

Definition 2.2. [20] Let $\mathcal{A}_\Sigma(\mathrm{R})$ interpret the class of functions of the form (1), defined on $|z|<\mathrm{R}$, for which the inverse function has an analytic continuation to $|z|<\mathrm{R}$ where f^{-1} is given by (2). We call the functions in $\mathcal{A}_\Sigma(\mathrm{R})$ bi-analytic in $|z|<\mathrm{R}$. We abbreviate $\mathcal{A}_\Sigma(1)=\mathcal{A}_\Sigma$ and we note that $\mathcal{A}_\Sigma$ is a proper subclass of $\mathcal{A}$.

Definition 2.3. [20] Let $0\le\alpha\le 1$. We say that $f\in\mathcal{A}_\Sigma$ presented by (1) is strongly bi-close-to-convex of order α if there exist bi-convex functions $\phi,\psi\in\mathcal{C}(0)$ so that

$$|arg\,(f'(z)/\phi'(z))|<\frac{\alpha\pi}{2}\qquad(z\in\mathbb{U}),$$

and

$$arg\,(g'(w)/\psi'(w))|<\frac{\alpha\pi}{2}\qquad(w\in\mathbb{U}),$$

where

$$\phi(z)=z+c_2z^2+c_3z^3+c_4z^4+\cdots,$$

$$\phi^{-1}(w)=\psi(w)=w-c_2w^2+(2c_2^2-c_3)w^3-(5c_2^3-5c_2c_3+c_4)w^4+\cdots,$$

and g is the analytic continuation presented by (2). The category of strongly bi-close-to-convex functions of order α denoted by $\mathcal{K}_\Sigma[\alpha]$.

Definition 2.4. [20] Let $0\le\beta<1$. A function $f\in\mathcal{A}_\Sigma$ given by (1) be so that $f'(z)\neq 0$ on $\mathbb{U}$. Then we say f is bi-close-to convex of order β if there exist bi-convex functions $\phi,\psi\in\mathcal{C}(0)$ such that

$$Re\left(f'(z)/\phi'(z)\right)>\beta\qquad(z\in\mathbb{U}),$$

and

$$Re\left(g'(w)/\psi'(w)\right)>\beta\qquad(w\in\mathbb{U}),$$

where g is the analytic continuation presented by (2). The category of functions bi-close-to-convex of order β denoted by $\mathcal{K}_\Sigma(\beta)$.

Lemma 2.5. [1] Let $\mathrm{u}(z)$ is analytic in $\mathbb{U}$ satisfying $\mathrm{u}(0)=0$, $|\mathrm{u}(z)|<1$, and assume that

$$u(z)=\sum_{n=1}^{\infty}p_nz^n\qquad(z\in\mathbb{U}).$$

Then $|p_n|\le 1$ for all $n=1,2,3,\cdots$.

Lemma 2.6. [12] Let $\omega(z)=\sum_{n=1}^{\infty}\omega_n z^n \in \mathcal{A}$ is a Schwarz function such that $|\omega(z)|<1$ for $|z|<1$. If $\gamma \geq 0$ then

$$|\omega_2+\gamma\omega_1^2| \leq 1+(\gamma-1)|\omega_1^2|.$$

3. MAIN RESULTS

First, the subclass $\mathcal{K}_\Sigma(\varphi)$ is introduced and investigated then coefficients $|a_n|$ are estimated for functions in this category.

Definition 3.1. We say that $f \in \Sigma$ presented by (1) is in the class $\mathcal{K}_\Sigma(\varphi)$ if the following condition is considered:

$$\frac{f'(z)}{\phi'(z)} \prec \varphi(z) \qquad (z \in \mathbb{U}), \tag{4}$$

and

$$\frac{g'(w)}{\psi'(w)} \prec \varphi(w) \qquad (w \in \mathbb{U}), \tag{5}$$

where $\phi(z)=z+\sum_{n=2}^{\infty}c_n z^n$, $\psi(w)=w+\sum_{n=2}^{\infty}d_n w^n$ belong to $\mathcal{C}(0)$ and g is presented by (2).

Remark 3.2. Since every starlike function is a close-to-convex function, so for the class $\mathcal{K}_\Sigma(\varphi)$ we can write $\mathcal{S}^*_\Sigma(\varphi)$ class as

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \quad (z \in \mathbb{U}) \qquad \text{and} \qquad \frac{wg'(w)}{g(w)} \prec \varphi(w) \quad (w \in \mathbb{U}). \tag{6}$$

There are several elections of φ, ϕ and ψ, which supply interesting subclasses of $\mathcal{K}_\Sigma(\varphi)$.

Remark 3.3. For $\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}$ where $0<\alpha\leq 1$ and $\phi^{-1}(w)=\psi(w)$, the class $\mathcal{K}_\Sigma(\varphi)$ convert to class $\mathcal{K}_\Sigma[\alpha]$ in Definition 2.3.

Remark 3.4. For $\varphi(z)=\frac{1+(1-2\beta)z}{1-z}$ where $0\leq\beta<1$ and $\phi^{-1}(w)=\psi(w)$, the class $\mathcal{K}_\Sigma(\varphi)$ reduce to class $\mathcal{K}_\Sigma(\beta)$ in Definition 2.4.

Remark 3.5. For $\phi(z)=z$, $\psi(w)=w$ and $\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}$ where $0\le\alpha\le 1$, the class $\mathcal{K}_{\Sigma}(\varphi)$ reduce to a class $\mathcal{H}_{\Sigma}^{\alpha}$ which defined by Srivastava et al. [7, Definition 1].

Remark 3.6. For $\phi(z)=z$, $\psi(w)=w$ $\varphi(z)=\frac{1+(1-2\beta)z}{1-z}$, where $0\le\beta<1$, the class $\mathcal{K}_{\Sigma}(\varphi)$ reduce to a class $\mathcal{H}_{\Sigma}(\beta)$ which defined by Srivastava et al. [7, Definition 2].

Remark 3.7. For $\phi(z)=z$, $\psi(w)=w$, the class $\mathcal{K}_{\Sigma}(\varphi)$ reduce to a class $\mathcal{H}_{\Sigma}(\varphi)$ which defined by Ali et al. [3, page 345].

Theorem 3.8. Suppose $f\in\mathcal{K}_{\Sigma}(\varphi)$ be given by (1). If $a_k=c_n=d_n=0$ for $2\le k\le n-1$, then

$$|a_n|\le 1+\frac{B_1}{n} \qquad n\ge 3. \tag{7}$$

Proof. Let function $f\in\mathcal{K}_{\Sigma}(\varphi)$, by the definition of subordination for two Schwarz functions $u,v:\mathbb{U}\to\mathbb{U}$ with

$$u(z)=\sum_{n=1}^{\infty}p_n z^n \quad \text{and} \quad v(z)=\sum_{n=1}^{\infty}q_n z^n \qquad (z\in\mathbb{U}).$$

we have

$$\frac{f'(z)}{\phi'(z)}=\varphi(u(z)), \tag{8}$$

and

$$\frac{g'(w)}{\psi'(w)}=\varphi(v(w)). \tag{9}$$

So from (8) and (9) we get

$$\begin{aligned} &1+\sum_{n=2}^{\infty}na_n z^{n-1}=f'(z)=\phi'(z)\varphi(u(z))\\ &=[1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}B_k D_n^k(p_1,p_2,\cdots,p_n)z^n][1+\sum_{n=1}^{\infty}(n+1)c_{n+1}z^n], \end{aligned} \tag{10}$$

and

$$
\begin{aligned}
1+\sum_{n=2}^{\infty} nb_n w^{n-1} &= 1+\sum_{n=2}^{\infty} n\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\cdots,a_n)w^{n-1} = g'(w)=\varphi(v(w)) \\
&= [1+\sum_{n=1}^{\infty}\sum_{k=1}^{n} B_k D_n^k(q_1,q_2,\cdots,q_n)w^n][1+\sum_{n=1}^{\infty}(n+1)d_{n+1}w^n].
\end{aligned} \tag{11}
$$

By considering the corresponding coefficients of (10), we get

$$
na_n = nc_n + \sum_{t=1}^{n-1}\left[(n-t)c_{n-t}\sum_{k=1}^{t} B_k D_t^k(p_1,p_2,\cdots,p_t)\right] \quad (c_1=1). \tag{12}
$$

Similarly, by considering the corresponding coefficients of (11), we find that

$$
nb_n = nd_n + \sum_{t=1}^{n-1}\left[(n-t)d_{n-t}\sum_{k=1}^{t} B_k D_t^k(q_1,q_2,\cdots,q_t)\right] \quad (d_1=1). \tag{13}
$$

Now, from $a_k = c_n = d_n = 0$ for $2 \le k \le n-1$, we have

$$
a_n = B_1 p_{n-1} + nc_n, \tag{14}
$$

and

$$
-na_n = nb_n = B_1 q_{n-1} + nd_n. \tag{15}
$$

Now solving the absolute values of either of two relations mentioned above and using $|p_{n-1}| \le 1, |q_{n-1}| \le 1$ and $|c_n| \le 1, |d_n| \le 1$, we obtain

$$
|a_n| = \frac{|B_1 p_{n-1} + nc_n|}{n} = \frac{|B_1 q_{n-1} + nd_n|}{n} \le 1 + \frac{B_1}{n}.
$$

This concludes the bound as presented in equation (7) and this completes the proof.

Corollary 3.9. Suppose $f \in \mathcal{K}_{\Sigma}\left(\left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots\right)$ where $0 < \alpha \le 1$ be given by (1). If $a_k = c_n = d_n = 0$ for $2 \le k \le n-1$, then

$$
|a_n| \le 1 + \frac{2\alpha}{n} \qquad n \ge 3.
$$

Corollary 3.10. ([11, Theorem 2.1]) For $0 \le \beta < 1$, Suppose the function $f \in \mathcal{K}_{\Sigma}\left(\frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots\right)$ be given by (1). If $a_k = c_n = d_n = 0$ for $2 \le k \le n-1$, then

$$|a_n| \leq 1 + \frac{2(1-\beta)}{n} \qquad n \geq 3.$$

Corollary 3.11. ([12, Theorem 2.1]) Suppose $f \in \mathcal{S}^*_\Sigma\left(\frac{1+Az}{1+Bz} = 1 + (A-B)z + \cdots\right)$ where $-1 \leq B < A \leq 1$ be presented by equation (1). If $a_k = 0$ for $2 \leq k \leq n-1$, then

$$|a_n| \leq \frac{A-B}{n-1} \qquad n \geq 3.$$

Proof. By proof of Theorem 3.8 and from (14) and (15), we will have

$$na_n = B_1 p_{n-1} + a_n,$$

and

$$-na_n = nb_n = B_1 q_{n-1} + b_n = B_1 q_{n-1} - a_n.$$

Now solving the absolute values of two relations mentioned above, using $B_1 = A - B$ we obtain result and this completes the proof.

Theorem 3.12. Suppose $f \in \Sigma$ presented by (1) be in the subclass $\mathcal{K}_\Sigma(\varphi)$, $\phi^{-1}(w) = \psi(w) = w - c_2 w^2 + (2c_2^2 - c_3)w^3 - (5c_2^3 - 5c_2c_3 + c_4)w^4 + \cdots$ and $B_2 = \alpha B_1$, $0 < \alpha \leq 1$. Then

$$|a_2| \leq \sqrt{1 + B_1} \tag{16}$$

and

$$|a_3| \leq 1 + B_1. \tag{17}$$

Proof. With respect to the equations (12) and (13) for n = 2 and n = 3, we have respectively,

$$2a_2 = 2c_2 + B_1 p_1 \tag{18}$$

$$3a_3 = 3c_3 + 2B_1 c_2 p_1 + B_1 p_2 + \alpha B_1 p_1^2 \tag{19}$$

$$-2a_2 = -2c_2 + B_1 q_1 \tag{20}$$

$$6a_2^2 - 3a_3 = 3(2c_2^2 - c_3) - 2B_1 c_2 q_1 + B_1 q_2 + \alpha B_1 q_1^2. \tag{21}$$

Also from adding (19) and (21), we have

$$6a_2^2 = 6c_2^2 + 2B_1 c_2 (p_1 - q_1) + B_1\left[(p_2 + \alpha p_1^2) + (q_2 + \alpha q_1^2)\right].$$

Therefore, by taking absolute values for the above equation, we have

$$6|a_2^2| \leq 6|c_2^2| + 2B_1|c_2|(|p_1| + |q_1|) + B_1\left[|p_2 + \alpha p_1^2| + |q_2 + \alpha q_1^2|\right].$$

Since $0<\alpha\leq 1$, then by using Lemma 2.6, we get

$$\begin{aligned}6|a_2^2|&\leq 6|c_2^2|+4B_1|c_2|(|p_1|+|q_1|)+B_1\left[1+(\alpha-1)|p_1|^2+1+(\alpha-1)|q_1|^2\right]\\&\leq 6+4B_1+2B_1.\end{aligned}$$

So we get the desired estimate on $|a_2|$ in equation (16).

Finally, from equation (19) and using Lemma 2.6, we get that

$$3|a_3|\leq 3|c_3|+2B_1|c_2||p_1|+B_1\left[1+(\alpha-1)|p_1|^2\right]\leq 3+2B_1+B_1.$$

Therefore, we obtain inequality equation (17) and this completes the proof.

Remark 3.13. Taking $\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}$ where $0<\alpha\leq 1$in Theorem 3.12 we have the next corollary which is the results presented by Sivasubramanian et al. in [20, Theorem 2.1].

Corollary 3.14. Suppose $f\in\Sigma$ presented by (1) be in the class $\mathcal{K}_{\Sigma}\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$, where $0<\alpha\leq 1$. Then

$$|a_2|\leq\sqrt{1+2\alpha}$$

and

$$|a_3|\leq 1+2\alpha.$$

Remark 3.15. By taking $\varphi(z)=\frac{1+(1-2\beta)z}{1-z}$ where $0\leq\beta<1$ in Theorem 3.12 we conclude the next corollary which is the results presented by Sivasubramanian et al. in [20, Theorem 3.1].

Corollary 3.16. Suppose $f\in\Sigma$ presented by (1) be in the class $\mathcal{K}_{\Sigma}\left(\frac{1+(1-2\beta)z}{1-z}\right)$, where $0\leq\beta<1$. Then

$$|a_2|\leq\sqrt{1+2(1-\beta)}$$

and

$$|a_3|\leq 1+2(1-\beta).$$

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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