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Computational Complexity Comparison of a New Linear Block Approach and Modified Taylor Series Approach for Developing k-Step Third Derivative Block Methods

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Abstract

This article introduces two approaches to develop block methods for solving second order ordinary differential equations directly. Both approaches, namely a new linear block approach and the modified Taylor series approach are capable of producing a family of methods that will simultaneously approximate the solutions of any ordinary differential equation at the respective grid points of the block method. The computational complexities of both approaches are examined, and the results show the new linear block approach require less computations compared to the modified Taylor series approach.

1. INTRODUCTION

The initial and most conventional approach for solving second order ordinary differential equations was the concept of reducing the differential equation to a system of first order ordinary differential equations. After this, any known numerical method can be applied. This approach of reduction was discussed by several authors such as [1-4]. This approach was saddled with drawbacks such as inability to maximize computational time and low level of accuracy as discussed in [5]. Direct application of linear multi-step methods to differential equations was then adopted in an attempt to improve on these setbacks [6-7]. In a bid to bypass the disadvantages of wastage in computational time and methods giving low level of accuracy that was still attached to direct solution methods such as predictor-corrector methods, the development of block methods came to light.

[5] have presented two approaches to develop block methods which are capable of computing the solution of second order ordinary differential equations directly. The approaches considered include integration and collocation approaches. However, one of the drawbacks mentioned in the work was the inability to present a generalized algorithm for the collocation approach, while the integration approach was said to be complicated in derivation of the block methods. Hence, this article takes up this challenge by introducing less complicated generalized approaches for developing block methods to directly solve second order ordinary differential equations.

This involves the proposition of a new approach (linear block approach) and an extension of the conventional Taylor series approach (modified Taylor series). The conventional Taylor series approach dates back to [4], however, it was not adopted for the development of higher derivative block methods. To determine which approach requires less computational burden and thus more suitable for developing block methods for the solution of second order ordinary differential equations, the approaches are compared in terms of computational complexity.

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2. DERIVATION OF k-STEP THIRD DERIVATIVE BLOCK METHODS (TDBM) USING MODIFIED TAYLOR SERIES APPROACH

Consider the form for developing the initial discrete multi-step scheme as

$$y_{n+k} = \sum_{j=0}^{k-1} \alpha_{j_v} y_{n+j} + \sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=0}^{k} \lambda_j g_{n+j} \, , \tag{1}$$

where $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j})$, $g_{n+j} = \frac{df(x_{n+j}, y_{n+j}, y'_{n+j})}{dx}$ and $v = m$ (m is the order of the differential equation which is 2). Hence, the expected α_{j_v} 's are α_{j_1} and α_{j_2}. Note that the α_{j_v} -values can take ${}^kC_v$ forms and j_v -values not chosen will be used as evaluation points when developing the additional methods needed to form the block method subsequently.

Expanding individual terms in (1) using Taylor series expansion and substituting back in (1) gives

$$\begin{aligned}
& y(x_n) + khy'(x_n) + \frac{(kh)^2}{2!} y''(x_n) + \cdots = \alpha_{j_1}\left[y(x_n) + j_1 hy'(x_n) + \frac{(j_1 h)^2}{2!} y''(x_n) + \cdots \right] \\
& + \alpha_{j_2}\left[y(x_n) + j_2 hy'(x_n) + \frac{(j_2 h)^2}{2!} y''(x_n) + \cdots \right] + \beta_0 \left[y^{(m)}(x_n) \right] \\
& + \beta_1 \left[y^{(m)}(x_n) + hy^{(m+1)}(x_n) + \frac{h^2}{2!} y^{(m+2)}(x_n) + \cdots \right] + \cdots \\
& + \beta_{(k-1)} \left[y^{(m)}(x_n) + (k-1) hy^{(m+1)}(x_n) + \frac{((k-1)h)^2}{2!} y^{(m+2)}(x_n) + \cdots \right] \\
& + \beta_k \left[y^{(m)}(x_n) + khy^{(m+1)}(x_n) + \frac{(kh)^2}{2!} y^{(m+2)}(x_n) + \cdots \right] \\
& + \lambda_1 \left[y^{(m+1)}(x_n) + hy^{(m+2)}(x_n) + \frac{h^2}{2!} y^{(m+3)}(x_n) + \cdots \right] + \cdots \\
& + \lambda_{(k-1)} \left[y^{(m+1)}(x_n) + (k-1) hy^{(m+2)}(x_n) + \frac{((k-1)h)^2}{2!} y^{(m+3)}(x_n) + \cdots \right] \\
& + \lambda_k \left[y^{(m+1)}(x_n) + khy^{(m+2)}(x_n) + \frac{(kh)^2}{2!} y^{(m+3)}(x_n) + \cdots \right] .
\end{aligned} \tag{2}$$

Rewriting (2) in matrix form $Ax = B$ by equating coefficients of $y^{(m)} x_n$ yields

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
j_1 h & j_2 h & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\frac{(j_1 h)^2}{2!} & \frac{(j_2 h)^2}{2!} & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
\frac{(j_1 h)^3}{3!} & \frac{(j_2 h)^3}{3!} & 0 & h & \cdots & kh & 1 & 1 & \cdots & 1 \\
\frac{(j_1 h)^4}{4!} & \frac{(j_2 h)^4}{4!} & 0 & \frac{h^2}{2!} & \cdots & \frac{(kh)^2}{2!} & 0 & h & \cdots & kh \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\frac{(j_1 h)^{(2k+3)}}{(2k+3)!} & \frac{(j_2 h)^{(2k+3)}}{(2k+3)!} & 0 & \frac{h^{(2k+1)}}{(2k+1)!} & \cdots & \frac{(kh)^{(2k+1)}}{(2k+1)!} & 0 & \frac{h^{(2k)}}{(2k)!} & \cdots & \frac{(kh)^{(2k)}}{(2k)!}
\end{pmatrix}
\begin{pmatrix}
\alpha_{j_q} \\ \alpha_{j_2} \\ \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \\ \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_k
\end{pmatrix}
=
\begin{pmatrix}
1 \\ k \\ \frac{(kh)^2}{2!} \\ \frac{(kh)^3}{3!} \\ \frac{(kh)^4}{4!} \\ \vdots \\ \frac{(kh)^{(2k+3)}}{(2k+3)!}
\end{pmatrix}.
$$

Adopting matrix inverse method, the values of the coefficients $\alpha_{j_1},\ \alpha_{j_2},\ \beta_0,\ \beta_1,\ \cdots,\ \beta_{k-1},\ \beta_k$ and $\lambda_0,\ \lambda_1,\ \cdots,\ \lambda_{k-1},\ \lambda_k$ are obtained and upon substitution back in (1) gives the desired initial discrete multi-step method.

However, additional schemes need to be derived at the remaining $j-$ grid points that α-values were not selected at, that is, $j_3,\ j_4,\ \cdots,\ j_k$. Hence additional schemes

$$
\begin{aligned}
y_{n+j_3} &= \sum_{j=0}^{k-1} \alpha_{j_n} y_{n+j} + \sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=0}^{k} \lambda_j g_{n+j}, \\
&\vdots \\
y_{n+j_k} &= \sum_{j=0}^{k-1} \alpha_{j_n} y_{n+j} + \sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=0}^{k} \lambda_j g_{n+j},
\end{aligned}
\tag{3}
$$

are derived using the same approach as the initial discrete multi-step method.

Similarly, first derivative multi-step methods are also derived at all grid points $x_n,\ x_{n+1},\ \cdots,\ x_{n+k}$, which produce the following schemes

$$
\begin{aligned}
y'_n &= \sum_{j=0}^{k-1} \alpha_{j_n} y_{n+j} + \sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=0}^{k} \lambda_j g_{n+j}, \\
y'_{n+1} &= \sum_{j=0}^{k-1} \alpha_{j_n} y_{n+j} + \sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=0}^{k} \lambda_j g_{n+j}, \\
&\vdots \\
y'_{n+k} &= \sum_{j=0}^{k-1} \alpha_{j_n} y_{n+j} + \sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=0}^{k} \lambda_j g_{n+j}.
\end{aligned}
\tag{4}
$$

All schemes derived in (1), (2) and (4) are then combined to a matrix equation form $Ax = B$ where the vector x is $\left(y_{n+1},\ y_{n+2},\ \cdots,\ y_{n+k}, y'_{n+1},\ y'_{n+2},\ \cdots,\ y'_{n+k}\right)^T$. This matrix is likewise solved using matrix inverse method and the resultant is the desired block method.

3. DERIVATION OF k-STEP THIRD DERIVATIVE BLOCK METHODS (TDBM) USING NEW LINEAR BLOCK APPROACH

The linear block approach for developing third derivative block methods for solving second order ordinary differential equations is given as

$$y_{n+\xi}=\sum_{i=0}^{1}\frac{(\xi h)^{i}}{i!}y_n^{(i)}+\sum_{i=0}^{k}\left(\phi_{\xi i}f_{n+i}+\tau_{\xi i}g_{n+i}\right),\ \xi=1,\ 2,\ \cdots,\ k, \tag{5}$$

with first derivative

$$y'_{n+\xi}=y'_n+\sum_{i=0}^{k}\left(\omega_{\xi i}f_{n+i}+\varphi_{\xi i}g_{n+i}\right),\ \xi=1,\ 2,\ \cdots,\ k,$$

$\left(\phi_{\xi 0},\ \phi_{\xi 1},\ \cdots,\phi_{\xi k},\ \tau_{\xi 0},\ \tau_{\xi 1},\ \cdots,\ \tau_{\xi k}\right)^T=A^{-1}B$ and

$\left(\omega_{\xi 0},\ \omega_{\xi 1},\ \cdots,\omega_{\xi k},\ \varphi_{\xi 0},\ \varphi_{\xi 1},\ \cdots,\ \varphi_{\xi k}\right)^T=A^{-1}D$.

$$A=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & h & 2h & \cdots & kh & 1 & 1 & 1 & \cdots & 1 \\ 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & \cdots & \frac{(kh)^2}{2!} & 0 & h & 2h & \cdots & kh \\ 0 & \frac{(h)^3}{3!} & \frac{(2h)^3}{3!} & \cdots & \frac{(kh)^3}{3!} & 0 & \frac{(h)^2}{2!} & \frac{(2h)^2}{2!} & \cdots & \frac{(kh)^2}{2!} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \frac{(h)^k}{k!} & \frac{(2h)^k}{k!} & \cdots & \frac{(kh)^k}{k!} & 0 & \frac{(h)^{(k-1)}}{(k-1)!} & \frac{(2h)^{(k-1)}}{(k-1)!} & \cdots & \frac{(kh)^{(k-1)}}{(k-1)!} \end{pmatrix},\ B=\begin{pmatrix} \frac{(\xi h)^m}{m!} \\ \frac{(\xi h)^{(m+1)}}{(m+1)!} \\ \frac{(\xi h)^{(m+2)}}{(m+2)!} \\ \vdots \\ \frac{(\xi h)^{(2k+3)}}{(2k+3)!} \end{pmatrix} \text{ and } D=\begin{pmatrix} \frac{(\xi h)^{(m-1)}}{(m-1)!} \\ \frac{(\xi h)^m}{m!} \\ \frac{(\xi h)^{(m+1)}}{(m+1)!} \\ \vdots \\ \frac{(\xi h)^{(2k+2)}}{(2k+2)!} \end{pmatrix}.$$

Substituting the coefficients $\phi_{\xi 0},\ \phi_{\xi 1},\ \cdots,\phi_{\xi k}$; $\tau_{\xi 0},\ \tau_{\xi 1},\ \cdots,\tau_{\xi k}$; $\omega_{\xi 0},\ \omega_{\xi 1},\ \cdots,\omega_{\xi k}$; and $\varphi_{\xi 0},\ \varphi_{\xi 1},\ \cdots,\ \varphi_{\xi k}$ back in (5) gives the required block method.

4. COMPUTATIONAL COMPLEXITY COMPARISON

In order to compare the computational complexity, the algorithms for both approaches will be the displayed and the number of calculations involved will be counted using the Big(O) notation for describing complexity of an algorithm. Algorithm 1 shows the steps involved in adopting the modified Taylor series approach for deriving the TDBM while Algorithm 2 shows the steps involved in implementing the new linear block approach.

Algorithm 1:

START

Step 1: Obtain the coefficients of the initial discrete multi-step scheme

$$y_{n+k}=\sum_{j=0}^{k-1}\alpha_{j_n}y_{n+j}+\sum_{j=0}^{k}\beta_j f_{n+j}+\sum_{j=0}^{k}\lambda_j g_{n+j}$$, where k is the step-number and $n=m$

Step 2: Obtain the coefficients of the additional schemes

$$y_{n+j_3}=\sum_{j=0}^{k-1}\alpha_{j_n}y_{n+j}+\sum_{j=0}^{k}\beta_j f_{n+j}+\sum_{j=0}^{k}\lambda_j g_{n+j},$$

$\vdots$

$$y_{n+j_k}=\sum_{j=0}^{k-1}\alpha_{j_n}y_{n+j}+\sum_{j=0}^{k}\beta_j f_{n+j}+\sum_{j=0}^{k}\lambda_j g_{n+j}.$$

Step 3: Derive the coefficients of the first derivative schemes

$$y'_n = \sum_{j=0}^{k-1} \alpha_{j_n} y_{n+j} + \sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=0}^{k} \lambda_j g_{n+j},$$
$$y'_{n+1} = \sum_{j=0}^{k-1} \alpha_{j_n} y_{n+j} + \sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=0}^{k} \lambda_j g_{n+j},$$
$$\vdots$$
$$y'_{n+k} = \sum_{j=0}^{k-1} \alpha_{j_n} y_{n+j} + \sum_{j=0}^{k} \beta_j f_{n+j} + \sum_{j=0}^{k} \lambda_j g_{n+j}.$$

Step 4: Combine schemes obtained in *Steps 1, 2, 3* above to form a system of equations with matrix form equivalent $Ax = B$ where $x = \left(y_{n+1},\ y_{n+2},\ \cdots,\ y_{n+k}, y'_{n+1},\ y'_{n+2},\ \cdots,\ y'_{n+k}\right)^T$.

Step 5: Adopt matrix inverse approach to the system of equations in *Step 4* to obtain the expected block method.

STOP

Table 1. *Computational complexity of algorithm 1*

Step	Computational complexity in Big(O) notation
Step 1	$O\{[2(k+2)]^2\} + O\{[2(k+2)]^3\}$
Step 2	$(k-2)(O\{[2(k+2)]^2\} + O\{[2(k+2)]^3\})$
Step 3	$(k+1)(O\{[2(k+2)]^2\} + O\{[2(k+2)]^3\})$
Step 4	-
Step 5	$k(O\{[2(k+1)]^2\} + O\{[2(k+1)]^3\})$
$\sum =$	$2k(O\{[2(k+2)]^2\} + O\{[2(k+2)]^3\}) + k(O\{[2(k+1)]^2\} + O\{[2(k+1)]^3\})$

Algorithm 2:

START

Step 1: Obtain the block method from the given expression

$$y_{n+\xi} = \sum_{i=0}^{1} \frac{(\xi h)^i}{i!} y_n^{(i)} + \sum_{i=0}^{k} \left(\phi_{\xi i} f_{n+i} + \tau_{\xi i} g_{n+i}\right),\ \xi = 1,\ 2,\ \cdots,\ k.$$

Step 2: Obtain the first derivative schemes of the block method from

$y'_{n+\xi} = y'_n + \sum_{i=0}^{k} \left(\omega_{\xi i} f_{n+i} + \varphi_{\xi i} g_{n+i}\right),\ \xi = 1,\ 2,\ \cdots,\ k$, where

$\left(\phi_{\xi 0},\ \phi_{\xi 1},\ \cdots, \phi_{\xi k},\ \tau_{\xi 0},\ \tau_{\xi 1},\ \cdots,\ \tau_{\xi k}\right)^T = A^{-1}B$ and

$\left(\omega_{\xi 0},\ \omega_{\xi 1},\ \cdots, \omega_{\xi k},\ \varphi_{\xi 0},\ \varphi_{\xi 1},\ \cdots,\ \varphi_{\xi k}\right)^T = A^{-1}D$. Matrices A, B and D are as given in Section 3 above.

STOP

Table 2. *Computational complexity of algorithm 2*

Step	Computational complexity in Big(O) notation
Step 1	$k(O\{[2(k+1)]^2\}+O\{[2(k+1)]^3\})$
Step 2	$k(O\{[2(k+1)]^2\}+O\{[2(k+1)]^3\})$
$\sum=$	$2k(O\{[2(k+1)]^2\}+O\{[2(k+1)]^3\})$

Note that the computational complexities of both algorithms were calculated using the knowledge of the computational complexity of taking the inverse of a $n \times n$ matrix being $O\left(n^3\right)$, while the computational complexity of the matrix multiplication of a $n \times s$ matrix with one $s \times p$ matrix is $O\left(nsp\right)$. Therefore, the computational complexity of developing third derivative block methods for solving second order differential equations using the modified Taylor series approach is $O\left\{\left[2\left(k+2\right)\right]^3\right\}$ while the computational complexity using the new linear block approach is $O\left\{\left[2\left(k+1\right)\right]^3\right\}$. This is obtained from the size of the largest matrices involved when adopting either of the approaches. Hence, it can be deduced that for any suitable value of k chosen, the computational complexity associated with adopting the modified Taylor series approach is greater than when using the new linear block approach.

5. DISCUSSION AND CONCLUSION

This paper has presented two different approaches for developing third derivative block methods for the solution of second order ordinary differential equations. The computational complexity test has shown that the modified Taylor series approach is more rigorous to adopt which is also evident by observation in the lengthy steps involved in the development of the methods. Hence, it can be said that the new linear block approach is more suitable when developing block methods of this form. Likewise, another novel contribution is the fact that this new linear block approach will develop any step third derivative block methods directly unlike the approach in [5] which is limited by specific interpolation points. Future work considers the adoption of the less computations approach for developing TDBM to solve second order ODEs directly. The obtained block method can be adopted to approximate both initial and boundary value problems.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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