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The Generalized Odd Lomax Generated Family of Distributions with Applications

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Abstract

Through this article, a new generated family of distributions under the name of "The generalized odd Lomax-G family" by adding three additional parameters to generalize any continuous baseline distribution is provided. For the generalized odd Lomax-G family main properties, stochastic orderings, entropy measures have been studied. Three special models have been discussed for the new family. By using the maximum likelihood method, The model parameters are estimated. Simulation is carried out for one of the sub-models to check the asymptotic behavior of the maximum likelihood estimates. We explained the efficiency of the new family by using four applications to the real world.

1. INTRODUCTION

Through dealing with the real world in several areas, it was found that the classical available distributions were not enough to model some types of data. So, there was a need to extend of these distributions. For these purposes, focus has been on new methods for building meaningful distributions see Lee et al. [1]. The Lomax distribution [2] counted as the most distributions for model lifetime data and has many practical applications in different fields, including business failure data, income and wealth data, reliability problems. Also, it has been applied as a replacement to some distributions when the data are heavy tailed. The Lomax distribution was used for modelling size spectra data in aquatic ecology [3]. Depending on the Lomax random variable a new class of distributions suggested by Cordeiro et al. [4] under the name of the Lomax generator of distributions. Recently, a very modern family has been proposed by Cordero et al. [5] with additional two parameters based on the Lomax random variable named the odd Lomax generator of distributions. Our motivations of this manuscript is to provide and study a new wider family of distributions by adding one shape parameter to the odd Lomax family [5] to generate a heavy-tailed distributions with left-skewed, right-skewed or reversed-J shape, and introduce distributions have all types of hazard rate functions which express of the nature and characteristics of the real life survival data and in general for modeling real-world data better than other models that were generated under the same baseline distribution.

The remainder of this paper has been arranged as follows. The new family of distributions is defined in Section 2. Three models from the new family are presented in Section 3. Main characteristics of the new family are presented in Section 4. Section 5, is devoted to the stochastic orderings. The Rènyi and the Shannon entropies presented In Sec. 6. The model parameters are estimated using the maximum likelihood method in Section 7. In Section 8, to evaluate the performance of maximum likelihood estimators, a

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simulation is stated. In Section 9, four applications are provided to prove the importance of the generalized odd Lomax-G family. Finally, Section 10, provided some concluding remarks.

2. THE NEW FAMILY

Cordeiro et al. [5] proposed the odd Lomax-G family with CDF in (1) and its corresponding PDF in (2) as follows:

$$F(x;\alpha,\beta,\boldsymbol{\psi})=\alpha\beta^{\alpha}\int_{0}^{\frac{G(x:\boldsymbol{\psi})}{1-G(x:\boldsymbol{\psi})}}(\beta+t)^{-\alpha-1}dt=1-\beta^{\alpha}\left[\beta+\frac{G(x:\boldsymbol{\psi})}{1-G(x:\boldsymbol{\psi})}\right]^{-\alpha} \tag{1}$$

and

$$f(x;\alpha,\beta,\boldsymbol{\psi})=\alpha\beta^{\alpha}g(x;\boldsymbol{\psi})[1-G(x:\boldsymbol{\psi})]^{-2}\left[\beta+\frac{G(x:\boldsymbol{\psi})}{1-G(x:\boldsymbol{\psi})}\right]^{-(\alpha+1)},t\geq 0. \tag{2}$$

Such that $\beta>0$ and $\alpha>0$ are the scale and shape parameter, respectively, and $g(x,\boldsymbol{\psi})$ and $G(x,\boldsymbol{\psi})$ are the probability and cumulative functions of the baseline distribution with parameter vector $\boldsymbol{\psi}$.

In this section, we will propose a generalization of the above family by adding one shape parameter $\theta>0$. Thus, the CDF and its corresponding PDF for the proposed family are shown as

$$F(x;\theta,\alpha,\beta,\boldsymbol{\psi})=1-\left[1+\frac{G(x;\boldsymbol{\psi})^{\theta}}{\beta(1-G(x;\boldsymbol{\psi})^{\theta})}\right]^{-\alpha}, \tag{3}$$

and

$$f(x;\theta,\alpha,\beta,\boldsymbol{\psi})\ =\alpha\theta\beta^{-1}g(x;\boldsymbol{\psi})G(x;\boldsymbol{\psi})^{\theta-1}\left[1-G(x;\boldsymbol{\psi})^{\theta}\right]^{-2}\left[1+\frac{G(x;\boldsymbol{\psi})^{\theta}}{\beta[1-G(x;\boldsymbol{\psi})^{\theta}]}\right]^{-\alpha-1}, \tag{4}$$

respectively.

This family will be known as the generalized odd Lomax-G (GOLx-G) family. Sometimes, we will delete the parameters and write $G(x)^{\theta}$ instead of $G(x;\boldsymbol{\psi})^{\theta}$, $g(x)$ instead of $g(x;\boldsymbol{\psi})$, $F(x)$ instead of $F(x;\theta,\alpha,\beta,\boldsymbol{\psi})$ and $f(x)$ instead of $f(x;\theta,\alpha,\beta,\boldsymbol{\psi})$, also, we write $X\sim GOLx-G(\theta,\alpha,\beta,\boldsymbol{\psi})$ if X has the PDF (4). The following some choices of specified values of the parameters yields five particular cases of the GOLx-G family as seen through the Table 1. For the GOLx-G family, we find the hazard rate function (hrf) is

$$h(x;\theta,\alpha,\beta,\boldsymbol{\psi})=\frac{\alpha}{\beta+(1-\beta)G(x)^{\theta}}\times\tau(x;\theta,\boldsymbol{\psi}), \tag{5}$$

where $\tau(x;\theta,\boldsymbol{\psi})$ is the hrf of the exp-G distribution and $\frac{\alpha}{\beta+(1-\beta)G(x)^{\theta}}$ is a correction factor. We can get the quantile function of the new family by inverting (3), and hence

$$Q_X(u)=Q_G\left[\left(\frac{\beta\left[1-(1-u)^{1/\alpha}\right]}{\beta-(\beta-1)(1-u)^{1/\alpha}}\right)^{\frac{1}{\theta}}\right], \tag{6}$$

where $Q_G(u)=G^{-1}(u)$ is the qf of the parent distribution and $u\in(0,1)$.

Table 1. *Some Special Models*

α	β	θ	Reduced Model	Authors
-	-	1	The OLx-G family	[5]
1	-	-	The MO exp-G family	-
1	-	1	The MO-G family	[6]
1	1	-	The (phr) model	[7]
-	1	1	The (prhr) model	[8]
1	1	1	The baseline model G	-

3. SOME SPECIAL DISTRIBUTIONS

3.1. The Generalized Odd Lomax Uniform Distribution (GOLxU)

Let the parent distribution be uniform with PDF and CDF functions given by $g(x;\gamma)=1/\gamma$, and $G(x;\gamma)=x/\gamma$, respectively, where $\gamma>0$. Then, the PDF of the GOLxU distribution given as follows

$$f(x;\gamma,\theta,\alpha,\beta)=\frac{\alpha\theta\beta^{\alpha}x^{\theta-1}\left[1-(x/\gamma)^{\theta}\right]^{\alpha-1}}{\gamma^{\theta}[(x/\gamma)^{\theta}+\beta(1-(x/\gamma)^{\theta})]^{1+\alpha}}.$$

Figure 1 shows a wealth of possible shapes of the distribution once different choices of the parameters are made. For example, the shape can be left skewed, right skewed, reversed-J shape or symmetrical. Also, Figure 2 reveals that the hrf of the GOLxU distribution can be increasing, bathtub, upside-down or upside-down and bathtub shapes.

3.2. The Generalized Odd Lomax Lomax Distribution (GOLxLx)

If the parent distribution is the Lomax distribution with PDF and CDF functions $g(x;\gamma,\lambda)=\frac{\lambda}{\gamma}(x/\gamma+1)^{-\lambda-1}$, and $G(x;\gamma,\lambda)=1-(x/\gamma+1)^{-\lambda}$, respectively, where $\lambda,\gamma>0$. Then, the PDF of the GOLxLx distribution is

$$f(x;\gamma,\lambda,\theta,\alpha,\beta)=\frac{\alpha\theta\lambda\beta^{\alpha}\left(\frac{x}{\gamma}+1\right)^{-(\lambda+1)}\left[1-\left[1-\left(\frac{x}{\gamma}+1\right)^{-\lambda}\right]^{\theta}\right]^{\alpha-1}\left[1-\left(\frac{x}{\gamma}+1\right)^{-\lambda}\right]^{\theta-1}}{\gamma\left[\beta-(\beta-1)\left[1-\left(\frac{x}{\gamma}+1\right)^{-\lambda}\right]^{\theta}\right]^{1+\alpha}}.$$

Figure 3 shows that the density of GOLxLx can be right skewed or reversed-J shape. Also, Figure 4 shows that the hrf can be decreasing or upside-down bathtub and unimodal shape.

3.3. The Generalized Odd Lomax Weibull Distribution (GOLxW)

If the parent distribution is the Weibull distribution with PDF and CDF functions $g(x;\gamma,\lambda)=\lambda\gamma x^{\gamma-1}e^{-\lambda x^{\gamma}}$ and $G(x;\gamma,\lambda)=1-e^{-\lambda x^{\gamma}}$, respectively, where $\lambda,\gamma>0$ and $x\geq 0$. Then, the GOLxW distribution has PDF given by

$$f(x;\gamma,\lambda,\theta,\alpha,\beta)=\frac{\alpha\theta\lambda\gamma\,\beta^{\alpha}x^{\gamma-1}e^{-\lambda x^{\gamma}}\left[1-e^{-\lambda x^{\gamma}}\right]^{\theta-1}\left[1-\left(1-e^{-\lambda x^{\gamma}}\right)^{\theta}\right]^{\alpha-1}}{[\beta(1-(1-e^{-\lambda x^{\gamma}})^{\theta})+(1-e^{-\lambda x^{\gamma}})^{\theta}]^{\alpha+1}}.$$

Figure 5 shows that the density of GOLxW can be right skewed, left skewed, reversed-J or symmetrical shape. Figure 6 shows that the hrf can be decreasing, increasing or upside-down bathtub and unimodal shape. Seventeen special models are listed in Table 2 for the GOLxW model including well known distributions discussed and studied in the literature.

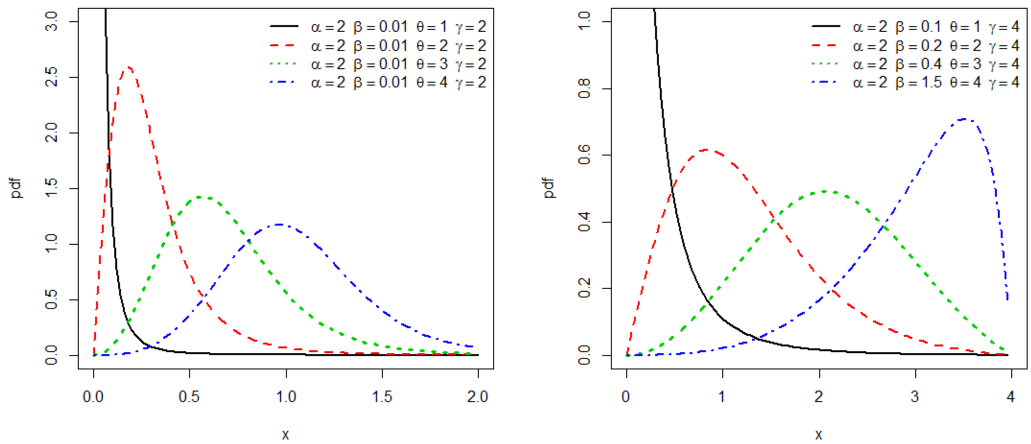


Figure 1. *PDF plots of the GOLxU distribution*

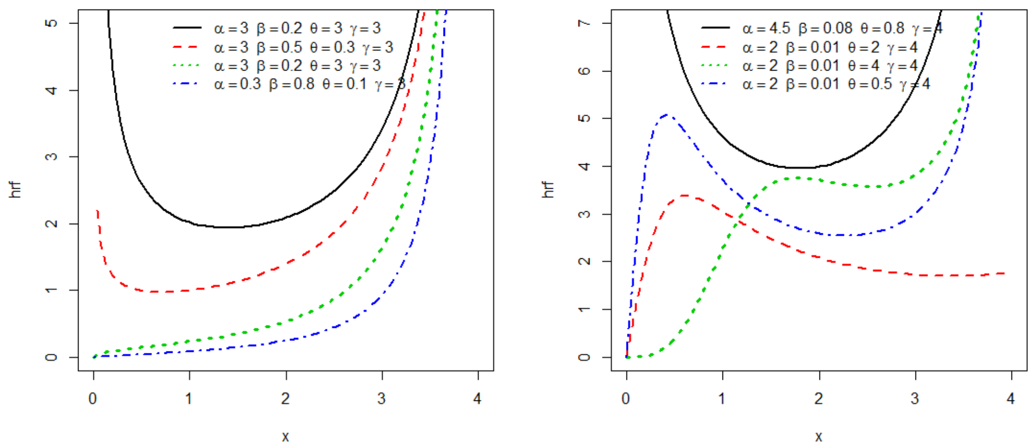


Figure 2. *Hazard rate plots of the GOLxU distribution*

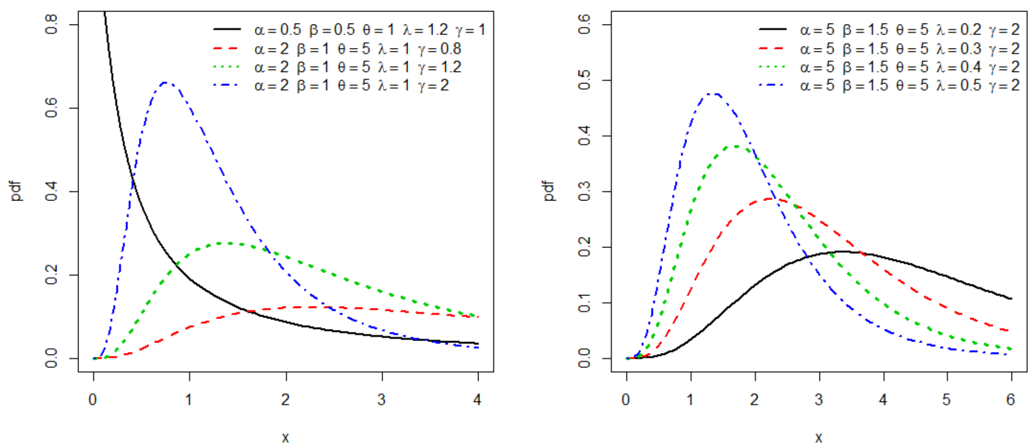


Figure 3. *PDF plots of the GOLxLx distribution*

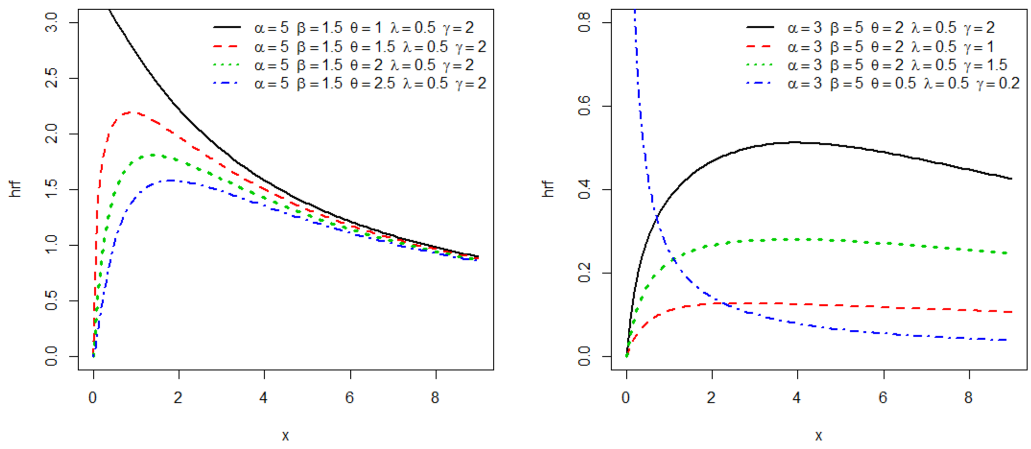


Figure 4. *Hazard rate plots of the GOLxLx distribution*

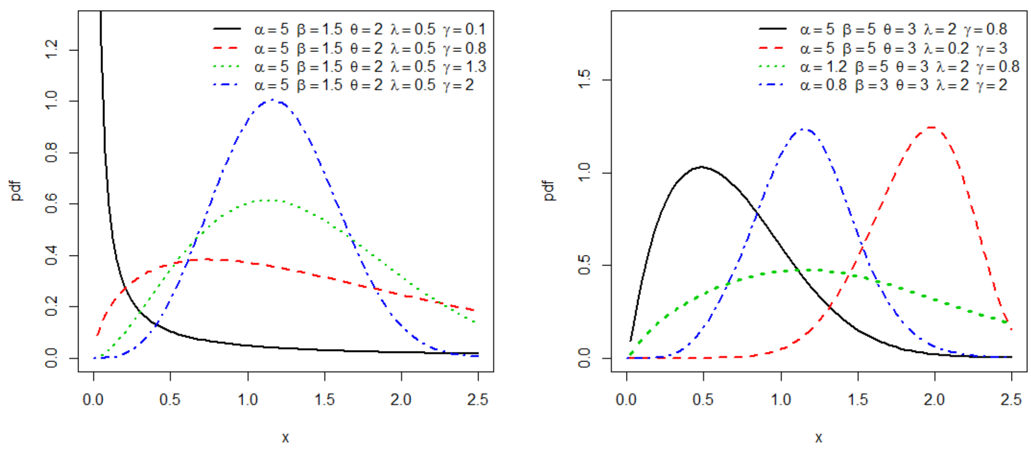


Figure 5. *PDF plots of the GOLxW distribution*

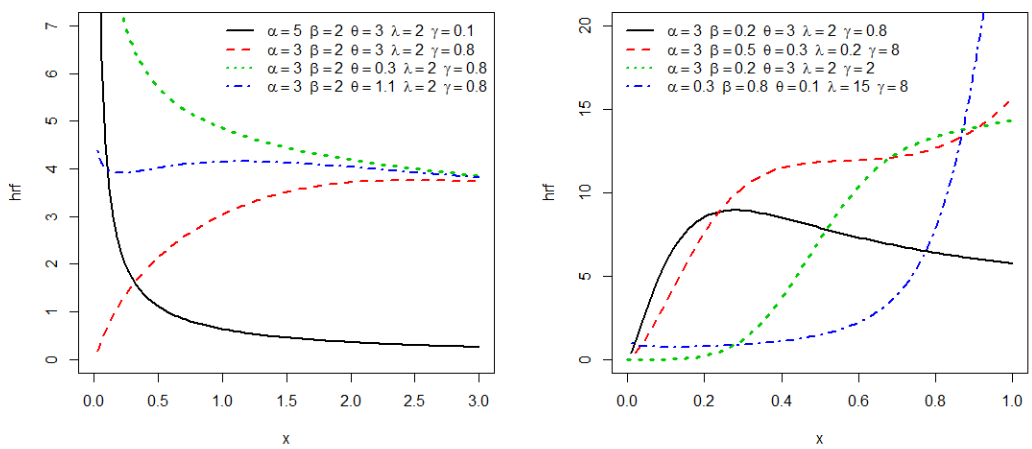


Figure 6. *Hazard rate plots of the GOLxW distribution*

4. MAIN PROPERTIES

4.1. Asymptotics

Proposition 1. If $x \to -\infty$, then

$F(x) \sim 1 - \left(1 + G(x)^{\theta}/\beta\right)^{-\alpha}$.

$f(x) \sim \alpha\theta\beta^{\alpha} g(x) G(x)^{\theta-1} / \left(\beta + G(x)^{\theta}\right)^{1+\alpha}$.

$h(x) \sim \alpha\theta G(x)^{\theta-1} g(x) / \left(\beta + G(x)^{\theta}\right)$.

Proposition 2. If $x \to \infty$, then

$1 - F(x) \sim \alpha\beta\bar{G}(x)^{\theta}$.

$f(x) \sim \alpha\theta\beta g(x) G(x)^{\theta-1}$.

$h(x) \sim \theta g(x) G(x)^{\theta-1} / \bar{G}(x)^{\theta}$.

Table 2. *Sub-models of the GOLxW*$(\theta, \alpha, \beta, \lambda, \gamma)$

S.N.	α	β	θ	λ	γ	Reduced Distribution	Authors
1	-	-	-	-	2	The generalized OLxRay	New
2	-	-	-	-	1	The generalized OLxExp	New
3	-	-	1	-	-	The OLxW	[5]
4	-	-	1	-	2	The OLxRay	[5]
5	-	-	1	-	1	The OLxExp	[5]
6	1	-	-	-	-	The MO-EEW	[9]
7	1	-	-	-	2	The MO- EGRay	-
8	1	-	-	-	1	The MO-EGExp	-
9	1	-	1	-	-	The MO-EW	[10]
10	1	-	1	-	2	The MO-ERay	[10]
11	1	-	1	-	1	The MO-EExp	[10]
12	1	1	-	-	-	The exp-W	[11]
13	1	1	-	-	2	The GRay or the Burr Type X	[12]
14	1	1	-	-	1	The GExp	[13]
15	1	1	1	-	-	The W distribution	[14]
16	1	1	1	-	2	The Ray distribution	[15]
17	1	1	1	-	1	The Exp distribution	-

4.2. Linear Representation

From the binomial expansion, we can write the PDF (4) as follows

$$f(x) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{j}\alpha\theta\beta^{-i-1}\binom{-(i+2)}{j}\binom{-(\alpha+1)}{i} g(x) G(x)^{\theta(1+i+j)-1}, \tag{7}$$

since

$$G(x)^{\theta(1+i+j)-1} = \sum_{k=0}^{\infty}\sum_{l=0}^{k}(-1)^{k+l}\binom{k}{l}\binom{\theta(1+i+j)-1}{k} G(x)^{l}, \tag{8}$$

By changing the sums over l and k in (8), and inserting (8) in (7), then we can write the PDF of the GOLx-G family as below:

$$f(x)=\sum_{l=0}^{\infty} a_l\, h_{l+1}(x), \tag{9}$$

where

$$a_l=\sum_{i,j=0}^{\infty}\sum_{k=l}^{\infty}\frac{(-1)^{k+j+l}\alpha\theta\binom{k}{l}\binom{-(i+2)}{j}\binom{-(1+\alpha)}{i}\binom{\theta(1+i+j)-1}{k}}{\beta^{i+1}(l+1)},$$

and

$$h_{l+1}(x)=(l+1)g(x)\,G(x)^{l}.$$

By integrating Eq. (9), the CDF of X becomes

$$F(x)=\sum_{l=0}^{\infty} a_l\, H_{l+1}(x), \tag{10}$$

where $H_{l+1}(x)=G(x)^{l+1}$.

4.3. Ordinary and Incomplete Moments

Suppose $X_{l+1}\sim h_{l+1}(x)$, then the nth moment for random variable X obtained as

$$\mu_n'=E(X^n)=\sum_{l=0}^{\infty} a_l\; E(X_{l+1}^{n}). \tag{11}$$

The 2nd alternative formula of μ_n' follows from Eq. (9) in expression of the baseline qf. $Q_G(u)$ is

$$\mu_n'=\sum_{l=0}^{\infty}(l+1)a_l\,\tau(n,l), \tag{12}$$

where $\tau(n,l)=\int_0^1 Q_G(u)^n u^l du$.

Cordeiro and Nadarajah [16] determined $\tau(n,l)$ for some known and important distributions and therefore can be used. By the incomplete moment we can get many measurements. From Eq. (9) the rth incomplete moment of X is

$$\eta_r(y)=E(X^r|X<y)=\sum_{l=0}^{\infty}(l+1)a_l\int_0^{G(y,\psi)} Q_G(u)^r\, u^l du\,. \tag{13}$$

We can compute the above integral in (13) analytically or numerically.

4.4. Moment Generating Function

In this subsection different forms will be provided for the (MGF) $M(t)=E(e^{tX})$ of the random variable X. The first one is

$$M(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mu_n' , \tag{14}$$

where μ_n' is calculated from (11) or (12). The second form for $M(t)$ comes from (9) as

$$M(t) = \sum_{k=0}^{\infty} a_l \, M_{X_{l+1}}(t), \tag{15}$$

where $M_{X_{l+1}}(t)$ is the mgf of the random variable X_{l+1}~exp-G $(l+1)$.

The 3rd form for $M(t)$ employed from (9) as

$$M(t) = \sum_{l=0}^{\infty} (l+1) a_l \int_{-\infty}^{\infty} e^{tx} g(x) \, G(x)^l dx = \sum_{l=0}^{\infty} (l+1) a_l \, \rho(t,l), \tag{16}$$

where

$$\rho(t,l) = \int_0^1 exp\big(tQ_G(u)\big) \, u^l \, du,$$

and it be calculated from the baseline qf $Q_G(u) = G^{-1}(u)$. Table 3 displays mean, variance, skewness and kurtosis of the GOLxW distribution for some choices values of the parameters.

Table 3. *The calculated values of mean, variance, skewness and kurtosis for the GOLxW distribution with* $\lambda = 1.25$, $\gamma = 1.5$ *and different values of* θ, α *and* β

θ	α	β	Mean	Variance	Skewness	Kurtosis
1.5	1	1	0.1536	0.1241	2.3876	8.2356
		1.5	0.1560	0.1370	2.4423	8.3805
		3	0.1501	0.1524	2.6606	9.3948
		6	0.1333	0.1559	3.0460	11.6147
		9	0.1201	0.1520	3.3568	13.6759
		20	0.0914	0.1333	4.1846	20.2181
		50	0.0606	0.1018	5.5814	34.5906
		100	0.0419	0.0772	7.0700	54.4495
2	1.5	1	0.5798	0.2079	0.3880	2.6462
		1.5	0.5995	0.2594	0.3877	2.3776
		3	0.6105	0.3609	0.4828	2.1191
		6	0.5919	0.4669	0.6729	2.1228
		9	0.5687	0.5233	0.8177	2.2557
		20	0.5038	0.6065	1.1610	2.8365
		50	0.4129	0.6406	1.6404	4.1382
		100	0.3431	0.6237	2.0666	5.7217

We note from Table 3 that the skewness of the GOLxW distribution is always possitive, whereas the kurtosis of the GOLxW distribution varies only in the interval (2.12, 54.45).

4.5. Order Statistics

By taking a random sample say, $X_1, \ X_2, \dots, X_n$ from GOLx-G family and $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ are the order statistics for this sample. Let $X_{i:n}$ is ith order statistic, then the PDF of $X_{i:n}$ is

$$f_{i:n}(x) = \sum_{j=0}^{n-i} K(j)\, f(x) F(x)^{j+i-1}, \tag{17}$$

where $K(j) = \frac{(-1)^j\, n!}{j!(i-1)!(n-i-j)!}$.

From (3) and (4) and applying the generalized binomial expansion, we find that

$$F(x)^{i+j-1}\, f(x) = \sum_{k=0}^{i+j-1} \sum_{r,s=0}^{\infty} (-1)^{k+s} \theta\alpha\beta^{-r-1} \binom{i+j-1}{k} \binom{-r-2}{s} \binom{-1-\alpha(k+1)}{r} \times g(x) G(x)^{\theta(1+r+s)-1}. \tag{18}$$

By inserting (18) in (17), we find the probability density function of $X_{i:n}$ is

$$f_{i:n}(x) = \sum_{r,s=0}^{\infty} \gamma_{r,s}\, h_{\theta(r+s+1)}(x), \tag{19}$$

where $h_{\theta(r+s+1)}(x) = \theta(r+s+1) g(x) G(x)^{\theta(r+s+1)-1}$, and

$$\gamma_{r,s} = \sum_{j=0}^{n-i} \sum_{k=0}^{i+j-1} \frac{(-1)^{k+s+j} \alpha\, n! \binom{i+j-1}{k} \binom{-r-2}{s} \binom{-\alpha(k+1)-1}{r}}{\beta^{r+1} (r+s+1)\, j!\, (i-1)!\, (n-i-j)!}.$$

Based on (19), we can get many statistical properties of the order statistic.

5. STOCHASTIC ORDERINGS

It is said that the random variable X is less than the random variable Y in the

1. Usual stochastic order, if $F_X(x) \geq F_Y(x)\ \forall x$, and denoted by $X \leq_{st} Y$.
2. Hazard rate order, if $h_X(x) \geq h_Y(x)\ \forall x$, and denoted by $X \leq_{hr} Y$.
3. Reversed hazard rate order, if $F_X(x)/F_Y(x)$ is decreases in x, and denoted by $X \leq_{rh} Y$.
4. Mean residual life order, if $m_X(x) \leq m_Y(x)\ \forall x$, and denoted by $X \leq_{mrl} Y$.
5. Likelihood ratio order, if $f_X(x)/f_Y(x)$ is decreases in x, and this is expressed by the symbol $X \leq_{lr} Y$.

For all the previous orders we have the following chains of implications:

- $X \leq_{lr} Y$ *imply that* $X \leq_{hr} Y$ *imply that* $X \leq_{st} Y$.
- $X \leq_{lr} Y$ *imply that* $X \leq_{rh} Y$ *imply that* $X \leq_{st} Y$.
- $X \leq_{lr} Y$ *imply that* $X \leq_{mrl} Y$.

For our family, the following Theorem and corollaries provides the stochastic comparison results with respect to the above orderings.

Theorem 1 : Let $X \sim GOLx - G(\theta, \alpha_1, \beta_1)$ and $Y \sim GOLx - G(\theta, \alpha_2, \beta_2)$. If $\alpha_1 \geq \alpha_2$ and $\beta_2 \geq \beta_1$, then $X \leq_{lr} Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \left(\frac{\alpha_1 {\beta_1}^{\alpha_1}}{\alpha_2 {\beta_2}^{\alpha_2}}\right)\left(\frac{\left(\beta_2 \bar{G}(x)^\theta + G(x)^\theta\right)^{\alpha_2+1}}{(\beta_1 \bar{G}(x)^\theta + G(x)^\theta)^{\alpha_1+1}}\right)\left(1 - G(x)^\theta\right)^{\alpha_1-\alpha_2},$$

where $\bar{G}(x)^\theta = 1 - G(x)^\theta$.

Thus,

$$log\left(\frac{f_X(x)}{f_Y(x)}\right) = log\left(\frac{\alpha_1 {\beta_1}^{\alpha_1}}{\alpha_2 {\beta_2}^{\alpha_2}}\right) + (\alpha_2 + 1)log\left(G(x)^\theta + \beta_2 \bar{G}(x)^\theta\right) - (\alpha_1 + 1)log\left(G(x)^\theta + \beta_1 \bar{G}(x)^\theta\right) + (\alpha_1 - \alpha_2)log\left(1 - G(x)^\theta\right).$$

By differentiating the last Eq. and after some simplifications, we get

$$\frac{d}{dx}\left[log\left(\frac{f_X(x)}{f_Y(x)}\right)\right] = \left[(\alpha_2 - \alpha_1)G(x)^\theta + (\beta_1 - \beta_2)\bar{G}(x)^\theta + (\alpha_2\beta_1 - \alpha_1\beta_2)\bar{G}(x)^\theta\right] \times \frac{\theta g(x)G(x)^{-1+\theta}}{(G(x)^\theta + \beta_1 \bar{G}(x)^\theta) \times \bar{G}(x)^\theta \times (G(x)^\theta + \beta_2 \bar{G}(x)^\theta)},$$

therefore, when $\alpha_1 \geq \alpha_2$ and $\beta_2 \geq \beta_1$, then $\frac{d}{dx}\left[log\left(\frac{f_X(x)}{f_Y(x)}\right)\right] < 0$ for all x. This implies that $X \leq_{lr} Y$.

Corollary 1 : Let $X \sim GOLx - G(\theta, \alpha, \beta_1)$ and $Y \sim GOLx - G(\theta, \alpha, \beta_2)$. If $\beta_2 \geq \beta_1$, then $X \leq_{lr} Y$.

Corollary 2 : Let $X \sim GOLx - G(\theta, \alpha_1, \beta)$ and $Y \sim GOLx - G(\theta, \alpha_2, \beta)$. If $\alpha_1 \geq \alpha_2$, then $X \leq_{lr} Y$.

From the previous evidence, we find that the GOLx-G family has the strongest ordering (likelihood ratio order) under some restrictions for the parameters.

6. RENYI AND SHANNON ENTROPY

For the random variable X the Rènyi entropy is defined as

$$I_R(\delta) = \frac{1}{1-\delta} log \int_{-\infty}^{\infty} f(x)^\delta dx, \tag{20}$$

where $\delta > 0$ and $\delta \neq 1$.

Based on the expansion of the generalized binomial, we have

$$f(x)^\delta = \sum_{k,l=0}^{\infty} \Pi_{k,l}\, g(x)^\delta G(x)^{(\delta+k+l)\theta-\delta}, \tag{21}$$

where

$$\Pi_{l,k} = (-1)^l (\alpha\theta)^\delta \beta^{-(k+\delta)} \binom{-k-2\delta}{l}\binom{-\delta(\alpha+1)}{k}.$$

By combining (20) and (21), we can get of the Rènyi entropy of the GOLx-G family in the form of

$$I_R(\delta) = \frac{1}{1-\delta} log \left[\sum_{k,l=0}^{\infty} \Pi_{k,l} \int_{-\infty}^{\infty} g(x)^{\delta} G(x)^{\theta(\delta+k+l)-\delta} dx \right].$$

The Shannon entropy can be obtained by $\lim_{r\to 1} I_R(r)$.

7. ESTIMATION

Let $x_1, x_2, \dots, x_n$ are the observed values from the GOLx-G family with parameters θ, α, β and $\boldsymbol{\psi}$. Let $\boldsymbol{\Phi} = (\theta, \alpha, \beta, \boldsymbol{\psi})^T$ be the parameters vector, then $l(\boldsymbol{\Phi}) = \log L(\boldsymbol{\Phi})$ where $L(\boldsymbol{\Phi})$ is the likelihood function is given by

$$l(\boldsymbol{\Phi}) = n\, log\alpha + n\, log\theta + \alpha n\, log\beta + \sum_{i=1}^{n} log g(x_i;\boldsymbol{\psi}) + (\theta-1)\sum_{i=1}^{n} log G(x_i;\boldsymbol{\psi}) + (\alpha-1)\sum_{i=1}^{n} log\big(1-G(x_i;\boldsymbol{\psi})^{\theta}\big) - \sum_{i=1}^{n} (\alpha+1) log\big(\beta + (1-\beta)G(x_i;\boldsymbol{\psi})^{\theta}\big).$$

The score vector is $U(\boldsymbol{\Phi}) = \left(U_\theta = \frac{\partial l(\boldsymbol{\Phi})}{\partial\theta}, U_\alpha = \frac{\partial l(\boldsymbol{\Phi})}{\partial\alpha}, U_\beta = \frac{\partial l(\boldsymbol{\Phi})}{\partial\beta}, U_{\boldsymbol{\psi_k}} = \frac{\partial l(\boldsymbol{\Phi})}{\partial\boldsymbol{\psi_k}}\right)^T$, therefore its components are given by

$$U_\theta = \frac{n}{\theta} + \sum_{i=1}^{n} log G(x_i;\boldsymbol{\psi}) - (\alpha-1)\sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\psi})^{\theta}\, log G(x_i;\boldsymbol{\psi})}{1-G(x_i;\psi)^{\theta}} - (\alpha+1)\sum_{i=1}^{n} \frac{(1-\beta)G(x_i;\boldsymbol{\psi})^{\theta} log G(x_i;\boldsymbol{\psi})}{\beta+(1-\beta)G(x_i;\boldsymbol{\psi})^{\theta}}.$$

$$U_\alpha = \frac{n}{\theta} + \alpha n\, log\beta + \sum_{i=1}^{n} log\big(1-G(x_i;\boldsymbol{\psi})^{\theta}\big) - \sum_{i=1}^{n} log\big(\beta+(1-\beta)G(x_i;\boldsymbol{\psi})^{\theta}\big).$$

$$U_\beta = \frac{\alpha n}{\theta} - \sum_{i=1}^{n} \frac{(\alpha+1)\big(1-G(x_i;\boldsymbol{\psi})^{\theta}\big)}{\beta+(1-\beta)G(x_i;\boldsymbol{\psi})^{\theta}}.$$

and

$$U_{\boldsymbol{\psi_k}} = \sum_{i=1}^{n} \frac{\partial g(x_i,\boldsymbol{\psi})/\partial\boldsymbol{\psi_k}}{g(x_i;\boldsymbol{\psi})} + (\theta-1)\sum_{i=1}^{n} \frac{\partial G(x_i,\boldsymbol{\psi})/\partial\boldsymbol{\psi_k}}{G(x_i;\psi)} - \theta\left[\sum_{i=1}^{n} \frac{(\alpha-1)G(x_i;\boldsymbol{\psi})^{\theta-1}\,\partial G(x_i,\boldsymbol{\psi})/\partial\boldsymbol{\psi_k}}{1-G(x_i;\boldsymbol{\psi})^{\theta}} + \sum_{i=1}^{n} \frac{(1-\beta)(\alpha+1)G(x_i;\boldsymbol{\psi})^{\theta-1}\,\partial G(x_i,\boldsymbol{\psi})/\partial\boldsymbol{\psi_k}}{\beta+(1-\beta)G(x_i;\boldsymbol{\psi})^{\theta}}\right].$$

By equating $U_\alpha, U_\beta, U_\theta$ and $U_{\boldsymbol{\psi}}$ to zero and solve the equations simultaneously we can get the MLEs $\widehat{\boldsymbol{\Phi}} = \big(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \widehat{\boldsymbol{\psi}}\big)^T$. We can use the statistical software to solve these equations.

8. SIMULATION STUDY

Through current section, a numerical study will be implemented to calculate the performance of the MLEs of GOLxW parameters. The random observations from GOLxW distribution are generated by using the quantile function that obtained from the cdf of GOLxW distribution. 1000 samples of size n =20, 50, 150 and 300 are generated. Performance of estimates is evaluated based on their bias of the MLEs of the model parameters, the mean squared error (MSE) of the MLEs. The empirical study was conducted with software Mathematica 2010 and the results are given in Table 4. It is observed from Table 4 and Figure 7 that the biases and MSEs decrease as n increases. The simulation study shows that the maximum likelihood method is appropriate for estimating the parameters of the GOLxW distribution.

Table 4. *Biases and MSEs for the MLEs of the parameters of the GOLxW distribution*

n	Par.	Initial value	Bias	MSE	Initial value	Bias	MSE
20	α	1	0.7802	0.0682	0.5	0.6989	0.0679
	β	2.5	1.0693	0.0600	1.5	1.6180	0.0552
	θ	1.5	1.8286	0.0594	0.5	1.9912	0.0808
	λ	1.5	1.1411	0.0612	2.5	0.6511	0.0686
	γ	2.5	1.4106	0.0691	2	0.6666	0.0691
50	α	1	0.5404	0.0422	0.5	0.6631	0.0382
	β	2.5	0.8098	0.0219	1.5	1.5894	0.0414
	θ	1.5	1.8123	0.0370	0.5	1.7858	0.0504
	λ	1.5	1.0739	0.0438	2.5	0.6188	0.0429
	γ	2.5	1.2021	0.0636	2	0.6174	0.0446
150	α	1	0.5021	0.0163	0.5	0.5131	0.0203
	β	2.5	0.7999	0.0223	1.5	1.6027	0.0150
	θ	1.5	1.7688	0.0327	0.5	1.6833	0.0343
	λ	1.5	1.0083	0.0189	2.5	0.6238	0.0311
	γ	2.5	1.0652	0.0229	2	0.6422	0.0183
300	α	1	0.4638	0.0082	0.5	0.5901	0.0054
	β	2.5	0.7146	0.0100	1.5	1.5699	0.0098
	θ	1.5	1.5566	0.0194	0.5	1.6863	0.0203
	λ	1.5	0.8341	0.0112	2.5	0.4823	0.0079
	γ	2.5	0.5733	0.0079	2	0.6400	0.0084

9. DATA ANALYSIS

In this section, we prove the efficiency and flexibility of the GOLx-G family empirically by providing three real data sets. Our special model GOLxW, GOLxLx and GOLxU will compared with some competitive models (LxMW) by [17], (WLx) by [18] and (ELx) by [19]. The parameters of each model are estimated by the method of maximum likelihood using (L-BFGS-B) method and the goodness-of-fit statistics AIC (Akaike information criterion), BIC (Bayesian information criterion), A* (Anderson-Darling), W* (Cramér–von Mises), KS (Kolmogorov Smirnov with its p-value (PV)) are used to compare the five models. Where $\hat{l}$ is the maximized log-likelihood. The smaller values of these statistics are the best to the data. The R-script has been used to implement all computations. The first data is the taxes in Egypt from 2006 to 2010. This data used in [20]. The second data is the strengths of 1.5 cm glass fibres and obtained by the UK (NPL) and has been used by [21]. The third data set was analyzed in [22]. This data is a carbon fibre tensile resistance that have been tested with measured lengths 20 mm. Tables 5, 7 and 9 lists the maximum likelihood estimates of the model parameters and its corresponding standard errors. The numerical values of the AIC, BIC, A* and W* statistics are displayed in Tables 6, 8 and 10. The results indicate that the GOLx-G family provides the best fit as compared to the other models. Moreover, we apply the considered sub-models of the family to the fourth censored data set, where the data consist of death times (in weeks) of patients with cancer of tongue with aneuploidy DNA profile (see [23]). As it can be seen from Table 11, the family models are good competitor to the compared models. Noting that goodness-of-fit statistics computations have not been developed much for censored data, but the quality of fit can be checked by Akaike and Bayesian information criteria (AIC and BIC), see [24].

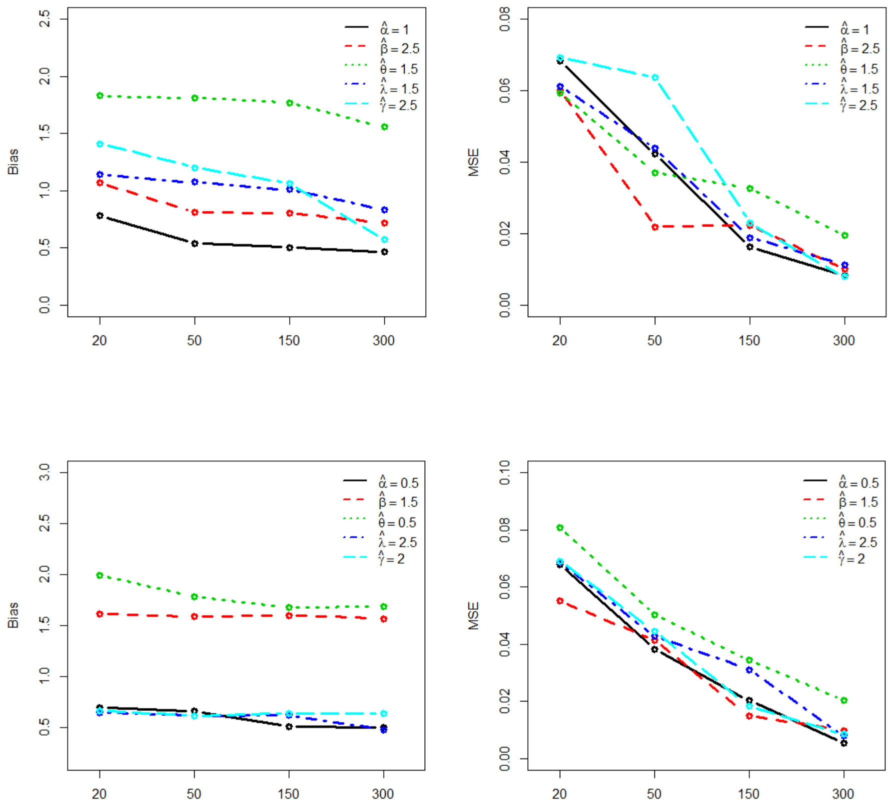


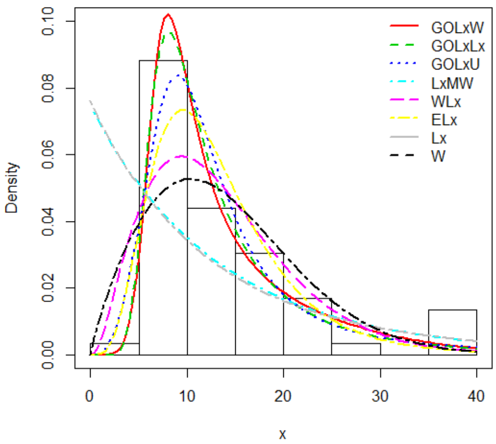
Figure 7. *Graph for Biases and MSEs for the MLEs of the parameters of the GOLxW distribution given in Table 4*

Table 5. *The parameters estimation and its standard errors for data 1*

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}$
GOLxW	0.3738 (0.1675)	0.0583 (0.0311)	5.8193 (4.7474)	0.0446 (0.0221)	1.4807 (0.5640)
GOLxLx	0.5379 (0.2140)	0.3838 (0.1951)	08.1610 (03.1414)	0.0215 (0.0201)	15.0326 (2.1580)
GOLxU	0.6850 (0.0545)	0.0029 (0.0010)	3.7238 (0.1005)	-	46.0177 (23.4322)
LxMW	0.0001 (0.0023)	0.0002 (0.0064)	0.0092 (0.0020)	13.7060 (0.0064)	0.0072 (0.0009)
WLx	10.4253 (12.6150)	3.7294 (1.4547)	0.1962 (0.1018)	1.8943 (2.7799)	-
ELx	12.3552 (5.9394)	40.9410 (7.8352)	4.9398 (1.3514)	-	-
Lx	-	-	-	0.0063 (0.0025)	12.0624 (4.8309)
W	-	-	-	0.0064 (0.0025)	1.8493 (0.1310)

Table 6. *Comparison criteria for data set 1*

Model	$\hat{l}$	AIC	BIC	W*	A*	KS	KS(PV)
GOLxW	187.4011	384.8022	395.1899	0.0254	0.1963	0.0515	0.9976
GOLxLx	187.9596	385.9193	396.3069	0.0318	0.2109	0.0630	0.9730
GOLxU	188.8160	385.6320	393.9422	0.0815	0.4639	0.0651	0.9636
LxMW	212.8455	435.6911	446.0787	0.1919	1.1918	0.30504	3.409e-05
WLx	194.0453	396.0906	404.4008	0.2129	1.3343	0.1290	0.2798
ELx	190.9645	387.9290	394.1616	0.1370	0.8218	0.1113	0.4571
Lx	214.1132	432.2264	436.3814	0.1771	1.0888	0.3060	0.0001
W	197.2924	398.5847	402.7398	0.2933	1.8721	0.1434	0.1763



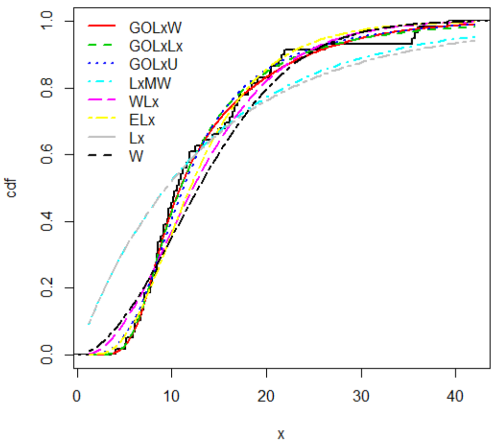


Figure 8. *Estimated PDFs and CDFs for data 1, respectively*

Table 7. *The parameters estimation and its standard errors for data 2*

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}$
GOLxW	0.9341 (0.7710)	17.0563 (10.6305)	0.6338 (0.6094)	0.4966 (0.3313)	3.6720 (2.2389)
GOLxLx	3.2629 (1.5132)	25.9333 (1.9546)	20.5535 (2.3212)	0.0514 (0.0302)	100.1415 (6.0016)
GOLxU	19.5563 (7.2504)	0.7128 (2.2032)	5.6814 (1.0024)	2.9251 (1.3559)	-
LxMW	6.3237 (2.4224)	0.0285 (0.0161)	6.7887 (0.9486)	16.9657 (13.0057)	0.0318 (0.0354)
WLx	0.0195 (0.0389)	3.2897 (1.3828)	12.0107 (40.3310)	12.6730 (42.1120)	-
ELx	18.1615 (4.0606)	32.2211 (3.1215)	30.1213 (5.0132)	-	-
Lx	-	-	-	0.0100 (0.0072)	66.3221 (7.9671)
W	-	-	-	0.0597 (0.0204)	5.7796 (0.5751)

Table 8. *Comparison criteria for data set 2*

Model	$\hat{l}$	AIC	BIC	W*	A*	KS	KS(P-V)
GOLxW	11.9153	33.8306	44.5462	0.0878	0.5051	0.0968	0.5959
GOLxLx	13.1865	36.3730	47.0887	0.1637	0.8986	0.1287	0.2472
GOLxU	15.2077	38.4154	46.9879	0.2350	1.2939	0.1517	0.1097
LxMW	15.1195	40.2391	50.9548	0.1301	0.7591	0.1300	0.2368

WLx	14.4110	36.8221	45.3947	0.1841	1.0280	0.1406	0.1651
ELx	31.9050	69.8101	76.2395	0.7990	4.3550	0.2286	0.0027
Lx	89.2816	182.5633	186.8496	0.5735	3.1452	0.4167	6.28*10^-10
W	15.5068	38.5136	48.6999	0.2372	1.3038	0.1523	0.1075

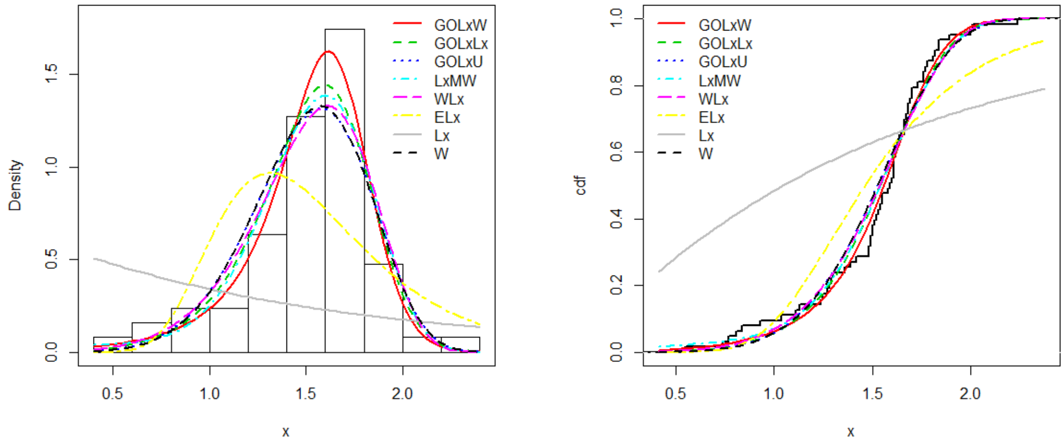


Figure 9. *Estimated PDFs and CDFs for data 2, respectively*

Table 9. *The parameters estimation and its standard errors for data 3*

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}$
GOLxW	1.9601 (0.4918)	47.6841 (24.1313)	4.9272 (2.8580)	3.3305 (0.6326)	0.8714 (1.7225)
GOLxLx	1.9122 (7.0625)	56.5855 (37.0700)	2.5998 (3.1526)	0.0256 (0.0026)	113.9693 (55.4095)
GOLxU	6.3391 (1.2900)	0.5893 (1.2174)	3.1638 (0.6121)	3.4600 (2.0106)	-
LxMW	6.4021 (7.7873)	0.1446 (0.0323)	3.6086 (5.2244)	16.9331 (6.3424)	0.0239 (0.0113)
WLx	3.6725 (0.4244)	2.7313 (0.9682)	4.8209 (2.4422)	15.4613 (11.9462)	-
ELx	91.1149 (11.0110)	45.3534 (8.5950)	7.0179 (1.0392)	-	-
Lx	-	-	-	0.0113 (0.0081)	60.8846 (8.6734)
W	-	-	-	0.2098 (0.0465)	3.2487 (0.3065)

Table 10. *Comparison criteria for data set 3*

Model	$\hat{l}$	AIC	BIC	W*	A*	KS	KS(P-V)
GOLxW	48.5338	107.0678	118.2383	0.0146	0.1366	0.0376	1.000
GOLxLx	48.6139	107.2278	118.3984	0.0152	0.1363	0.0384	1.000
GOLxU	48.9634	105.9269	114.8634	0.0284	0.2420	0.0460	0.9970
LxMW	48.9777	107.9585	119.1269	0.0186	0.1579	0.0437	0.9976
WLx	48.9789	107.7580	118.6944	0.0253	0.2195	0.0488	0.9955
ELx	57.0067	120.0136	126.7159	0.2135	1.4454	0.1131	0.3400
Lx	95.2008	194.4017	198.8699	0.1266	0.8894	0.3613	2.97*10^-8
W	49.0005	107.9001	119.4693	0.0260	0.2202	0.0437	0.9969

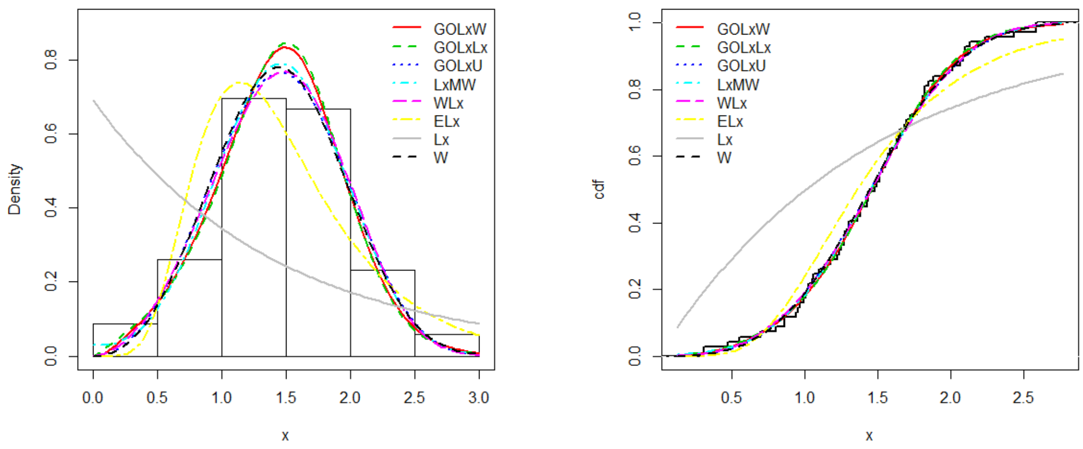


Figure 10. *Estimated PDFs and CDFs for data 3, respectively*

Table 11. *MLEs and their standard errors with AIC and BIC for censored data set*

Distribution	Parameter and (S.E)	-L	AIC	BIC
GOLxW	$\hat{\alpha} = 0.4791(0.3833)$ $\hat{\beta} = 2.9403(0.8849)$ $\hat{\theta} = 4.4319(0.6092)$ $\hat{\lambda} = 0.7231(0.6603)$ $\hat{\gamma} = 0.3675(0.1460)$	181.5575	370.1150	375.9687
GOLxLx	$\hat{\alpha} = 1.3361(1.1460)$ $\hat{\beta} = 20.9897(3.5253)$ $\hat{\theta} = 13.6143(0.1012)$ $\hat{\lambda} = 7.0102\ (0.2643)$ $\hat{\gamma} = 0.8261(0.2240)$	181.7975	371.1150	376.9687
GOLxU	$\hat{\alpha} = 0.4639(0.2431)$ $\hat{\beta} = 0.0598(0.0840)$ $\hat{\theta} = 1.1396(0.2864)$ $\hat{\lambda} = 418.7976(29.6408)$	182.7187	373.4373	380.2423
WLx	$\hat{\alpha} = 0.0685(0.0099)$ $\hat{\beta} = 0.4343(0.1061)$ $\hat{\theta} = 2.0572(\ 0.6190)$ $\hat{\lambda} = 8.1386(0.8779)$	183.7187	375.4373	383.2423
ELx	$\hat{\alpha} = 0.7443(0.3257)$ $\hat{\beta} = 39.2425(38.6966)$ $\hat{\theta} = 1.2541(0.4195)$	182.5575	372.1150	376.9687
Lx	$\hat{\lambda} = 0.3471(0.0924)$ $\hat{\gamma} = 13.5801(6.0837)$	185.7654	375.5309	379.4334
W	$\hat{\lambda} = 0.0170(0.0096)$ $\hat{\gamma} = 0.8176\ (0.1175)$	182.4754	372.0507	376.4394

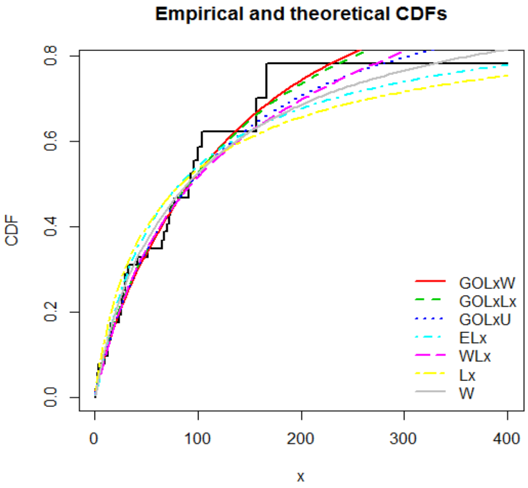


Figure 11. *Plots of estimated cdfs of the models compared in censored data set*

10. CONCLUSIONS

In this paper one shape parameter is added to the odd Lomax-G family to provide the generalized odd Lomax-G family. the main properties of the generalized odd Lomax-G family and other properties associated with the area of reliability are discussed. It has been noted that the distributions generated by the generalized odd Lomax-G family are highly flexible in data modeling where we used three members to fit four real data to illustrate the importance of this family. These members provided consistently better fits than the other comparative distributions.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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