



SPHERICAL INDICATRICES OF A BERTRAND CURVE IN THREE DIMENSIONAL LIE GROUPS

ALI ÇAKMAK AND SEZAI KIZILTUĞ

ABSTRACT. In this paper, new representations of a Bertrand curve pair in three dimensional Lie groups with bi-invariant metric are given. Besides, the spherical indicatrices of a Bertrand curve pair are obtained and the relations between the spherical indicatrices and new representations of Bertrand curve pair are shown.

1. INTRODUCTION

The theory of curves have an important place in differential geometry. Many authors made a significant contribution to this subject from past to present. Among these contributions, some of the relationships among the curve pairs are particularly interesting. The curve pairs for which there exists some relationships between their Frenet vectors or curvatures are examples to special space curves. In particular, involute-evolute curves, Bertrand curves, Mannheim curves are well-known examples of curve pairs and have been studied by many authors [1], [6]-[11].

It is known that the curve α , which is given by a parametrized regular curve, is called a Bertrand curve if there exists another curve β such that the principal normal lines to α and β are equal at all points. The curve β is called a Bertrand mate of α [15]. These curves were first investigated by J. Bertrand. He proved that the curve α for which there exists a linear relation between curvature and torsion: $a\kappa + b\tau = c$, (a, b, c are non-zero constants) admits a Bertrand mate.

The degenerate semi-Riemannian geometry of a Lie group are examined in [2]. The general helices in three dimensional Lie groups with a bi-invariant metric are introduced and a generalization of Lancret's theorem is obtained in [4]. The some special curves in three dimensional Lie groups using harmonic curvature function are recharacterized and Bertrand and Mannheim curves are investigated via harmonic curvature function in three dimensional Lie groups in [7, 12]. The some relations between slant helices and their involutes, spherical images in three dimensional

Received by the editors: March 29, 2018; Accepted: March 12, 2019.

2010 *Mathematics Subject Classification.* 53A04, 22E15.

Key words and phrases. Lie group, Bertrand curve, spherical indicatrix.

Lie groups are given in [13]. The some features of the spherical indicatrices of a Bertrand curve and its mate curve are presented in Euclidean 3-space in [14].

In this paper, we have done a study on spherical indicatrices of Bertrand curve pair in three dimensional Lie groups with a bi-invariant metric. As far as we know, spherical indicatrices of a Bertrand curve and its mate curve have not been shown in three dimensional Lie groups. Hence, this study is intended to fill this gap.

2. PRELIMINARIES

Suppose that G is a Lie group with a bi-invariant metric such that \langle, \rangle is a bi-invariant metric on G . If the Lie algebra of G is given by g , the Lie algebra g is isomorphic to T_eG , where e is neutral element of G . Since \langle, \rangle is a bi-invariant metric on G , we get

$$\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle, \tag{1}$$

and

$$D_X Y = \frac{1}{2}[X, Y], \tag{2}$$

where $X, Y, Z \in g$ and D is the Levi-Civita connection of G . Let us assume that $\alpha : I \subset \mathbb{R} \rightarrow G$ is an arc length parameterized curve and $\{X_1, X_2, \dots, X_n\}$ is an orthonormal basis of g . Here, we consider that any two vector fields W and Z along the curve α as $W = \sum_{i=1}^n w_i X_i$ and $Z = \sum_{i=1}^n z_i X_i$ such that $w_i : I \rightarrow \mathbb{R}$ and $z_i : I \rightarrow \mathbb{R}$ are smooth functions. It is well-known that Lie bracket of W and Z can be written as

$$[W, Z] = \sum_{i=1}^n w_i z_i [X_i, X_j],$$

and $D_{\alpha'} W$ is obtained as

$$D_{\alpha'} W = \dot{W} + \frac{1}{2}[T, W], \tag{3}$$

where $T = \alpha'$, $\dot{W} = \sum_{i=1}^n \dot{w}_i X_i$ and $D_{\alpha'} W$ is the covariant derivative of W along α . In this case, $\dot{W} = 0$ under the condition that W is the left-invariant vector field to α [3].

Suppose that G is a Lie group and (T, N, B, κ, τ) is the Frenet apparatus of α . Hence, the Frenet formulas of the curve α can be written as

$$D_T T = \kappa N, \quad D_T N = -\kappa T + \tau B, \quad D_T B = -\tau N,$$

where $\kappa = \left\| \dot{T} \right\|$ [3].

Proposition 1. [13] *Suppose that the curve $\alpha(s)$ is a curve in Lie group G such that the parameter s is the arc length parameter of $\alpha(s)$ and the Frenet apparatus of $\alpha(s)$ are (T, N, B, κ, τ) . Then the following equalities hold*

$$\begin{cases} [T, N] = \langle [T, N], B \rangle B = 2\tau_G B, \\ [T, B] = \langle [T, B], N \rangle N = -2\tau_G N. \end{cases} \tag{4}$$

Suppose that $\alpha(s)$ is a curve in three dimensional Lie group G such that the parameter s is the arc length parameter of $\alpha(s)$. Then from the Eq.(3) and Proposition 1, Frenet formulas are found as follows:

$$\begin{pmatrix} \frac{dT}{ds} \\ \frac{dN}{ds} \\ \frac{dB}{ds} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau - \tau_G \\ 0 & -(\tau - \tau_G) & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where $\{T, N, B\}$ is the Frenet frame, $\tau_G = \frac{1}{2} \langle [T, N], B \rangle$ and κ, τ are curvature and torsion of α in G , respectively [13].

Definition 2. [13] *Suppose that $\alpha(s)$ is a curve in G such that the parameter s is the arc length parameter of $\alpha(s)$ and the Frenet apparatus of $\alpha(s)$ are (T, N, B, κ, τ) . Then the harmonic curvature function of α can be given by*

$$H = \frac{\tau - \tau_G}{\kappa}. \quad (5)$$

Theorem 3. [4] *Suppose that $\alpha(s)$ is a curve in Lie group G such that s is the arc length parameter of $\alpha(s)$ and the Frenet apparatus of $\alpha(s)$ are (T, N, B, κ, τ) . The curve α is a general helix if and only if $\tau = c\kappa + \tau_G$, where $c \in \mathbb{R}$.*

Definition 2 and Theorem 3 immediately give the following corollary:

Corollary 4. [12] *Suppose that α is a curve in G . Being a general helix in G of α is a necessary and sufficient condition of being $H = \text{constant}$.*

Theorem 5. [13] *Suppose that α is an arc length parameterized curve with the Frenet apparatus (T, N, B, κ, τ) in G . Then α is a slant helix if and only if the function*

$$\sigma = \frac{\kappa(1 + H^2)^{3/2}}{H'}, \quad (6)$$

is a constant.

Theorem 6. [12] *Suppose that the curve $\alpha(s)$ is a Bertrand curve in G such that the parameter s is the arc length parameter of $\alpha(s)$ and the Frenet apparatus of $\alpha(s)$ are (T, N, B, κ, τ) . For all $s \in I$, α satisfy the equation $\lambda\kappa(s) + \mu\kappa(s)H(s) = 1$, where λ and μ are constants. If the curve $\beta(s)$ is given by $\beta(s) = \alpha(s) + \lambda N(s)$ for all $s \in I$, then (α, β) is the Bertrand curve pair.*

Theorem 7. [12] *Suppose that (α, β) is a Bertrand curve pair in G . Then, being slant helix of α is a necessary and sufficient condition of being a slant helix of β .*

Remark 8. [5] *Suppose that G is a Lie group with a bi-invariant metric such that \langle, \rangle is a bi-invariant metric on G . Hence, the following items can be written in various Lie groups:*

- i) If G is abelian group, $\tau_G = 0$.*
- ii) If G is SO^3 , $\tau_G = \frac{1}{2}$.*
- iii) If G is SU^3 , $\tau_G = 1$.*

3. SPHERICAL INDICATRICES OF A BERTRAND CURVE IN THREE DIMENSIONAL LIE GROUPS

In this chapter, the spherical indicatrices of a Bertrand curve with respect to its partner curve in G with a bi-invariant metric are presented and given some significant results by using the features of the curves.

Theorem 9. *Suppose that $(\alpha, \tilde{\alpha})$ is a Bertrand curve pair with arc-length parameter s and s^* , respectively and the Frenet invariants of α and $\tilde{\alpha}$ are denoted by $\{T, N, B, \kappa, \tau - \tau_G\}$ and $\{\tilde{T}, \tilde{N}, \tilde{B}, \tilde{\kappa}, \tilde{\tau} - \tilde{\tau}_G\}$, respectively. Then the relationship between the Frenet invariants of α and $\tilde{\alpha}$ is given as follows:*

$$T = \frac{-1}{\sqrt{1 + \rho^2}} \{\tilde{T} - \rho \tilde{B}\}, \quad N = \epsilon \tilde{N}, \quad B = \frac{-\epsilon}{\sqrt{1 + \rho^2}} \{\rho \tilde{T} + \tilde{B}\}$$

and

$$\kappa = \frac{-\epsilon \tilde{\kappa} (1 + \tilde{H} \rho) (\rho - \tilde{H})}{\tilde{H} (1 + \rho^2)}, \quad \tau - \tau_G = \frac{\tilde{\kappa} (\rho - \tilde{H})^2}{\tilde{H} (1 + \rho^2)},$$

where \tilde{H} is a harmonic curvature function of the curve $\tilde{\alpha}$, $\rho = \frac{(\tilde{\tau} - \tilde{\tau}_G)'}{\tilde{\kappa}'}$ and $\epsilon = \pm 1$.

Proof. Since α and $\tilde{\alpha}$ are the Bertrand curves, then

$$\alpha(s) = \tilde{\alpha}(s^*) - \lambda \epsilon \tilde{N} \tag{7}$$

Differentiating the Eq. (7) according to s^* and using the Eq. (3), we have

$$T \frac{ds}{ds^*} = \tilde{T} - \lambda \epsilon (D_{\tilde{T}} \tilde{N} - \frac{1}{2} [\tilde{T}, \tilde{N}]) \tag{8}$$

By using the Frenet formulas, we get

$$T \frac{ds}{ds^*} = \tilde{T} (1 + \lambda \epsilon \tilde{\kappa}) - \lambda \epsilon (\tilde{\tau} - \tilde{\tau}_G) \tilde{B} \tag{9}$$

which gives us

$$\frac{ds^*}{ds} = \frac{1}{\sqrt{(1 + \lambda \epsilon \tilde{\kappa})^2 + \lambda^2 (\tilde{\tau} - \tilde{\tau}_G)^2}}. \tag{10}$$

From the Eq. (9) and (10), we have

$$T = \frac{1}{\sqrt{(1 + \lambda \epsilon \tilde{\kappa})^2 + \lambda^2 (\tilde{\tau} - \tilde{\tau}_G)^2}} [\tilde{T} (1 + \lambda \epsilon \tilde{\kappa}) - \lambda \epsilon (\tilde{\tau} - \tilde{\tau}_G) \tilde{B}] \tag{11}$$

Since $B = T \times N$, from (11) it is obtained that

$$B = \frac{1}{\sqrt{(1 + \lambda \epsilon \tilde{\kappa})^2 + \lambda^2 (\tilde{\tau} - \tilde{\tau}_G)^2}} [\tilde{T} (\lambda \epsilon (\tilde{\tau} - \tilde{\tau}_G)) + (1 + \lambda \epsilon \tilde{\kappa}) \tilde{B}] \tag{12}$$

By taking the derivative of Eq.(11) according to s^* and considering Frenet formulas, we have

$$\begin{aligned}
 T' \frac{ds}{ds^*} &= \left(\lambda \epsilon \tilde{\kappa}' \right) \left((1 + \lambda \epsilon \tilde{\kappa})^2 + \lambda^2 (\tilde{\tau} - \tilde{\tau}_G)^2 \right) \\
 &\quad - (1 + \lambda \epsilon \tilde{\kappa}) \left[(1 + \lambda \epsilon \tilde{\kappa}) \lambda \epsilon \tilde{\kappa}' + \lambda^2 (\tilde{\tau} - \tilde{\tau}_G)' (\tilde{\tau} - \tilde{\tau}_G) \right] \tilde{T} + \\
 &\quad \left[(\tilde{\kappa} (1 + \lambda \epsilon \tilde{\kappa}) + \lambda \epsilon (\tilde{\tau} - \tilde{\tau}_G)^2) \left((1 + \lambda \epsilon \tilde{\kappa})^2 + \lambda^2 (\tilde{\tau} - \tilde{\tau}_G)^2 \right) \right] \tilde{N} + \\
 &\quad \left[\begin{aligned} & \left(-\lambda \epsilon (\tilde{\tau} - \tilde{\tau}_G)' \right) \left((1 + \lambda \epsilon \tilde{\kappa})^2 + \lambda^2 (\tilde{\tau} - \tilde{\tau}_G)^2 \right) + \\ & \left[(1 + \lambda \epsilon \tilde{\kappa}) \left[(\lambda \epsilon (\tilde{\tau} - \tilde{\tau}_G)) \lambda \epsilon \tilde{\kappa}' + \lambda^2 (\tilde{\tau} - \tilde{\tau}_G)' (\tilde{\tau} - \tilde{\tau}_G) \right] \right] \end{aligned} \right] \tilde{B}
 \end{aligned}$$

we know that N and \tilde{N} are linearly dependent. Then from the last equation we have

$$\begin{aligned}
 & \left[(\lambda \epsilon \tilde{\kappa}') \left((1 + \lambda \epsilon \tilde{\kappa})^2 + \lambda^2 (\tilde{\tau} - \tilde{\tau}_G)^2 \right) \right] \\
 & - \left[(1 + \lambda \epsilon \tilde{\kappa}) \left[(1 + \lambda \epsilon \tilde{\kappa}) \lambda \epsilon \tilde{\kappa}' + \lambda^2 (\tilde{\tau} - \tilde{\tau}_G)' (\tilde{\tau} - \tilde{\tau}_G) \right] \right] = 0.
 \end{aligned} \tag{13}$$

From the last equation, we get

$$\lambda = \frac{-\epsilon \rho}{\tilde{\kappa}(\rho - \tilde{H})} \tag{14}$$

where $\rho = \frac{(\tilde{\tau} - \tilde{\tau}_G)'}{\tilde{\kappa}'}$ and $\tilde{H} = \frac{\tilde{\tau} - \tilde{\tau}_G}{\tilde{\kappa}}$.

If substituting the Eq. (14) in the Eq. (11) and (12), we have

$$T = \frac{-1}{\sqrt{1 + \rho^2}} \{ \tilde{T} - \rho \tilde{B} \}, \tag{15}$$

$$B = \frac{-\epsilon}{\sqrt{1 + \rho^2}} \{ \rho \tilde{T} + \tilde{B} \} \tag{16}$$

By using Frenet formulas and with the help of the Proposition 2.1, we get $T' = \kappa N$. By considering the Eq. (15) we obtain

$$\kappa = \frac{-\epsilon \tilde{\kappa} (1 + \tilde{H} \rho) (\rho - \tilde{H})}{\tilde{H} (1 + \rho^2)}. \tag{17}$$

Since $\tau' = \langle B', N \rangle$, we get $\tau - \tau_G = \frac{\tilde{\kappa}(\rho - \tilde{H})^2}{\tilde{H}(1 + \rho^2)}$. Hence, our theorem is proved. \square

Thus, the geodesic curvature of the principal image of the principal normal indicatrix of α is given by

$$\Gamma = \frac{-\tilde{\kappa}(\rho - \tilde{H})}{\tilde{\kappa}^2 (1 + \tilde{H}^2)^{\frac{3}{2}}}. \tag{18}$$

Corollary 10. *Suppose that $(\alpha, \tilde{\alpha})$ is a Bertrand curve pair with arc-length parameter s and s^* , respectively. Then the relationship between s and s^* is given by*

$$s = \int \frac{\tilde{H}\sqrt{1+\rho^2}}{\rho - \tilde{H}} ds^*. \tag{19}$$

3.1. Tangent indicatrix $\alpha_t = T$ of the Bertrand curve in G .

Definition 11. *Suppose that the curve α is a regular curve in G . The curve α_t is called the tangent indicatrix of the curve α and $\alpha_t : I \subset S^2 \rightarrow G$ is defined by*

$$\alpha_t(s_t) = T(s).$$

Then the tangent indicatrix of a Bertrand curve in G can be given by

$$\alpha_t = \frac{-1}{\sqrt{1+\rho^2}} \{\tilde{T} - \rho\tilde{B}\}. \tag{20}$$

Thus, we can give the following characterizations in the view of above equation.

Theorem 12. *Suppose that the curve α is a regular curve in G and the Frenet apparatus of the tangent indicatrix $\alpha_t = T$ of the Bertrand curve are denoted by $\{T_t, N_t, B_t, \kappa_t, (\tau - \tau_G)_t\}$. Then we have*

$$T_t = -\tilde{N}, N_t = \frac{1}{\sqrt{1+\tilde{H}^2}} \{\tilde{T} - \tilde{H}\tilde{B}\}, B_t = \frac{1}{\sqrt{1+\tilde{H}^2}} \{\tilde{H}\tilde{T} + \tilde{B}\} \tag{21}$$

and

$$s_t = - \int \frac{\tilde{\kappa}(\rho - \tilde{H})^2}{\tilde{H}(1+\rho)^2} ds, \kappa_t = \frac{\sqrt{1+\rho^2}\sqrt{1+\tilde{H}^2}}{\tilde{H} - \rho}, (\tau - \tau_G)_t = \frac{-\tilde{\kappa}'\sqrt{1+\rho^2}}{\tilde{\kappa}^2(1+\tilde{H}^2)}. \tag{22}$$

and s_t is a natural representation of the tangent indicatrix of α . The geodesic curvature of the principal image of the principal normal indicatrix of $\alpha_t(s)$ is given by

$$\Gamma_t = \frac{-\tilde{\kappa}^3(1+\tilde{H}^2)^{\frac{3}{2}}(\rho - \tilde{H})^2[\tilde{\kappa}''\tilde{\kappa}(1+\tilde{H}^2) - 3\tilde{\kappa}'^2(1+\rho\tilde{H})]}{\sqrt{1+\rho^2}(\tilde{\kappa}(1+\tilde{H}^2)^3 + \tilde{\kappa}'^2(\tilde{H} - \rho)^2)^{\frac{3}{2}}} \frac{ds^*}{ds_t} \tag{23}$$

where $\frac{ds^*}{ds_t} = \frac{\sqrt{1+\rho^2}}{\tilde{\kappa}(\tilde{H}-\rho)}$.

As a result of the above calculations, the following theorems can be given:

Theorem 13. *Suppose that $(\alpha, \tilde{\alpha})$ is a non-helical and non-planar Bertrand curve pair in G . Then the following properties hold:*

- i) If the curve α is a slant helix, then its necessary and sufficient condition is $\alpha_t =$ spherical helix.*
- ii) If the curve $\tilde{\alpha}$ is a slant helix, then its necessary and sufficient condition is $\alpha_t =$ spherical helix.*

Further the following theorem can be given:

Theorem 14. *Suppose that $(\alpha, \tilde{\alpha})$ is a non-helical and non-planar Bertrand curve pair in G . If the tangent indicatrix of α is a spherical helix, then its necessary and sufficient condition is*

$$\tilde{\kappa}'' \tilde{\kappa}(1 + \tilde{H}^2) - 3\tilde{\kappa}'^2(1 + \rho\tilde{H}) = 0. \tag{24}$$

3.2. Principal normal indicatrix $\alpha_n = N(s)$ of the Bertrand curve in G .

Definition 15. *Suppose that the curve α is a regular curve in G . The curve α_n is called the principal normal indicatrix of the curve α and $\alpha_n : I \subset S^2 \rightarrow G$ is defined by*

$$\alpha_n(s_n) = N(s). \tag{25}$$

From here, we can write the principal normal indicatrix $\alpha_n = N(s)$ as follows:

$$\alpha_n = \epsilon\tilde{N}.$$

Theorem 16. *Suppose that the curve α is a regular curve in G and the Frenet apparatus of the normal indicatrix $\alpha_n = \epsilon\tilde{N}$ of the Bertrand curve are denoted by $\{T_n, N_n, B_n, \kappa_n, (\tau - \tau_G)_n\}$. Then we have*

$$T_n = \frac{-\epsilon\{\tilde{T} - \tilde{H}\tilde{B}\}}{\sqrt{1 + \tilde{H}^2}}, \tag{26}$$

$$N_n = \frac{-\epsilon\{\tilde{H}\tilde{\kappa}'(\rho - \tilde{H})\tilde{T} - \tilde{\kappa}^2(1 + \tilde{H}^2)^2\tilde{N} - \tilde{\kappa}'(\tilde{H} - \rho)\tilde{B}\}}{\sqrt{\tilde{\kappa}'^2(\tilde{H} - \rho)^2 + \tilde{\kappa}^4(1 + \tilde{H}^2)^3}\sqrt{1 + \tilde{H}^2}} \tag{27}$$

$$B_n = \frac{\tilde{\kappa}^2\tilde{H}(1 + \tilde{H}^2)\tilde{T} + \tilde{\kappa}'(\rho - \tilde{H})\tilde{N} + \tilde{\kappa}^2(1 + \tilde{H}^2)\tilde{B}}{\sqrt{\tilde{\kappa}'^2(\tilde{H} - \rho)^2 + \tilde{\kappa}^4(1 + \tilde{H}^2)^3}} \tag{28}$$

and

$$s_n = \int \frac{\tilde{\kappa}(\rho - \tilde{H})\sqrt{1 + \tilde{H}^2}}{\tilde{H}\sqrt{1 + \rho^2}} ds, \quad \kappa_n = \frac{\sqrt{\tilde{\kappa}'^2(\tilde{H} - \rho)^2 + \tilde{\kappa}^4(1 + \tilde{H}^2)^3}}{\tilde{\kappa}^2(1 + \tilde{H}^2)^{\frac{3}{2}}}, \tag{29}$$

$$(\tau - \tau_G)_n = \frac{\epsilon(\tilde{H} - \rho)[(3\tilde{\kappa}'^2 - \tilde{\kappa}''\tilde{\kappa})(1 + \rho^2) - 3\tilde{\kappa}'^2\rho(\tilde{H} - \rho)]}{\tilde{\kappa}'^2(\tilde{H} - \rho)^2 + \tilde{\kappa}^4(1 + \tilde{H}^2)^3}. \tag{30}$$

and s_n is a natural representation of the principal normal indicatrix of α .

Further the following theorem can be given:

Theorem 17. *Suppose that $(\alpha, \tilde{\alpha})$ is a non-helical and non-planar Bertrand curve pair in G . If the the principal normal indicatrix of α is planar, then its necessary and sufficient condition is*

$$(3\tilde{\kappa}'^2 - \tilde{\kappa}''\tilde{\kappa})(1 + \rho^2) - 3\tilde{\kappa}'^2\rho(\tilde{H} - \rho) = 0. \tag{31}$$

3.3. Binormal indicatrix $\alpha_b = B(s)$ of the Bertrand curve in G .

Definition 18. Suppose that the curve α is a regular curve in G . The curve α_b is called the binormal indicatrix of the curve α and $\alpha_b : I \subset S^2 \rightarrow G$ is defined by

$$\alpha_b(s_b) = B(s). \tag{32}$$

Thus, the binormal indicatrix of the Bertrand curve in G is given by

$$\alpha_b = \frac{-\epsilon}{\sqrt{1 + \rho^2}} \{\rho \tilde{T} + \tilde{B}\}. \tag{33}$$

Theorem 19. Suppose that the curve α is a regular curve in G and the Frenet apparatus of the binormal indicatrix $\alpha_b = B$ of the Bertrand curve are denoted by $\{T_b, N_b, B_b, \kappa_b, (\tau - \tau_G)_b\}$. Then we have

$$T_b = \epsilon \tilde{N} \tag{34}$$

$$N_b = \frac{-\epsilon}{\sqrt{1 + \tilde{H}^2}} \{\tilde{T} - \tilde{H} \tilde{B}\} \tag{35}$$

$$B_b = \frac{1}{\sqrt{1 + \tilde{H}^2}} \{\tilde{H} \tilde{T} + \tilde{B}\} \tag{36}$$

and

$$s_b = - \int \frac{\tilde{\kappa}(\rho - \tilde{H})^2}{\tilde{H}(1 + \rho^2)} ds, \quad \kappa_b = \frac{\sqrt{(1 + \tilde{H}^2)(1 + \rho^2)}}{\rho - \tilde{H}}, \quad (\tau - \tau_G)_b = \frac{\epsilon \tilde{\kappa}' \sqrt{1 + \rho^2}}{\tilde{\kappa}^2 (1 + \tilde{H}^2)}. \tag{37}$$

and s_b is a natural representation of the binormal indicatrix of α .

The geodesic curvature of the principal image of the principal normal indicatrix of $\alpha_b(s)$ is given by

$$\Gamma_b = \frac{-\tilde{\kappa}^3 (1 + \tilde{H}^2)^{\frac{3}{2}} (\rho - \tilde{H})^2 [\tilde{\kappa}'' \tilde{\kappa} (1 + \tilde{H}^2) - 3\tilde{\kappa}'^2 (1 + \rho \tilde{H})]}{\sqrt{1 + \rho^2} (\tilde{\kappa} (1 + \tilde{H}^2)^3 + \tilde{\kappa}'^2 (\tilde{H} - \rho)^2)^{\frac{3}{2}}} \frac{ds^*}{ds_b} \tag{38}$$

where $\frac{ds^*}{ds_b} = \frac{\sqrt{1 + \rho^2}}{\tilde{\kappa}(\tilde{H} - \rho)}$.

From Theorem 3.11, we obtain the following corollaries:

Corollary 20. Suppose that the curve α is a Bertrand curve in G . Since $\Gamma_t = \Gamma_b$, the spherical images of the tangent and binormal indicatrices of α are the curves with same curvature and same torsion.

Corollary 21. Suppose that $(\alpha, \tilde{\alpha})$ is a non-helical and non-planar Bertrand curve pair in G . If α is a slant helix, then its necessary and sufficient condition is $\alpha_b =$ spherical helix.

Corollary 22. Suppose that $(\alpha, \tilde{\alpha})$ is a non-helical and non-planar Bertrand curve pair in G . If $\tilde{\alpha}$ is a slant helix, then its necessary and sufficient condition is $\alpha_b =$ spherical helix.

REFERENCES

- [1] Ahmat, T. A., New special curves and their spherical indicatrices, *Global Journal of Advanced Research on Classical and Modern Geometries*, 1 (2), (2012), 28-38.
- [2] Cöken, A. C. and Çiftçi, Ü., A note on the geometry of Lie groups, *Nonlinear Analysis*, TMA68 (2008), 2013-2016.
- [3] Crouch, P. and Leite, F. S., The dynamic interpolation problem: on Riemannian manifolds, Lie groups and symmetric spaces, *J. Dyn. Control Syst.* 1(2), (1995), 177-202.
- [4] Çiftçi, Ü., A generalization of Lancert's theorem, *J. Geom. Phys.* 59 (2009), 1597-1603.
- [5] Do Espírito-Santo, N., Fornari, S., Frensel, K. and Ripoll, J., Constant mean curvature hypersurfaces in a Lie group with a bi-invariant metric, *Manuscripta Math.* 111 (4), (2003), 459-470.
- [6] Ekmekci, N. and İlarıslan, K., On Bertrand curves and their characterization, *Differ. Geom.Dyn.Syst.* 3 (2001), no. 2, 17-24.
- [7] Gök, İ., Okuyucu, O. Z., Ekmekci, N. and Yaylı, Y., On Mannheim partner curves in three dimensional Lie groups, *Miskolc Mathematical Notes*, 17(2), (2017), 999-1010.
- [8] Izumiya, S. and Takeuchi, N., Generic properties of helices and Bertrand curves, *J. Geom.* 74 (2002), 97-109.
- [9] Izumiya, S. and Takeuchi, N., New special curves and developable surfaces, *Turkish J. Math.* 28 (2), (2004)153-163.
- [10] Kula, L. and Yaylı, Y., On slant helix and its spherical indicatrix, *Applied Mathematics and Computation*, 169 (2005), 600-6007.
- [11] Matsuda, H. and Yorozu, S., Notes on Bertrand curves, *Yokohama Mathematical Journal*, 50 (2003), 41-58.
- [12] Okuyucu, O. Z., Gök, İ., Yaylı, Y. and Ekmekci, N., Bertrand curves in three dimensional Lie groups, *Miskolc Mathematical Notes*, 17(2), (2017), 999-1010.
- [13] Okuyucu, O.Z., Gök, İ., Yaylı, Y. and Ekmekçi, F.N., Slant helices in three dimensional Lie groups, *Applied Mathematics and Computation*, 221 (2013), 672-683.
- [14] Tunçer, Y. and Ünal, S., New representations of Bertrand pairs in Euclidean 3-space, *Applied Mathematics and Computation*, 219 (2012), 1833-1842.
- [15] Whittemore, J.K., Bertrand curves and helices, *Duke Math. J.* 6(1), (1940), 235-245.

Current address: Department of Mathematics, Faculty of Arts and Sciences Bitlis Eren University Bitlis, Turkey

E-mail address: alicakmak@yahoo.com

ORCID Address: <http://orcid.org/0000-0002-2783-9311>

Current address: Department of Mathematics, Faculty of Arts and Sciences Erzincan University Erzincan, Turkey

E-mail address: skiziltug@erzincan.edu.tr

ORCID Address: <http://orcid.org/0000-0003-3899-4189>