THE MATRIX SEQUENCE IN TERMS OF BI-PERIODIC FIBONACCI NUMBERS

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ABSTRACT. In this paper, we define the bi-periodic Fibonacci matrix sequence that represent bi-periodic Fibonacci numbers. Then, we investigate generating function, Binet formula and summations of bi-periodic Fibonacci matrix sequence. After, we say that some behaviours of bi-periodic Fibonacci numbers can be obtained via the properties of this new matrix sequence. Finally, we express that well-known matrix sequences such as Fibonacci, Pell, k-Fibonacci matrix sequences are special cases of this generalized matrix sequence.

1. INTRODUCTION AND PRELIMINARIES

The special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal and their properties have been investigated in many articles and books (see, for example [1, 3, 4, 6, 7, 9, 14, 16, 17] and the references cited therein). The Fibonacci numbers have attracted the attention of mathematicians because of their intrinsic theory and applications. The ratio of two consecutive of these numbers converges to the Golden section $\alpha = \frac{1+\sqrt{5}}{2}$. It is also clear that the ratio has so many applications in, specially, Physics, Engineering, Architecture, etc.[8].

After the study of Fibonacci numbers started in the beginning of 13. century, many authors have generalized this sequence in different ways. One of those generalizations was published in 2009 by Edson et al. in [3]. In this reference, the authors defined the bi-periodic Fibonacci $\{q_n\}_{n \in \mathbb{N}}$ sequence as

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \qquad (1.1)$$

where $q_0 = 0$, $q_1 = 1$ and a, b are nonzero real numbers.

Also, in [11], it is defined a sequence $\{q_m\}_{m \in \mathbb{N}}$, for all $m \geq 2$, $q_m = a_t q_{m-1} + q_{m-2}$ with initial conditions $q_0 = 0$, $q_1 = 1$ where $m \equiv t(mod r)$ and $a_0, a_1, \ldots, a_{r-1}$

positive numbers. For $r = 2$, the author called the sequence $\{q_m\}$ as the generalized Fibonacci sequences and it is studied in [3]. But, it remains open to find a closed form of the generating function for general $\{q_m\}$. In this paper, the author solve this open problem, that is, he find a closed form of the generating function for $\{q_m\}$ in terms of the continuant. In addition, in [10], this general conditional sequences are generalized as non-homogeneous conditional sequences and developed some techniques for the solution.

On the other hand, the matrix sequences have taken so much interest for different type of numbers ([2, 5, 12, 13, 15, 18]). In [5], the authors defined (s, t)-Pell and (s, t)-Pell–Lucas sequences and (s, t)-Pell and (s, t)-Pell–Lucas matrix sequences, also gave their some properties. Yazlik et al., in [18], establish generalized (s, t)-matrix sequences and present some important relationships among (s, t)-Fibonacci and (s, t)-Lucas sequences and their matrix sequences.

The goal of this paper is to define the related *matrix sequence* for bi-periodic Fibonacci numbers as the first time in the literature. Then, it will be given the generating function, Binet formula and summation formulas for this new generalized matrix sequence. Thus, some fundamental properties of bi-periodic Fibonacci numbers can be obtained by taking into account this generalized matrix sequence and its properties. By using the results in Sections 2, we have a great opportunity to obtain some new properties over this matrix sequence.

2. The matrix representation of bi-periodic Fibonacci numbers

In this section, we mainly focus on the matrix sequence of bi-periodic Fibonacci numbers to get some important results. In fact, we also present the generating function, Binet formula and summations for the matrix sequence.

Hence, in the following, we firstly define the bi-periodic Fibonacci matrix sequence.

Definition 2.1. *For $n \in \mathbb{N}$ and $a, b \in \mathbb{R} - \{0\}$, the bi-periodic Fibonacci matrix sequence $\mathcal{F}_n(a, b)$ is defined by*

$$\mathcal{F}_n(a, b) = \begin{cases} a\mathcal{F}_{n-1}(a, b) + \mathcal{F}_{n-2}(a, b), & n \text{ even} \\ b\mathcal{F}_{n-1}(a, b) + \mathcal{F}_{n-2}(a, b), & n \text{ odd} \end{cases} \tag{2.1}$$

with initial conditions $\mathcal{F}_0(a, b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathcal{F}_1(a, b) = \begin{pmatrix} b & \frac{b}{a} \\ 1 & 0 \end{pmatrix}$.

In Definition 2.1, the matrix $\mathcal{F}_1$ is analogue to the Fibonacci Q-matrix which exists for Fibonacci numbers.

In the following theorem, we give the nth general term of the matrix sequence in (2.1) via bi-periodic Fibonacci numbers.

Theorem 2.2. *For any integer $n \geq 0$, we have the matrix sequence*

$$\mathcal{F}_n(a, b) = \begin{pmatrix} \left(\frac{b}{a}\right)^{\varepsilon(n)} q_{n+1} & \frac{b}{a} q_n \\ q_n & \left(\frac{b}{a}\right)^{\varepsilon(n)} q_{n-1} \end{pmatrix}, \tag{2.2}$$

where $\varepsilon(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$.

Proof. We know that $q_2 = a$, $q_{-1} = q_1 = 1$ and $q_0 = 0$. So, the first and second steps of the induction is obtained as follows:

$$\mathcal{F}_0(a,b) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q_1 & \frac{b}{a}q_0 \\ q_0 & q_{-1} \end{pmatrix},$$

$$\mathcal{F}_1(a,b) = \begin{pmatrix} b & \frac{b}{a} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{b}{a}q_2 & \frac{b}{a}q_1 \\ q_1 & \frac{b}{a}q_0 \end{pmatrix}.$$

Actually, by assuming the equation in (2.2) holds for all $n = k \in \mathbb{Z}^+$, we can end up the proof if we manage to show that the case also holds for $n = k + 1$:

$$
\begin{aligned}
\mathcal{F}_{k+1}(a,b) &= a^{\varepsilon(k)} b^{1-\varepsilon(k)} \mathcal{F}_k(a,b) + \mathcal{F}_{k-1}(a,b) \\
&= a^{\varepsilon(k)} b^{1-\varepsilon(k)} \begin{pmatrix} \left(\frac{b}{a}\right)^{\varepsilon(k)} q_{k+1} & \frac{b}{a} q_k \\ q_k & \left(\frac{b}{a}\right)^{\varepsilon(k)} q_{k-1} \end{pmatrix} \\
&\quad + \begin{pmatrix} \left(\frac{b}{a}\right)^{\varepsilon(k-1)} q_k & \frac{b}{a} q_{k-1} \\ q_{k-1} & \left(\frac{b}{a}\right)^{\varepsilon(k-1)} q_{k-2} \end{pmatrix} \\
&= \begin{cases} \begin{pmatrix} \frac{b}{a} q_{k+2} & \frac{b}{a} q_{k+1} \\ q_{k+1} & \frac{b}{a} q_k \end{pmatrix}, & k \text{ even} \\[2ex] \begin{pmatrix} q_{k+2} & \frac{b}{a} q_{k+1} \\ q_{k+1} & q_k \end{pmatrix}, & k \text{ odd} \end{cases}.
\end{aligned}
$$

By combining this partial function, we obtain

$$\mathcal{F}_{k+1}(a,b) = \begin{pmatrix} \left(\frac{b}{a}\right)^{\varepsilon(k+1)} q_{k+2} & \frac{b}{a} q_{k+1} \\ q_{k+1} & \left(\frac{b}{a}\right)^{\varepsilon(k+1)} q_k \end{pmatrix}.$$

$\square$

Theorem 2.3. *Let $\mathcal{F}_n(a,b)$ be as in (2.2). Then the following equality is valid for all positive integers:*

$$\det(\mathcal{F}_n(a,b)) = \left(-\frac{b}{a}\right)^{\varepsilon(n)}$$

Proof. By using the induction on n, we get

$$\det(\mathcal{F}_n(a,b)) = \begin{cases} -\frac{b}{a}, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$$

which is desired.

$\square$

In [3], the authors obtained the Cassini identity for bi-periodic Fibonacci numbers. With a different approach, so as a consequence of Theorem 2.2 and Theorem 2.3, we can obtain this identity. In fact, in the following corollary, we just compare the determinants.

Corollary 2.4. *Cassini identity for bi-periodic Fibonacci sequence can also be obtained using bi-periodic Fibonacci matrix sequence. That is, by using Theorem 2.2 and Theorem 2.3, we can write*

$$\left(\frac{b}{a}\right)^{2\varepsilon(n)} q_{n+1}q_{n-1} - \frac{b}{a}q_n^2 = \left(-\frac{b}{a}\right)^{\varepsilon(n)}.$$

Thus, we obtain

$$a^{1-\varepsilon(n)}b^{\varepsilon(n)}q_{n+1}q_{n-1} - a^{\varepsilon(n)}b^{1-\varepsilon(n)}q_n^2 = a\left(-1\right)^n.$$

Theorem 2.5. *For bi-periodic Fibonacci matrix sequence, we have the generating function*

$$\sum_{i=0}^{\infty} \mathcal{F}_i\left(a,b\right)x^i = \frac{1}{1-\left(ab+2\right)x^2+x^4}\begin{pmatrix} 1+bx-x^2 & \frac{b}{a}x+bx^2-\frac{b}{a}x^3 \\ x+ax^2-x^3 & 1-\left(ab+1\right)x^2+bx^3 \end{pmatrix}.$$

Proof. Assume that $G(x)$ is the generating function for the sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$. Then, we have

$$G\left(x\right) = \sum_{i=0}^{\infty} \mathcal{F}_i\left(a,b\right)x^i = \mathcal{F}_0\left(a,b\right) + \mathcal{F}_1\left(a,b\right)x + \sum_{i=2}^{\infty} \mathcal{F}_i\left(a,b\right)x^i.$$

Note that

$$-bxG\left(x\right) = -bx\sum_{i=0}^{\infty} \mathcal{F}_i\left(a,b\right)x^i = -bx\mathcal{F}_0\left(a,b\right) - b\sum_{i=2}^{\infty} \mathcal{F}_{i-1}\left(a,b\right)x^i$$

and

$$-x^2 G\left(x\right) = -\sum_{i=2}^{\infty} \mathcal{F}_{i-2}\left(a,b\right)x^i.$$

Thus, we can write

$$\begin{aligned}
\left(1-bx-x^2\right)G\left(x\right) \;=\;& \mathcal{F}_0\left(a,b\right) + x\left(\mathcal{F}_1\left(a,b\right) - b\mathcal{F}_0\left(a,b\right)\right) \\
& + \sum_{i=2}^{\infty}\left(\mathcal{F}_i\left(a,b\right) - b\mathcal{F}_{i-1}\left(a,b\right) - \mathcal{F}_{i-2}\left(a,b\right)\right)x^i.
\end{aligned}$$

Since $\mathcal{F}_{2i+1}\left(a,b\right) = b\mathcal{F}_{2i}\left(a,b\right) + \mathcal{F}_{2i-1}\left(a,b\right)$, we get

$$\begin{aligned}
\left(1-bx-x^2\right)G\left(x\right) \;=\;& \mathcal{F}_0\left(a,b\right) + x\left(\mathcal{F}_1\left(a,b\right) - b\mathcal{F}_0\left(a,b\right)\right) \\
& + \sum_{i=1}^{\infty}\left(\mathcal{F}_{2i}\left(a,b\right) - b\mathcal{F}_{2i-1}\left(a,b\right) - \mathcal{F}_{2i-2}\left(a,b\right)\right)x^{2i} \\
\;=\;& \mathcal{F}_0\left(a,b\right) + x\left(\mathcal{F}_1\left(a,b\right) - b\mathcal{F}_0\left(a,b\right)\right) \\
& + \left(a-b\right)x\sum_{i=1}^{\infty} \mathcal{F}_{2i-1}\left(a,b\right)x^{2i-1}.
\end{aligned}$$

Now, let

$$g(x) = \sum_{i=1}^{\infty} \mathcal{F}_{2i-1}(a,b) \, x^{2i-1}.$$

Since

$$
\begin{aligned}
\mathcal{F}_{2i+1}(a,b) &= b\mathcal{F}_{2i}(a,b) + \mathcal{F}_{2i-1}(a,b) \\
&= b(a\mathcal{F}_{2i-1}(a,b) + \mathcal{F}_{2i-2}(a,b)) + \mathcal{F}_{2i-1}(a,b) \\
&= (ab+1)\mathcal{F}_{2i-1}(a,b) + b\mathcal{F}_{2i-2}(a,b) \\
&= (ab+1)\mathcal{F}_{2i-1}(a,b) + \mathcal{F}_{2i-1}(a,b) - \mathcal{F}_{2i-3}(a,b) \\
&= (ab+2)\mathcal{F}_{2i-1}(a,b) - \mathcal{F}_{2i-3}(a,b),
\end{aligned}
$$

we have

$$
\begin{aligned}
\left(1 - (ab+2)\,x^2 + x^4\right) g(x) &= \mathcal{F}_1(a,b)\,x + \mathcal{F}_3(a,b)\,x^3 - (ab+2)\mathcal{F}_1(a,b)\,x^3 \\
&\quad + \sum_{i=3}^{\infty} \left(\begin{array}{c} \mathcal{F}_{2i-1}(a,b) - (ab+2)\mathcal{F}_{2i-3}(a,b) \\ + \mathcal{F}_{2i-5}(a,b) \end{array} \right) x^{2i-1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
g(x) &= \frac{\mathcal{F}_1(a,b)\,x + \mathcal{F}_3(a,b)\,x^3 - (ab+2)\mathcal{F}_1(a,b)\,x^3}{1 - (ab+2)\,x^2 + x^4} \\
&= \frac{\mathcal{F}_1(a,b)\,x + (b\mathcal{F}_0(a,b) - \mathcal{F}_1(a,b))x^3}{1 - (ab+2)\,x^2 + x^4}
\end{aligned}
$$

and as a result, we get

$$
G(x) = \frac{\begin{array}{c}\mathcal{F}_0(a,b) + x\mathcal{F}_1(a,b) + x^2\left(a\mathcal{F}_1(a,b) - \mathcal{F}_0(a,b) - ab\mathcal{F}_0(a,b)\right) \\ + x^3\left(b\mathcal{F}_0(a,b) - \mathcal{F}_1(a,b)\right)\end{array}}{1 - (ab+2)\,x^2 + x^4}.
$$

which is desired equality. $\qquad\square$

Theorem 2.6. *For every $n \in \mathbb{N}$, we write the Binet formula for the bi-periodic Fibonacci matrix sequence as the form*

$$\mathcal{F}_n(a,b) = A_1(n)\left(\alpha^n - \beta^n\right) + B_1(n)\left(\alpha^{2\left\lfloor \frac{n}{2} \right\rfloor + 2} - \beta^{2\left\lfloor \frac{n}{2} \right\rfloor + 2}\right),$$

where

$$
A_1(n) = \frac{\left[\mathcal{F}_1(a,b) - b\mathcal{F}_0(a,b)\right]^{\varepsilon(n)} \left[a\mathcal{F}_1(a,b) - \mathcal{F}_0(a,b) - ab\mathcal{F}_0(a,b)\right]^{1-\varepsilon(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}(\alpha - \beta)},
$$

$$
B_1(n) = \frac{b^{\varepsilon(n)}\mathcal{F}_0(a,b)}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor + 1}(\alpha - \beta)},
$$

such that $\alpha = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2}, \beta = \frac{ab - \sqrt{a^2b^2 + 4ab}}{2}$ and $\varepsilon(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$.

Proof. We know that the generating function of bi-periodic Fibonacci matrix sequence is

$$G(x) = \frac{\begin{array}{c}\mathcal{F}_0(a,b) + x\mathcal{F}_1(a,b) + x^2(a\mathcal{F}_1(a,b) - \mathcal{F}_0(a,b) - ab\mathcal{F}_0(a,b)) \\ +x^3(b\mathcal{F}_0(a,b) - \mathcal{F}_1(a,b))\end{array}}{1 - (ab+2)x^2 + x^4}.$$

Using the partial fraction decomposition, we rewrite $G(x)$ as

$$G(x) = \frac{1}{\alpha - \beta}\left[\begin{array}{c}\dfrac{\begin{array}{c}x\{\alpha(b\mathcal{F}_0(a,b) - \mathcal{F}_1(a,b)) + b\mathcal{F}_0(a,b)\} \\ +\alpha(a\mathcal{F}_1(a,b) - \mathcal{F}_0(a,b) - ab\mathcal{F}_0(a,b)) \\ +a\mathcal{F}_1(a,b) - ab\mathcal{F}_0(a,b)\end{array}}{x^2-(\alpha+1)} \\ \\ +\dfrac{\begin{array}{c}x\{\beta(\mathcal{F}_1(a,b) - b\mathcal{F}_0(a,b)) - b\mathcal{F}_0(a,b)\} \\ +\beta(ab\mathcal{F}_0(a,b) + \mathcal{F}_0(a,b) - a\mathcal{F}_1(a,b)) \\ +ab\mathcal{F}_0(a,b) - a\mathcal{F}_1(a,b)\end{array}}{x^2-(\beta+1)}\end{array}\right].$$

Since the Maclaurin series expansion of the function $\frac{A-Bx}{x^2-C}$ is given by

$$\frac{A-Bx}{x^2-C} = \sum_{n=0}^{\infty} BC^{-n-1}x^{2n+1} - \sum_{n=0}^{\infty} AC^{-n-1}x^{2n},$$

the generating function $G(x)$ can be expressed as

$$G(x) = \frac{1}{\alpha - \beta}\left[\begin{array}{c}\sum_{n=0}^{\infty}\dfrac{\left\{\begin{array}{c}\alpha(\mathcal{F}_1(a,b) - b\mathcal{F}_0(a,b)) \\ -b\mathcal{F}_0(a,b)\end{array}\right\}(\beta+1)^{n+1} + \left\{\begin{array}{c}\beta(b\mathcal{F}_0(a,b) - \mathcal{F}_1(a,b)) \\ +b\mathcal{F}_0(a,b)\end{array}\right\}(\alpha+1)^{n+1}}{(\alpha+1)^{n+1}(\beta+1)^{n+1}}x^{2n+1} \\ \\ -\sum_{n=0}^{\infty}\dfrac{\begin{array}{c}\alpha(a\mathcal{F}_1(a,b) - \mathcal{F}_0(a,b) - ab\mathcal{F}_0(a,b)) \\ +a\mathcal{F}_1(a,b) - ab\mathcal{F}_0(a,b)\end{array}}{(\alpha+1)^{n+1}(\beta+1)^{n+1}}(\beta+1)^{n+1}x^{2n} \\ \\ -\sum_{n=0}^{\infty}\dfrac{\begin{array}{c}\beta(ab\mathcal{F}_0(a,b) + \mathcal{F}_0(a,b) - a\mathcal{F}_1(a,b)) \\ +ab\mathcal{F}_0(a,b) - a\mathcal{F}_1(a,b)\end{array}}{(\alpha+1)^{n+1}(\beta+1)^{n+1}}(\alpha+1)^{n+1}x^{2n}\end{array}\right].$$

By using properties of α and β, since we know that α and β are roots of equation $X^2 - abX - ab = 0$, we obtain

$$G(x) = \frac{1}{\alpha - \beta}\sum_{n=0}^{\infty}\left(\frac{1}{ab}\right)^{n+1}\left(\begin{array}{c}-ab(\mathcal{F}_1(a,b) - b\mathcal{F}_0(a,b))\beta^{2n+1} \\ -ab(b\mathcal{F}_0(a,b) - \mathcal{F}_1(a,b))\alpha^{2n+1} \\ -b\mathcal{F}_0(a,b)\beta^{2n+2} + b\mathcal{F}_0(a,b)\alpha^{2n+2}\end{array}\right)x^{2n+1}$$

$$+\frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\left(\frac{1}{ab}\right)^{n+1}\left(\begin{array}{c} -ab\left(\begin{array}{c} a\mathcal{F}_1\left(a,b\right)-\mathcal{F}_0\left(a,b\right)\\ -ab\mathcal{F}_0\left(a,b\right)\end{array}\right)\beta^{2n}\\ -ab\left(\begin{array}{c} ab\mathcal{F}_0\left(a,b\right)+\mathcal{F}_0\left(a,b\right)\\ -a\mathcal{F}_1\left(a,b\right)\end{array}\right)\alpha^{2n}\\ -\mathcal{F}_0\left(a,b\right)\beta^{2n+2}+\mathcal{F}_0\left(a,b\right)\alpha^{2n+2}\end{array}\right)x^{2n}.$$

$$\begin{aligned} G(x) \;=\; & \sum_{n=0}^{\infty}\left(\frac{1}{ab}\right)^{n}\frac{\left(\mathcal{F}_1\left(a,b\right)-b\mathcal{F}_0\left(a,b\right)\right)\left(\alpha^{2n+1}-\beta^{2n+1}\right)}{\alpha-\beta}x^{2n+1}\\ & +\sum_{n=0}^{\infty}\left(\frac{1}{ab}\right)^{n+1}\frac{b\mathcal{F}_0\left(a,b\right)\left(\alpha^{2n+2}-\beta^{2n+2}\right)}{\alpha-\beta}x^{2n+1}\\ & +\sum\left(\frac{1}{ab}\right)^{n}\frac{\left(ab\mathcal{F}_0\left(a,b\right)+\mathcal{F}_0\left(a,b\right)-a\mathcal{F}_1\left(a,b\right)\right)\left(\beta^{2n}-\alpha^{2n}\right)}{\alpha-\beta}x^{2n}\\ & +\sum_{n=0}^{\infty}\left(\frac{1}{ab}\right)^{n+1}\frac{\mathcal{F}_0\left(a,b\right)\left(\alpha^{2n+2}-\beta^{2n+2}\right)}{\alpha-\beta}x^{2n}. \end{aligned}$$

Combining the sums, we get

$$\begin{aligned} G(x) \;=\; & \sum_{n=0}^{\infty}\Bigg\{-\left[\mathcal{F}_1\left(a,b\right)-b\mathcal{F}_0\left(a,b\right)\right]^{\varepsilon(n)}\left[a\mathcal{F}_1\left(a,b\right)-\mathcal{F}_0\left(a,b\right)-ab\mathcal{F}_0\left(a,b\right)\right]^{1-\varepsilon(n)}\\ & \times\left(\frac{\alpha^n-\beta^n}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}(\alpha-\beta)}\right)+b^{\varepsilon(n)}\mathcal{F}_0\left(a,b\right)\left(\frac{\alpha^{2\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)}-\beta^{2\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)}}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor+1}(\alpha-\beta)}\right)\Bigg\}x^n. \end{aligned}$$

Therefore, for all $n \geq 0$, from the definition of generating function, we have

$$\mathcal{F}_n\left(a,b\right)=A_1\left(n\right)\left(\alpha^n-\beta^n\right)+B_1\left(n\right)\left(\alpha^{2\left\lfloor\frac{n}{2}\right\rfloor+2}-\beta^{2\left\lfloor\frac{n}{2}\right\rfloor+2}\right),$$

which is desired. $\qquad\square$

Now, for bi-periodic Fibonacci matrix sequence, we give the some *summations* by considering the Binet formula.

Theorem 2.7. *For $n \geq 1$, the following statements are true:*

(i)

$$\sum_{k=0}^{n-1}\mathcal{F}_k\left(a,b\right)=\frac{\begin{array}{c}a^{\varepsilon(n)}b^{1-\varepsilon(n)}\mathcal{F}_n\left(a,b\right)+a^{1-\varepsilon(n)}b^{\varepsilon(n)}\mathcal{F}_{n-1}\left(a,b\right)\\ -a\mathcal{F}_1\left(a,b\right)+ab\mathcal{F}_0\left(a,b\right)-b\mathcal{F}_0\left(a,b\right)\end{array}}{ab},$$

(ii)

$$\sum_{k=0}^{n} \mathcal{F}_k\left(a,b\right) x^{-k} = \frac{1}{1-(ab+2)x^2+x^4}\left\{\begin{array}{c} \dfrac{\mathcal{F}_{n-1}\left(a,b\right)}{x^{n-1}}-\dfrac{\mathcal{F}_{n+1}\left(a,b\right)}{x^{n-3}} \\ +\dfrac{\mathcal{F}_n\left(a,b\right)}{x^n}-\dfrac{\mathcal{F}_{n+2}\left(a,b\right)}{x^{n-2}} \\ +x^4\mathcal{F}_0\left(a,b\right)+x^3\mathcal{F}_1\left(a,b\right) \\ -x^2\left[(ab+1)\mathcal{F}_0\left(a,b\right)-a\mathcal{F}_1\left(a,b\right)\right] \\ -x\left(\mathcal{F}_1\left(a,b\right)-b\mathcal{F}_0\left(a,b\right)\right) \end{array}\right\}.$$

where $\varepsilon(n) = n - 2\left\lfloor\frac{n}{2}\right\rfloor$.

Proof. We omit the proof of (ii), because it can be done similarly as in the proof of (i). We investigate the situation according to the n is even or odd. Thus, for even n

$$\sum_{k=0}^{n-1} \mathcal{F}_k\left(a,b\right) = \sum_{k=0}^{\frac{n-2}{2}} \mathcal{F}_{2k}\left(a,b\right) + \sum_{k=0}^{\frac{n-2}{2}} \mathcal{F}_{2k+1}\left(a,b\right)$$

$$\begin{aligned}
\sum_{k=0}^{n-1} \mathcal{F}_k\left(a,b\right) &= \sum_{k=0}^{\frac{n-2}{2}} \frac{a\mathcal{F}_1\left(a,b\right)-\mathcal{F}_0\left(a,b\right)-ab\mathcal{F}_0\left(a,b\right)}{(ab)^k}\frac{\alpha^{2k}-\beta^{2k}}{\alpha-\beta} \\
&+\sum_{k=0}^{\frac{n-2}{2}} \frac{\mathcal{F}_0\left(a,b\right)}{(ab)^{k+1}}\frac{\alpha^{2k+2}-\beta^{2k+2}}{\alpha-\beta} \\
&+\sum_{k=0}^{\frac{n-2}{2}} \frac{\mathcal{F}_1\left(a,b\right)-b\mathcal{F}_0\left(a,b\right)}{(ab)^k}\frac{\alpha^{2k+1}-\beta^{2k+1}}{\alpha-\beta} \\
&+\sum_{k=0}^{\frac{n-2}{2}} \frac{b\mathcal{F}_0\left(a,b\right)}{(ab)^{k+1}}\frac{\alpha^{2k+2}-\beta^{2k+2}}{\alpha-\beta}.
\end{aligned}$$

In here, simplifying the last equality, we imply

$$\begin{aligned}
\sum_{k=0}^{n-1} \mathcal{F}_k\left(a,b\right) &= \frac{\begin{array}{c}a\mathcal{F}_1\left(a,b\right)-\mathcal{F}_0\left(a,b\right)\\-ab\mathcal{F}_0\left(a,b\right)\end{array}}{\alpha-\beta}\left\{\begin{array}{c}\frac{\alpha^n-(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\alpha^2-ab)}\\-\frac{\beta^n-(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\beta^2-ab)}\end{array}\right\} \\
&+\frac{\mathcal{F}_0\left(a,b\right)}{\alpha-\beta}\left\{\begin{array}{c}\frac{\alpha^{n+2}-\alpha^2(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}}(\alpha^2-ab)}\\-\frac{\beta^{n+2}-\beta^2(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}}(\beta^2-ab)}\end{array}\right\} \\
&+\frac{\mathcal{F}_1\left(a,b\right)-b\mathcal{F}_0\left(a,b\right)}{\alpha-\beta}\left\{\begin{array}{c}\frac{\alpha^{n+1}-\alpha(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\alpha^2-ab)}\\-\frac{\beta^{n+1}-\beta(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\beta^2-ab)}\end{array}\right\}
\end{aligned}$$

$$+\frac{b\mathcal{F}_0\left(a,b\right)}{\alpha-\beta}\left\{\begin{array}{c}\frac{\alpha^{n+2}-\alpha^2(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}}(\alpha^2-ab)}\\-\frac{\beta^{n+2}-\beta^2(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}}(\beta^2-ab)}\end{array}\right\}.$$

$$\sum_{k=0}^{n-1}\mathcal{F}_k\left(a,b\right)\ =\ \frac{a\mathcal{F}_1\left(a,b\right)-\mathcal{F}_0\left(a,b\right)-ab\mathcal{F}_0\left(a,b\right)}{\alpha-\beta}\left\{\begin{array}{c}-\frac{\alpha^{n-2}-\beta^{n-2}}{(ab)^{\frac{n}{2}}}\\+\frac{\alpha^n-\beta^n}{(ab)^{\frac{n}{2}+1}}\\-\frac{\alpha^2-\beta^2}{(ab)^2}\end{array}\right\}$$

$$+\frac{\mathcal{F}_0\left(a,b\right)}{\alpha-\beta}\left\{-\frac{\alpha^n-\beta^n}{(ab)^{\frac{n}{2}+1}}+\frac{\alpha^{n+2}-\beta^{n+2}}{(ab)^{\frac{n}{2}+2}}-\frac{\alpha^2-\beta^2}{(ab)^2}\right\}$$

$$+\frac{\mathcal{F}_1\left(a,b\right)-b\mathcal{F}_0\left(a,b\right)}{\alpha-\beta}\left\{-\frac{\alpha^{n-1}-\beta^{n-1}}{(ab)^{\frac{n}{2}}}+\frac{\alpha^{n+1}-\beta^{n+1}}{(ab)^{\frac{n}{2}+1}}\right\}$$

$$+\frac{b\mathcal{F}_0\left(a,b\right)}{\alpha-\beta}\left\{-\frac{\alpha^n-\beta^n}{(ab)^{\frac{n}{2}+1}}+\frac{\alpha^{n+2}-\beta^{n+2}}{(ab)^{\frac{n}{2}+2}}-\frac{\alpha^2-\beta^2}{(ab)^2}\right\}$$

and thus we get

$$\sum_{k=0}^{n-1}\mathcal{F}_k\left(a,b\right)\ =\ \frac{\begin{array}{c}\mathcal{F}_{n+1}\left(a,b\right)+\mathcal{F}_n\left(a,b\right)-\mathcal{F}_{n-1}\left(a,b\right)-\mathcal{F}_{n-2}\left(a,b\right)\\-a\mathcal{F}_1\left(a,b\right)+ab\mathcal{F}_0\left(a,b\right)-b\mathcal{F}_0\left(a,b\right)\end{array}}{ab}$$

$$=\ \frac{b\mathcal{F}_n\left(a,b\right)+a\mathcal{F}_{n-1}\left(a,b\right)-a\mathcal{F}_1\left(a,b\right)+ab\mathcal{F}_0\left(a,b\right)-b\mathcal{F}_0\left(a,b\right)}{ab}.$$

Similarly, for odd n, we obtain

$$\sum_{k=0}^{n-1}\mathcal{F}_k\left(a,b\right)\ =\ \sum_{k=0}^{\frac{n-1}{2}}\mathcal{F}_{2k}\left(a,b\right)+\sum_{k=0}^{\frac{n-3}{2}}\mathcal{F}_{2k+1}\left(a,b\right)$$

$$=\ \frac{a\mathcal{F}_n\left(a,b\right)+b\mathcal{F}_{n-1}\left(a,b\right)-a\mathcal{F}_1\left(a,b\right)+ab\mathcal{F}_0\left(a,b\right)-b\mathcal{F}_0\left(a,b\right)}{ab}.$$

As a result, we find

$$\sum_{k=0}^{n-1}\mathcal{F}_k\left(a,b\right)=\frac{\begin{array}{c}a^{\varepsilon(n)}b^{1-\varepsilon(n)}\mathcal{F}_n\left(a,b\right)+a^{1-\varepsilon(n)}b^{\varepsilon(n)}\mathcal{F}_{n-1}\left(a,b\right)\\-a\mathcal{F}_1\left(a,b\right)+ab\mathcal{F}_0\left(a,b\right)-b\mathcal{F}_0\left(a,b\right)\end{array}}{ab}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It is clear that the following result is correct for the bi-periodic Fibonacci matrix sequence as a consequence of the condition (ii) of Theorem 2.7.

Corollary 2.8. *For $k > 0$, we have*

$$\sum_{k=0}^{\infty} \mathcal{F}_k\left(a, b\right) x^{-k} = \frac{x}{1 - (ab+2)x^2 + x^4} \left[\begin{array}{cc} x^3 + bx^2 - x & \frac{b}{a}x^2 + bx - \frac{b}{a} \\ x^2 + ax - 1 & x^3 - (ab+1)\,x + b \end{array} \right].$$

Conclusion

In this paper, we define bi-periodic matrix sequence and give some properties of this new sequence. Thus, it is obtained a new genaralization for the matrix sequences and number sequences that have the similar recurrence relation in the literature. By taking into account this generalized matrix sequence and its properties, it also can be obtained properties of bi-periodic Fibonacci numbers. That is, if we compare the 2 nd row and $1st$ column entries of obtained equalities for matrix sequence in Section 2, we can get some properties for bi-periodic Fibonacci numbers. Also, some well-known matrix sequences, such as Fibonacci, Pell and k-Fibonacci are special cases of $\{\mathcal{F}_n\left(a, b\right)\}$ matrix sequence. That is, if we choose the different values of a and b, then we obtain the summations, generating functions, Binet formulas of the well-known matrix sequence in the literature:

- If we replace $a = b = 1$ in $\mathcal{F}_n\left(a, b\right)$, we obtain the generating function, Binet formula and summations for Fibonacci matrix sequence and Fibonacci numbers.
- If we replace $a = b = 2$ in $\mathcal{F}_n\left(a, b\right)$, we obtain the generating function, Binet formula and summations for Pell matrix sequence and Pell numbers.
- If we replace $a = b = k$ in $\mathcal{F}_n\left(a, b\right)$, we obtain the generating function, Binet formula and summations for k-Fibonacci matrix sequence and k-Fibonacci numbers.

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