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# LOXODROMES ON HELICOIDAL SURFACES AND TUBES WITH VARIABLE RADIUS IN $\mathbb{E}^4$

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Abstract. In this paper, we generalize the equations of loxodromes on helicoidal surfaces and canal surfaces in $\mathbb{E}^3$ to the case of 4-dimension ($\mathbb{E}^4$). Also we give some examples via Mathematica.

## 1. Introduction

A curve which cuts all meridians at a constant angle on the Earth's surface is called as loxodrome. Loxodromes don't need a change of course and thus, they are usually used in navigation. Noble [11] investigated the equations of loxodromes on the rotational surfaces in Euclidean 3-space $\mathbb{E}^3$. The orbit of a plane curve under a screw motion is called as helicoidal surface and it is a natural generalization of rotational surface. The equations of loxodromes on helicoidal surfaces in $\mathbb{E}^3$ were found by Babaarslan and Yayli [3].

Another generalization of rotational surfaces is canal surfaces and they are defined as envelope of a family spheres whose trajectory of centers lie on a space curve. When the radius of spheres is constant, the canal surfaces reduce to tubes with constant radius [12]. Also, if the centers of spheres lie on a straight line, then the canal surface is a rotational surface [9]. For example, the sphere is a special canal surface whose axis is a straight line. The differential equations of the loxodromes on canal surfaces in $\mathbb{E}^3$ were given by Babaarslan [4].

Rotational surfaces in Euclidean 4-space $\mathbb{E}^4$ was first introduced by Moore [10]. After that, a lot of authors studied on rotational surfaces in $\mathbb{E}^4$ (see [13], [14], [1], [2], [6]).

The parametrization of tube with variable radius in $\mathbb{E}^4$ was given by Gal and Pal [9]. Also, the definition and parametrization of helicoidal surface in $\mathbb{E}^4$ were given by Hieu and Thang [8].

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In this paper, we generalize the equations of loxodromes on helicoidal surfaces and canal surfaces in $\mathbb{E}^3$ to the case of 4-dimension ($\mathbb{E}^4$). Also, we give some examples by using Mathematica computer programme.

## 2. Preliminaries

In this section, we recall some important notions and also give some properties of curves and surfaces in $\mathbb{E}^4$.

Let $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ be vectors in $\mathbb{E}^4$. Then, the inner product of them is given by

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4. \tag{1}$$

The norm (length) of a vector $x \in \mathbb{E}^4$ is defined by $\|x\| = \sqrt{\langle x, x\rangle}$ and the vector is called as a unit vector if $\|x\| = 1$.

Also, the angle $\theta$ between $x$ and $y$ is given by

$$\cos\theta = \frac{\langle x, y\rangle}{\|x\|\,\|y\|}, \tag{2}$$

where $0 < \theta < \pi$.

Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ be a regular curve in $\mathbb{E}^4$. The arc-length of $\beta$ between $t_0$ and $t$ is

$$s(t) = \int_{t_0}^{t} \left\|\beta'(t)\right\| dt. \tag{3}$$

Then, the parameter $s \in J \subset \mathbb{R}$ is determined such as $\left\|\beta'(s)\right\| = 1$. Thus, $\beta$ is called a unit speed curve if $\left\|\beta'(s)\right\| = 1$.

Now, we give the definitions and parametrizations of rotational surfaces and helicoidal surfaces in $\mathbb{E}^4$.

Let $\beta : I \subset \mathbb{R} \to \Pi$ be a smooth curve in a hyperplane $\Pi \subset \mathbb{E}^4$ and $P$ be a 2-plane line in $\Pi$. If the profile curve $\beta$ is rotated about $P$, then the resulting surface is rotational surface in $\mathbb{E}^4$. Similarly, let us assume that when $\beta$ rotates about $P$, it simultaneously translates along a line $l$ parallel to $P$ so that the speed of the translation is proportional to the speed of rotation. Then, the resulting surface is a helicoidal surface in $\mathbb{E}^4$ (see [8]).

Let $x,\ y,\ z,\ w$ be the coordinates in $\mathbb{E}^4$. We assume that $\Pi$ is $xzw$-hyperplane, $P$ is $zw$-plane and $l$ is parallel to the $z$-axis. Then, the rotation which leaves the plane $P$ invariant is given by the following rotational matrix

$$\begin{bmatrix} \cos v & -\sin v & 0 & 0 \\ \sin v & \cos v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ 0 \leq v < 2\pi \tag{4}$$

[7].

We consider the profile curve $\beta(u) = (f(u), 0, g(u), h(u))$ in $\Pi$, where $u \in I \subset \mathbb{R}$ and $f(u) > 0$. Then, the parametrization of the helicoidal surface $M$ is

$$M(u,v) = \begin{bmatrix} \cos v & -\sin v & 0 & 0 \\ \sin v & \cos v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f(u) \\ 0 \\ g(u) \\ h(u) \end{bmatrix} + \lambda v \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

so

$$M(u,v) = (f(u)\cos v, f(u)\sin v, g(u) + \lambda v, h(u)), \tag{5}$$

where $\lambda > 0$. When $g$ is a constant function, the helicoidal surface is called the right helicoidal surface. When $\lambda = 0$, the helicoidal surfaces reduce to rotational surfaces in $\mathbb{E}^4$. Also, when $h$ is a constant function, the surface is just a helicoidal surface in $\mathbb{E}^3$ [8].

Also, we give the parametrizations of tubes with variable radius in $\mathbb{E}^4$.

We consider the spine curve $\beta(u) = (f(u), g(u), h(u), 0)$, where $u \in J \subset \mathbb{R}$, that is $\beta$ is parametrized by arc-length. Then, the Frenet formulae is given by

$$\beta'(u) = e_1(u),$$

$$e_1{}'(u) = \kappa(u)e_2(u),$$

$$e_2{}'(u) = -\kappa(u)e_1(u) + \tau(u)e_3(u),$$

$$e_3{}'(u) = -\tau(u)e_2(u),$$

$$e_4{}'(u) = 0,$$

where $\{e_1(u), e_2(u), e_3(u), e_4(u)\}$ is Frenet orthonormal basis of $\beta$, $\kappa(u)$ and $\tau(u)$ are the curvatures of $\beta$.

Then, the parametrization of tube with variable radius $C$ is

$$C(u,v) = \beta(u) + r(u)\left(e_3(u)\cos v + e_4(u)\sin v\right). \tag{6}$$

([9], [5]).

## 3. The Equations of Loxodromes on Helicoidal Surfaces

In this section, we find the equations of loxodromes on the helicoidal surfaces as well as rotational surfaces in $\mathbb{E}^4$. Also, we give an example to strengthen our main results.

**Definition 1.** *A curve on a helicoidal surface in $\mathbb{E}^4$ is called as a loxodrome if the curve cuts all meridians at a constant angle on the helicoidal surface.*

Let us consider the helicoidal surface $M$ which is given by Eq. (5). To simplify the calculations, we assume that $\beta$ is parametrized by arc-length, i.e., $f'^2(u) + g'^2(u) + h'^2(u) = 1$ for all $u \in J \subset \mathbb{R}$.

The tangent plane to $M$ at a point $p = M(u, v)$ is span$\{M_u, M_v\}$. A direct computation yields

$$M_u = (f'(u)\cos v, f'(u)\sin v, g'(u), h'(u)) \text{ and } M_v = (-f(u)\sin v, f(u)\cos v, \lambda, 0). \tag{7}$$

By using these equations, the coefficients of first fundamental form of $M$ are

$$E = \langle M_u, M_u\rangle = 1,\ F = \langle M_u, M_v\rangle = \lambda g'(u) \text{ and } G = \langle M_v, M_v\rangle = f^2(u)+\lambda^2. \tag{8}$$

Assume that $EG - F^2 = \lambda^2(1 - g'^2) + f^2 > 0$, that is, $M$ is regular.

The first fundamental form of $M$ is

$$ds^2 = du^2 + 2\lambda g'^2(u) + \lambda^2)dv^2. \tag{9}$$

Also, the arc-length of any curve on $M$ between $u_1$ and $u_2$ is given by

$$s = \left| \int_{u_1}^{u_2} \sqrt{1 + 2\lambda g'(u)\frac{dv}{du} + (f^2(u) + \lambda^2)(\frac{dv}{du})^2 du} \right|. \tag{10}$$

Suppose that $\alpha(t)$ is a curve on $M$. Then, we can write $\alpha(t) = M(u(t), v(t))$. With respect to the local base $\{M_u, M_v\}$, the vector $\alpha'(t)$ has the coordinates $(u', v')$ and the vector $M_u$ has the coordinates $(1, 0)$. At the point $p = M(u, v)$, where the loxodrome cuts the meridian at a constant angle $\theta$, we get

$$\cos\theta = \frac{du + \lambda g'(u)dv}{\sqrt{du^2 + 2\lambda g'^2(u)dudv + (f^2(u) + \lambda^2)dv^2}}. \tag{11}$$

Then, Eq. (11) can be expressed in the form:

$$\left(\cos^2\theta(f^2(u) + \lambda^2) - \lambda^2 g'^2(u)\right)(\frac{dv}{du})^2 - 2\lambda\sin^2\theta g'(u)\frac{dv}{du} = \sin^2\theta. \tag{12}$$

This is differential equation of the loxodromes on the helicoidal surfaces in $\mathbb{E}^4$.

Thus, the general solution of Eq. (12) becomes

$$v = v(u) = \int_{u_0}^{u} \frac{2\lambda\sin^2\theta g'(u) + \varepsilon\sqrt{\sin^2 2\theta\left(f^2(u) - \lambda^2(g'^2(u) - 1)\right)}}{2\cos^2\theta(f^2(u) + \lambda^2) - 2\lambda^2 g'^2(u)} du, \tag{13}$$

where $\varepsilon$ is plus or minus.

Then, we can give the following theorem.

**Theorem 1.** *The loxodromes on the helicoidal surfaces in* $\mathbb{E}^4$ *are*

$$\alpha(u) = (f(u)\cos v(u), f(u)\sin v(u), g(u) + \lambda v(u), h(u)),$$

*where* $v(u)$ *is given by Eq. (13).*

When $\lambda = 0$ in Eq. (13), we find the following general solution of differential equation of the loxodromes on the rotational surfaces in $\mathbb{E}^4$

$$v = v(u) = \varepsilon\tan\theta \int_{u_0}^{u} \frac{du}{f(u)}. \tag{14}$$

Thus, we have

**Theorem 2.** *The loxodromes on the rotational surfaces in $\mathbb{E}^4$ are*

$$\gamma(u) = \left(f(u)\cos v(u), f(u)\sin v(u), g(u), h(u)\right),$$

*where* $v(u) = \varepsilon \tan\theta \int_{u_0}^{u} \frac{du}{f(u)}$.

If $g$ is a constant function, then the arc-length of loxodrome on the right helicoidal surface in $\mathbb{E}^4$ is given by

$$s = \left| \frac{u_2 - u_1}{\cos\theta} \right|. \tag{15}$$

Similarly, the arc-length of the loxodrome on the rotational surface in $\mathbb{E}^4$ coincides with Eq. (15).

Now, we give the following example.

**Example 1.** *Let us consider the profile curve* $\beta(u) = \left(\cos\frac{u}{2}, 0, \sin\frac{u}{2}, \frac{\sqrt{3}}{2}u\right)$. *If we take* $\lambda = 1$, $\theta = \frac{\pi}{2}$ *and* $\varepsilon = 1$, *then we have the following helicoidal surface:*

$$M(u,v) = \left(\cos\frac{u}{2}\cos v, \cos\frac{u}{2}\sin v, \sin\frac{u}{2} + v, \frac{\sqrt{3}}{2}u\right).$$

*By using Eq. (13) and taking* $u_0 = 0$, *we get* $v(u) = 4\ln\left|\frac{1-\tan\frac{u}{4}}{1+\tan\frac{u}{4}}\right|$. *Taking* $u \in (-2, 2)$, *we have* $v \in (-4.9048, 4.9048)$.

*Then, the loxodrome is*

$$\alpha(u) = \left(\cos\frac{u}{2}\cos v(u), \cos\frac{u}{2}\sin v(u), \sin\frac{u}{2} + v(u), \frac{\sqrt{3}}{2}u\right),$$

*where* $v(u) = 4\ln\left|\frac{1-\tan\frac{u}{4}}{1+\tan\frac{u}{4}}\right|$. *Also, by using Eq. (10), the arc-length of the loxodrome is approximately equal to 12.0528.*

*Let us plot the graphs of the projections of helicoidal surface, loxodrome and meridian* ($v = 0$) *in* $\mathbb{E}^3$ *to see what they look like in* $\mathbb{E}^3$ *by using Mathematica plotting command*

*ParametricPlot3D[*$\{x(u,v), y(u,v), z(u,v) + w(u,v)\}$,$\{u, a, b\}$,$\{v, c, d\}$*];*

## 4. The Equations of Loxodromes on Tubes with Variable Radius

In this section, we find the differential equations of loxodromes on the tubes with variable radius in $\mathbb{E}^4$. Also, we give an example to strengthen our main results.

**Definition 2.** *A curve on a tube with variable radius in* $\mathbb{E}^4$ *is called as a loxodrome if the curve cuts all meridians at a constant angle on the tube with variable radius.*

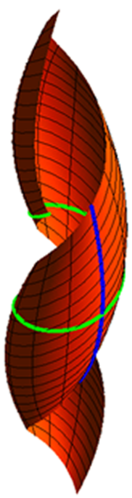

FIGURE 1. The projections of helicoidal surface, loxodrome (green), meridian (blue).

Let us consider the tube with variable radius $C$ which is given by Eq. (6). The tangent plane to $C$ at a point $p = C(u, v)$ is span$\{C_u, C_v\}$.

A direct computation yields

$$C_u = e_1(u) - r(u)\tau(u)\cos v e_2(u) + r'(u)\cos v e_3(u) + r'(u)\sin v e_4(u)$$

and

$$C_v = -r(u)\sin v e_3(u) + r(u)\cos v e_4(u).$$

By using these equations, the coefficients of first fundamental form of $C$ are

$$E = 1 + r'^2 + r^2\tau^2\cos^2 v,\ F = 0 \textit{ and } G = r^2. \tag{16}$$

Assume that $EG - F^2 = r^2 + r^2 r'^2 + r^4\tau^2\cos^2 v > 0$, that is, $C$ is regular.

The first fundamental form of $C$ is given by

$$ds^2 = (1 + r'^2 + r^2\tau^2\cos^2 v)du^2 + r^2 dv^2. \tag{17}$$

Also, the arc-length of any curve on $C$ between $u_1$ and $u_2$ is defined by

$$s = \left|\int_{u_1}^{u_2}\sqrt{1 + r'^2 + r^2\tau^2\cos^2 v + r^2(\frac{dv}{du})^2}du\right|. \tag{18}$$

Let us assume that $\sigma(t)$ is a curve on $C$. Then, we can write $\sigma(t) = C(u(t), v(t))$. With respect to the local base $\{C_u, C_v\}$ , the vector $\sigma'(t)$ has the coordinates $(u', v')$ and the vector $C_u$ has the coordinates $(1, 0)$. At the point $p = C(u, v)$, where the loxodrome cuts the meridian at a constant angle $\varphi$, we get

$$\cos\varphi = \frac{(1+r'^2+r^2\tau^2\cos^2 v)du}{\sqrt{(1+(r'^2+r^2\tau^2\cos^2 v)^2du^2+(1+r'^2+r^2\tau^2\cos^2 v)r^2dv^2}} . \tag{19}$$

Then, differential equation of the loxodromes on the tubes with variable radius in $\mathbb{E}^4$ is given by

$$(\frac{dv}{du})^2 = \tan^2\varphi\left(\frac{1+r'^2}{r^2} + \tau^2\cos^2 v\right). \tag{20}$$

Furthermore, we put the condition that the second curvature of $\beta$ has null value. This means that $\beta$ is a 2-plane curve. Then, the general solution of Eq. (20) becomes

$$v = v(u) = \varepsilon\tan\varphi\int_{u_0}^{u}\frac{\sqrt{1+r'^2}}{r}du, \tag{21}$$

where $\varepsilon$ is plus or minus.

Under this condition, we can give the following theorem.

**Theorem 3.** *The loxodromes on the tubes with variable radius in $\mathbb{E}^4$ are*

$$\sigma(u) = \beta(u) + r(u)\left(e_3(u)\cos v(u) + e_4(u)\sin v(u)\right),$$

*where $v(u) = \varepsilon\tan\varphi\int_{u_0}^{u}\frac{\sqrt{1+r'^2}}{r}du$.*

Now, we give the following example.

**Example 2.** *Let us consider the base curve $\beta(u) = (u, 0, 0, 0)$. Taking $r(u) = u$, $\varphi = \frac{\pi}{4}$ and $\varepsilon = 1$, we have the following tube with variable radius:*

$$C(u, v) = (u, 0, u\cos v, u\sin v).$$

*Using Eq. (21) and taking $u_0 = 1$, we get $v(u) = \sqrt{2}\ln|u|$. Taking $u \in (0.1, 9)$, we have $v \in (-3.2563, 3.1073)$. Then, the loxodrome is*

$$\sigma(u) = \left(u, 0, u\cos(\sqrt{2}\ln u), u\sin(\sqrt{2}\ln u)\right).$$

*Also, by using Eq. (18), the arc-length of the loxodrome is approximately equal to 17.8.*

*Let us plot the graphs of the projections of tube with variable radius, loxodrome and meridian ($v = 2$) in $\mathbb{E}^3$ to see what they look like in $\mathbb{E}^3$ by using Mathematica plotting command*

*ParametricPlot3D[$\{x(u,v) + y(u,v), z(u,v), w(u,v)\}$,$\{u, a, b\}$,$\{v, c, d\}$];*

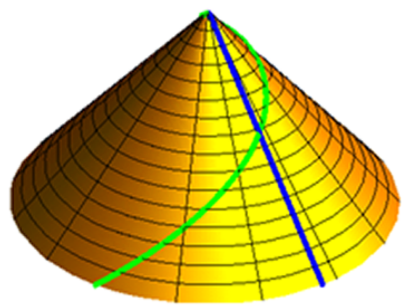

FIGURE 2. The projections of tubes with variable radius, loxodrome (green), meridian (blue).

## 5. Conclusion

Loxodromes are special curves which cut all meridians at a constant angle on the Earth's surface. They do not need a change of course. Thus, they are usually used in navigation. Loxodromes on rotational, helicoidal and canal surfaces in Euclidean 3-space $\mathbb{E}^3$ were studied by different authors (see [11], [3], [4]). In this paper, we investigate the equations of loxodromes on helicoidal surfaces as well as the tubes with variable radius in Euclidean 4-space $\mathbb{E}^4$, that is, we generalize the equations of loxodromes on helicoidal surfaces and canal surfaces in $\mathbb{E}^3$ to $\mathbb{E}^4$. The next time, we will investigate the differential equations of space-like and time-like loxodromes on the non-degenerate rotational surfaces, helicoidal surfaces and tubes with variable radius in Minkowski space-time $\mathbb{E}^4_1$.

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