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SOME HERMITE-HADAMARD TYPE INTEGRAL INEQUALITIES FOR S-CONVEX STOCHASTIC PROCESSES ON N-COORDINATES

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Abstract. In this study, we identified s-convexity of first and second sense for multidimensional stochastic processes. Concordantly, we verified Hermite-Hadamard type inequalities for these processes. Besides, we exemplified these results on two and three-dimensional stochastic processes. Ultimately, we compared our results with multidimensional harmonically convex stochastic processes in the literature. It must be known that the inequalities in our study are especially necessary to compare the maximum and minimum values of s-convex of first and second sense for multidimensional stochastic process with the expected value of stochastic processes. It is used mean-square integrability for the speciality of stochastic processes to obtain these inequalities in this study.

1. Introduction and Preliminaries

The function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, is referred convex [1], if the inequality satisfies for $x, y \in I$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y); \lambda \in [0,1].$$

It becomes famous the following Hermite-Hadamard inequality for convex functions [2]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}.$$

In terms of probability theory, this inequality gives a lower bound and an upper bound for the expectation value of a random variable X which distributed uniformly on $[a,b]$ [3]. Based on importance of convexity, many researchers studied on this issue, for examples Kumar [4], Gavrea [5]. In a sense, a stochastic process is a temporal parameterized family of random variables on a probability space [6]. In

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other words, let $(\Omega, \Im, P)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable, if it is $\Im-$measurable. Correspondingly, $X : I \subset \mathbb{R} \times \Omega \to \mathbb{R}$ is called a stochastic process, if $t \in I$ is considered of a time parameter of the random function $X(t, \omega)$ for all $\omega \in \Omega$. In this sense, researchers tackle many problems related convexity and inequality for stochastic processes privately [6]-[21]. Let us call up some basic notions related stochastic processes [6]. The process X is defined as
(i) continuous in probability on I, if $P - lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot)$ for $t_0 \in I$, where $P - lim$ defined limit in probability,
(ii) mean-square continuous at $t_0 \in I$, if $lim_{t \to t_0} E[X(t, \cdot) - X(t_0, \cdot)]^2 = 0$, where $E[X(t, \cdot)]$ is the expectation value of $X(t, \cdot)$
Also, a stochastic process is mean-square continuous(differentiable) on I, if it is continuous(differentiable) on the interval I,
(iii) increasing (decreasing) $t < s$ $(t > s)$ if $X(t, \cdot) \leq X(s, \cdot)$ $(X(t, \cdot) \geq X(s, \cdot))$for all $t, s \in I$,
(iv) monotonic if it is increasing or decreasing,
(v) mean-square differentiable at a point if $t \in I$ $X^{'}(t, \cdot) : I \times \Omega \to \mathbb{R}$ such that

$$X^{'}(t, \cdot) = P - \underset{t \to t_0}{lim} \frac{X(t, \cdot) - X(t_0, \cdot)}{t - t_0},$$

(vi) mean-square integrable on $[0, t] \subset I$ if

$$\underset{n \to \infty}{lim} E\left(\sum_{k=1}^{n} X(\ \Theta_k, \cdot).(t_k - t_{k-1}) - \eta(t, \cdot)\right)^2 = 0$$

where $\Theta_k \in [t_{k-1}, t_k]$ such that $[t_{k-1}, t_k]$, $k = 1, \ldots, n$ for $0 = t_0 < t_1 \ldots t_n = t$ is a partition of $[0, t]$. Then almost everywhere, it can be sometimes showed with (a.e.)

$$\int_0^t X(u, \cdot)\, du = \eta(t, \cdot).$$

Moreover, many applications of stochastic convexity were provided Shaked et al. [7]. According to Shaked and Shantikumar, there are various ways to define stochastic monotonicity and convexity for stochastic processes, and it is of great importance in optimization, especially in optimal designs, and also useful for numerical approximations when there exist probabilistic quantities. In 1980, Nikodem [8] described convex stochastic processes and some properties for them. Skowronski [9] introduced Jensen-convex, λ-convex stochastic processes in 1995. From 2010, Kotrys [6] obtained Hermite-Hadamard inequality for some convex stochastic processes. Accordingly, the stochastic process $X : I \subset \mathbb{R} \times \Omega \to \mathbb{R}$ is called convex, if the following inequality holds almost everywhere

$$X(\lambda t + (1 - \lambda) s, \cdot) \leq \lambda X(t, \cdot) + (1 - \lambda) X(s, \cdot) \tag{1}$$

for all $t, s \in I$ $\lambda \in [0,1]$. then almost everywhere

$$X\left(\frac{u+v}{2},\cdot\right) \leq \frac{1}{v-u}\int_u^v X(t,\cdot)\,dt \leq \frac{X(u,\cdot)+X(v,\cdot)}{2}.$$

Shortly, let us mention two types of s-convex stochastic processes in [10]-[11]. Let fix $s \in (0,1]$ and the stochastic process $X : [0,\infty) \times \Omega \to \mathbb{R}$ satisfy the following inequality almost everywhere for all $u, v \geq 0$

$$X(\alpha u + \beta v,\cdot) \leq \alpha^s X(u,\cdot) + \beta^s X(v,\cdot). \tag{2}$$

The stochastic process X is called s-convex of first or second sense, if $\alpha^s + \beta^s = 1$ or $\alpha + \beta = 1$ for $\alpha, \beta \geq 0$, respectively. Then almost everywhere respectively

$$X\left(\frac{u+v}{2},\cdot\right) \leq \frac{1}{v-u}\int_u^v X(t,\cdot)\,dt \leq \frac{X(u,\cdot)+sX(v,\cdot)}{s+1}, \tag{3}$$

$$2^{s-1}X\left(\frac{u+v}{2},\cdot\right) \leq \frac{1}{v-u}\int_u^v X(t,\cdot)\,dt \leq \frac{X(u,\cdot)+X(v,\cdot)}{s+1}. \tag{4}$$

Recently, Set et al. [12] proved Hermite-Hadamard inequalities for coordinated convex stochastic processes. Sarikaya et al. [13] studied on some types of stochastic processes and obtained new inequalities for related processes. Materano et al. [14] presented some estimates for quasi-convex stochastic processes. Barraaez et al. [15] also studied on h-convex stochastic processes. Moreover, Agahi et al. [16] worked on comonotonic stochastic processes and generalized mean-square stochastic integral. Recently, Okur et al. [17]-[18] investigated some convex stochastic processes and obtained also the Hermite-Hadamard type inequalities for a few of these processes both on real number set and on the rectangle plane.

Nowadays, Elahi [19] obtained Hadamard's inequality for s-convex function on n-coordinates. Viloria et al. [20] obtained Hermite-Hadamard type inequalities for harmonically convex functions on n-coordinates.

It is known that $\mathbb{N}(\mathbb{N}_0)$ is the set of all positive integers (non-negative integers) and $\mathbb{R}^n_+ := \{u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n : u_i > 0, i = 1, 2, \ldots, n\}$ Let $u < v$, $u, v \in \mathbb{R}^n_+$ be $u_i < v_i$ for each $i = 1, 2, \ldots, n$, and for $u_i < v_i$, $\Delta^n = \prod_{i=1}^n [u_i, v_i] \subseteq \mathbb{R}^n_+$ be n-dimensional closed interval [20].

Concordantly, Karahan et al. [21, 22, 23] investigated multidimensional convex, harmonically convex and φ-convex stochastic processes, and obtained Hermite-Hadamard type inequality for all of these processes. Okur et al. [24] derived Hermite-Hadamard type inequalities for multidimensional stochastic processes. Let us see the following results related multidimensional harmonically convex stochastic processes [22]:

The process $X : \Delta^n \subset \mathbb{R}^n_+ \times \Omega \to \mathbb{R}$ is defined harmonically convex on Δ^n, if the following inequality satisfies for all $t, s \in \Delta^n$ and $\lambda \in [0,1]$

$$X\left(\frac{ts}{\lambda t + (1-\lambda)s},\cdot\right) \leq \lambda X(s,\cdot) + (1-\lambda)X(t,\cdot).$$

Also, X is called a harmonically convex stochastic process, if the stochastic mappings of X, $X^{i}_{t_n}(t,\cdot) = X((t_1,\ldots,t_{i-1},t,t_{i+1},\ldots,t_n),\cdot)$ are harmonically convex almost everywhere on $[u_i,v_i]$. Then for $u,v\in\Delta^n$, almost everywhere

$$
\begin{aligned}
& \sum_{k=1}^{n-1} X\left(\left(t_1,\ldots,t_{k-1},\frac{2u_kv_k}{v_k+u_k},\frac{2u_{k+1}v_{k+1}}{v_{k+1}+u_{k+1}},\ldots,t_n\right),\cdot\right) \\
\leq\ & \sum_{k=1}^{n-1}\frac{u_kv_k}{v_k-u_k}\int_{u_k}^{v_k}\frac{X^{k}_{t_n}\left(\frac{2u_{k+1}v_{k+1}}{v_{k+1}+u_{k+1}},\cdot\right)}{t_k^2}dt_k \\
\leq\ & \sum_{k=1}^{n-1}\frac{u_kv_ku_{k+1}v_{k+1}}{(v_k-u_k)(v_{k+1}-u_{k+1})}\int_{u_k}^{v_k}\int_{u_{k+1}}^{v_{k+1}}\frac{X^{k+1}_{t_n}(t_{k+1},\cdot)}{(t_kt_{k+1})^2}dt_{k+1}dt_k \\
\leq\ & \sum_{k=1}^{n-1}\frac{u_kv_k}{v_k-u_k}\int_{u_k}^{v_k}\frac{X^{k}_{t_n}(u_{k+1},\cdot)+X^{k}_{t_n}(v_{k+1},\cdot)}{2t_k^2}dt_k \\
\leq\ & \frac{1}{4}\sum_{k=1}^{n-1}\left[\begin{array}{c} X((t_1,\ldots,t_{k-1},u_k,u_{k+1},\ldots,t_n),\cdot) \\ +X((t_1,\ldots,t_{k-1},v_k,u_{k+1},\ldots,t_n),\cdot) \\ +X((t_1,\ldots,t_{k-1},u_k,v_{k+1},\ldots,t_n),\cdot) \\ +X((t_1,\ldots,t_{k-1},v_k,v_{k+1},\ldots,t_n),\cdot); \end{array}\right]; \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
& X\left(\left(\frac{2u_1v_1}{v_1+u_1},\ldots,\frac{2u_{n-1}v_{n-1}}{v_{n-1}+u_{n-1}},\frac{2u_nv_n}{v_n+u_n}\right),\cdot\right) \\
\leq\ & \left(\prod_{1}^{n}\frac{u_iv_i}{v_i-u_i}\right)\int_{u_1}^{v_1}\cdots\int_{u_n}^{v_n}\frac{X^{n}_{t_n}(t_n,\cdot)}{\left(\prod_1^n t_i\right)^2}dt_n\ldots dt_1 \\
\leq\ & \frac{1}{2^n}P_n(X((u,v),\cdot)), \qquad (6)
\end{aligned}
$$

where $e(n) := \{\lambda\in\mathbb{N}_0^n : \lambda\leq 1 \textit{ and } |\lambda| \textit{ is even}\}$, $o(n) := \{\lambda\in\mathbb{N}_0^n : \lambda\leq 1 \textit{ and } |\lambda| \textit{ is odd}\}$,

$$
P_n(X((u,v),\cdot)) := \sum_{\lambda\in e(n)} X\left(\frac{uv}{\lambda u+(1-\lambda)v},\cdot\right) + \sum_{\lambda\in o(n)} X\left(\frac{uv}{\lambda u+(1-\lambda)v},\cdot\right)
$$

for all $u,v\in\Delta^n$. In the light of these, we defined s-convexity of first and second sense for multidimensional stochastic processes. Our supplemental claim is to obtain Hermite-Hadamard type inequalities for these processes.

2. Main Results

In this section, firstly, we investigated primarily s-convex of first (second) sense multidimensional stochastic processes. Additionally, this section contains two-subsections. In these sub-sections, we obtained some Hermite-Hadamard type inequalities for s-convex of first and second sense multidimensional stochastic processes, respectively.
Moreover, we adduced these results on two and three-dimensional stochastic processes. Finally, we compared our results with multidimensional harmonically convex stochastic processes.
Note that $\Delta^n := \prod_1^n [u_i, v_i] \subseteq \mathbb{R}_+^n$ be n-dimensional interval for $0 \leq u_i < v_i$, $i = 1, 2, \ldots, n$.

Definition 1. *The stochastic process $X : \Delta^n \times \Omega \to \mathbb{R}$ is called s-convex of first (second) sense on n-coordinates if the following stochastic mapping of X are s-convex of first (second) sense on $[u_i, v_i]$ almost everywhere*

$$X_{t_n}^i (t, \cdot) := X ((t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n), \cdot)$$

for all $t, t_i \geq 0$; $i = 1, 2, \ldots, n$.

Definition 2. *Let fix $s \in (0, 1]$ and the stochastic process $X : \Delta^n \times \Omega \to \mathbb{R}$ satisfy the following inequality almost everywhere for all $\tau, \theta \in \Delta^n$*

$$X (\alpha\tau + \beta\theta, \cdot) \leq \alpha^s X (\tau, \cdot) + \beta^s X (\theta, \cdot) .$$

The stochastic process X is called s-convex of first (second) sense on Δ^n, if $\alpha^s + \beta^s = 1$ $(\alpha^s + \beta^s = 1)$ for $\alpha, \beta \geq 0$, respectively. If the above inequality is reversed then X is said to be s-concave of first (second) sense on Δ^n.

Lemma 3. *Every s-convex of first (second) sense stochastic process on Δ^n is s-convex of first (second) sense on n-coordinates almost everywhere, but converse is not true.*

Proof. Let X be s-convex of first (second) sense on Δ^n. Using (**??**) almost everywhere

$$\begin{aligned}
&X_{t_n}^i (\alpha u + \beta v, \cdot) := X ((t_1, \ldots, t_{i-1}, \alpha u + \beta v, t_{i+1}, \ldots, t_n), \cdot) \\
&\leq \alpha^s X ((t_1, \ldots, t_{i-1}, u, t_{i+1}, \ldots, t_n), \cdot) + \beta^s X ((t_1, \ldots, t_{i-1}, v, t_{i+1}, \ldots, t_n), \cdot) \\
&= \alpha^s X_{t_n}^i (u, \cdot) + \beta^s X_{t_n}^i (v, \cdot)
\end{aligned}$$

for $u, v \geq 0$ and $\alpha, \beta \geq 0$ with $s \in (0, 1]$. Then, the stochastic process X is s-convex of first (second) sense on n-coordinates. On the contrary, let us see the following example:

Example 4. *Let us consider a stochastic process $X : [0, 1]^n \times \Omega \to \mathbb{R}$ defined as*

$$X_{t_n}^i (t, \cdot) := X ((t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n), \cdot) = t_1, \ldots, t_{i-1}, t_i^s, t_{i+1}, \ldots, t_n$$

for $s \in (0,1]$. It is s-convex of first (second) sense on n-coordinates as follows:

$$\begin{aligned}&X_{t_n}^i\left((\alpha u+\beta v),\cdot\right) := X\left((t_1,\ldots,t_{i-1},(\alpha u+\beta v),t_{i+1},\ldots,t_n),\cdot\right)\\&= t_1\ldots t_{i-1}(\alpha u+\beta v)^s t_{i+1}\ldots t_n\\&\le \alpha^s(t_1\ldots t_{i-1}ut_{i+1}\ldots t_n)+\beta^s(t_1\ldots t_{i-1}vt_{i+1}\ldots t_n)\\&= \alpha^s X_{t_n}^i(u,\cdot)+\beta^s X_{t_n}^i(v,\cdot).\end{aligned}$$

On the other hand, we verified for $u=(1,1,\ldots,0), v=(0,1,\ldots,1)\in[0,1]^n$

$$X\left((\alpha u+\beta v),\cdot\right) > \alpha^s X(u,\cdot)+\beta^s X(v,\cdot),$$

since $X\left((\alpha u+\beta v),\cdot\right) = X\left((\alpha,1,1,\ldots,\beta),\cdot\right)=\alpha\beta; \alpha^s X(u,\cdot)+\beta^s X(v,\cdot)=0$ So, X is not s-convex of first (second) sense on $[0,1]^n$.

□

3. Hermite-Hadamard type integral inequalities for s-convex of first sense multidimensional stochastic process

Let $X:\Delta^n\times\Omega\to\mathbb{R}$ be a s-convex of first sense stochastic process and be integrated in mean-square on Δ^n. Accordingly, the stochastic process $X_{t_n}^i:[u_i,v_i]\times\Omega\to\mathbb{R}$ is s-convex of first sense and integrated in mean-square on $[u_i,v_i]$ for each $i=1,2,\ldots,n, n\ge 2$. Using (3)

$$X_{t_n}^i\left(\frac{u_i+v_i}{2},\cdot\right)\le\frac{1}{v_i-u_i}\int_{u_i}^{v_i}X_{t_n}^i(t_i,\cdot)\,dt_i\le\frac{X_{t_n}^i(u_i,\cdot)+sX_{t_n}^i(v_i,\cdot)}{s+1},(a.e.). \tag{7}$$

Theorem 5. *Let $X:\Delta^n\times\Omega\to\mathbb{R}$ be a s-convex of first sense stochastic process and be integrated in mean-square on Δ^n. Then almost everywhere*

$$\begin{aligned}&\sum_{i=1}^{n-1}X\left(\left(t_1,\ldots,t_{i-1},\frac{u_i+v_i}{2},\frac{u_{i+1}+v_{i+1}}{2},\ldots,t_n\right),\cdot\right)\\ \le\ &\sum_{i=1}^{n-1}\frac{1}{v_i-u_i}\int_{u_i}^{v_i}X_{t_n}^{i+1}\left(\frac{u_{i+1}+v_{i+1}}{2},\cdot\right)dt_i\\ \le\ &\sum_{i=1}^{n-1}\frac{1}{(v_i-u_i)(v_{i+1}-u_{i+1})}\int_{u_i}^{v_i}\int_{u_{i+1}}^{v_{i+1}}X_{t_n}^{i+1}(t_{i+1},\cdot)\,dt_{i+1}dt_i\\ \le\ &\sum_{i=1}^{n-1}\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\frac{X_{t_n}^{i+1}(u_{i+1},\cdot)+sX_{t_n}^{i+1}(v_{i+1},\cdot)}{s+1}dt_i\\ \le\ &\sum_{i=1}^{n-1}\frac{1}{(s+1)^2}\left[\begin{array}{c}X\left((t_1,\ldots,t_{i-1},u_i,u_{i+1},\ldots,t_n),\cdot\right)\\+sX\left((t_1,\ldots,t_{i-1},v_i,u_{i+1},\ldots,t_n),\cdot\right)\\+sX\left((t_1,\ldots,t_{i-1},u_i,v_{i+1},\ldots,t_n),\cdot\right)\\+s^2X\left((t_1,\ldots,t_{i-1},v_i,v_{i+1},\ldots,t_n),\cdot\right).\end{array}\right].\end{aligned} \tag{8}$$

Proof. Using (7), we obtain the following inequality for $X_{t_n}^{i+1}$ almost everywhere

$$X_{t_n}^{i+1}\left(\frac{u_{i+1}+v_{i+1}}{2},\cdot\right) \le \frac{1}{v_{i+1}-u_{i+1}}\int_{u_{i+1}}^{v_{i+1}} X_{t_n}^{i+1}\left(t_{i+1},\cdot\right)dt_{i+1}$$
$$\le \quad \frac{X_{t_n}^{i+1}\left(u_{i+1},\cdot\right)+sX_{t_n}^{i+1}\left(v_{i+1},\cdot\right)}{s+1}.$$

All of sides of the above inequalities by integrating over $[u_i, u_i]$

$$\begin{aligned} &\frac{1}{v_{i+1}-u_{i+1}}\int_{u_i}^{v_i} X_{t_n}^{i+1}\left(\frac{u_{i+1}+v_{i+1}}{2},\cdot\right)dt_i \\ \le\ & \frac{1}{\left(v_i-u_i\right)\left(v_{i+1}-u_{i+1}\right)}\int_{u_i}^{v_i}\int_{u_{i+1}}^{v_{i+1}} X_{t_n}^{i+1}\left(t_{i+1},\cdot\right)dt_{i+1}dt_i \\ \le\ & \frac{1}{v_i-u_i}\int_{u_i}^{v_i}\frac{X_{t_n}^{i+1}\left(u_{i+1},\cdot\right)+sX_{t_n}^{i+1}\left(v_{i+1},\cdot\right)}{s+1}dt_i. \end{aligned} \tag{9}$$

Using Hermite-Hadamard inequality for the left of (9)

$$\begin{aligned} &X\left(\left(t_1,\ldots,t_{i-1},\frac{u_i+v_i}{2},\frac{u_{i+1}+v_{i+1}}{2},\ldots,t_n\right),\cdot\right) \\ \le\ & \frac{1}{v_i-u_i}\int_{u_i}^{v_i} X_{t_n}^{i+1}\left(\frac{u_{i+1}+v_{i+1}}{2},\cdot\right)dt_i \end{aligned} \tag{10}$$

for each $i \in 1, 2, \ldots, n-1$ and also taking into account the right of (9)

$$\begin{aligned} &\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\frac{X_{t_n}^{i+1}\left(u_{i+1},\cdot\right)+sX_{t_n}^{i+1}\left(v_{i+1},\cdot\right)}{s+1}dt_i \\ =\ & \frac{1}{s+1}\left[\frac{1}{v_i-u_i}\int_{u_i}^{v_i} X_{t_n}^{i+1}\left(u_{i+1},\cdot\right)dt_i+\frac{s}{v_i-u_i}\int_{u_i}^{v_i} X_{t_n}^{i+1}\left(v_{i+1},\cdot\right)dt_i\right] \\ \le\ & \frac{1}{s+1}\left[\begin{array}{c} \frac{X((t_1,\ldots,t_{i-1},u_i,u_{i+1},\ldots,t_n),\cdot)+sX((t_1,\ldots,t_{i-1},v_i,u_{i+1},\ldots,t_n),\cdot)}{s+1} \\ +\frac{sX((t_1,\ldots,t_{i-1},u_i,v_{i+1},\ldots,t_n),\cdot)+s^2X((t_1,\ldots,t_{i-1},v_i,v_{i+1},\ldots,t_n),\cdot)}{s+1} \end{array}\right] \\ =\ & \frac{1}{(s+1)^2}\left[\begin{array}{c} X\left(\left(t_1,\ldots,t_{i-1},u_i,u_{i+1},\ldots,t_n\right),\cdot\right) \\ +sX\left(\left(t_1,\ldots,t_{i-1},v_i,u_{i+1},\ldots,t_n\right),\cdot\right) \\ +sX\left(\left(t_1,\ldots,t_{i-1},u_i,v_{i+1},\ldots,t_n\right),\cdot\right) \\ +s^2X\left(\left(t_1,\ldots,t_{i-1},v_i,v_{i+1},\ldots,t_n\right),\cdot\right) \\ \end{array}\right], \end{aligned} \tag{11}$$

for each $i \in 1, 2, \hat{a}e|, n-1$. Using the inequalities (10) and (11) in (9) and taking summation from 1 to $n-1$, we have (8). □

Remark 6. *Under the assumptions of Theorem 5 for $n = 2$, Hermite-Hadamard inequality for s-convex of first sense two-dimensional stochastic process can be easily* *obtained almost everywhere as follows:*

$$
\begin{aligned}
& X\left(\left(\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2}, \cdot\right)\right) \\
& \leq \frac{1}{\left(v_1-u_1\right)\left(v_2-u_2\right)} \int_{u_1}^{v_1} \int_{u_2}^{v_2}\left(t_1, t_2, \cdot\right) d t_2 d t_1 \\
& \leq \frac{1}{(s+1)^2}\left[X\left(u_1, u_2, \cdot\right)+s X\left(u_1, v_2, \cdot\right)+s X\left(v_1, u_2, \cdot\right)+s^2 X\left(v_1, v_2, \cdot\right)\right] .
\end{aligned}
$$

Theorem 7. *Let $X : \Delta^n \times \Omega \rightarrow \mathbb{R}$ be a s-convex of first sense stochastic process and be integrated in mean-square on Δ^n. Then almost everywhere*

$$
\begin{aligned}
& X\left(\left(\frac{u_1+v_1}{2}, \ldots, \frac{u_{n-1}+v_{n-1}}{2}, \frac{u_n+v_n}{2}\right), \cdot\right) \\
& \leq\left(\prod_{i=1}^n \frac{1}{v_i-u_i}\right) \int_{u_1}^{v_1} \ldots \int_{u_n}^{v_n} X_{t_n}^n\left(t_n, \cdot\right) d t_n \ldots d t_1 \\
& \leq \frac{1}{(s+1)^n} \sum_{i=1}^{n+1} s^{i-1} \frac{1}{2^n} \sum_{k \in l_i(n)} X(\kappa u+(1-\kappa) v, \cdot),
\end{aligned} \tag{12}
$$

where $l_i(n) := \{\kappa \in \mathbb{N}_0^n\colon\ \kappa_i \leq 1,\ |\kappa| = n+1-i, i = 1, \ldots, n+1\}$; $|\kappa| := \kappa_1+\kappa_2+\cdots+\kappa_n \in \mathbb{N}$; $u\kappa := (\kappa_1 u_1, \ldots, \kappa_n u_n) \in \mathbb{N}_0^n$ for $u, v \in \Delta^n$.

Proof. Using 7, we get the following inequality for $X_{t_n}^n$ almost everywhere

$$
X_{t_n}^n\left(\frac{u_n+v_n}{2}, \cdot\right) \leq \frac{1}{v_n-u_n} \int_{u_n}^{v_n} X_{t_n}^n\left(t_n, \cdot\right) d t_n \leq \frac{X_{t_n}^n\left(u_n, \cdot\right)+s X_{t_n}^n\left(v_n, \cdot\right)}{s+1} .
$$

By integrating on $[u_{n-1}, v_{n-1}]$, we get

$$
\begin{aligned}
& \frac{1}{v_{n-1}-u_{n-1}} \int_{u_{n-1}}^{v_{n-1}} X_{t_n}^n\left(\frac{u_n+v_n}{2}, \cdot\right) d t_{n-1} \\
\leq\ & \frac{1}{\left(v_{n-1}-u_{n-1}\right)\left(v_n-u_n\right)} \int_{u_{n-1}}^{v_{n-1}} \int_{u_n}^{v_n} X_{t_n}^n\left(t_n, \cdot\right) d t_n d t_{n-1} \\
\leq\ & \frac{1}{v_{n-1}-u_{n-1}} \int_{u_{n-1}}^{v_{n-1}} \frac{X_{t_n}^n\left(u_n, \cdot\right)+s X_{t_n}^n\left(v_n, \cdot\right)}{s+1} d t_{n-1} .
\end{aligned} \tag{13}
$$

From (10), (11) respectively

$$
\begin{aligned}
& X\left(\left(t_1, \ldots, \frac{u_{n-1}+v_{n-1}}{2}, \frac{u_n+v_n}{2}\right), \cdot\right) \\
\leq\ & \frac{1}{v_{n-1}-u_{n-1}} \int_{u_{n-1}}^{v_{n-1}} X_{t_n}^n\left(\frac{u_n+v_n}{2}, \cdot\right) d t_{n-1} ; \qquad (14) \\
& \frac{1}{v_{n-1}-u_{n-1}} \int_{u_{n-1}}^{v_{n-1}} \frac{X_{t_n}^n\left(u_n, \cdot\right)+s X_{t_n}^n\left(v_n, \cdot\right)}{s+1} d t_{n-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{s+1}\left[\frac{1}{v_{n-1}-u_{n-1}}\int_{u_{n-1}}^{v_{n-1}} X_{t_n}^{n}\left(u_n,\cdot\right)dt_{n-1}\right. \\
&\quad \left.+\frac{s}{v_{n-1}-u_{n-1}}\int_{u_{n-1}}^{v_{n-1}} X_{t_n}^{n}\left(v_n,\cdot\right)dt_{n-1}\right] \\
&\le \frac{1}{(s+1)^2}\left[\begin{array}{c} X\left(\left(t_1,\ldots,u_{n-1},u_n\right),\cdot\right) \\ +sX\left(\left(t_1,\ldots,v_{n-1},u_n\right),\cdot\right) \\ +sX\left(\left(t_1,\ldots,u_{n-1},v_n\right),\cdot\right) \\ +s^2X\left(\left(t_1,\ldots,v_{n-1},v_n\right),\cdot\right). \end{array}\right]. \qquad (15)
\end{aligned}
$$

From (13)-(15)

$$
\begin{aligned}
& X\left(\left(t_1,\ldots,\frac{u_{n-1}+v_{n-1}}{2},\frac{u_n+v_n}{2}\right),\cdot\right) \\
\le\ & \frac{1}{\left(v_{n-1}-u_{n-1}\right)\left(v_n-u_n\right)}\int_{u_{n-1}}^{v_{n-1}}\int_{u_n}^{v_n} X_{t_n}^{n}\left(t_n,\cdot\right)dt_n dt_{n-1} \\
\le\ & \frac{1}{(s+1)^2}\left[\begin{array}{c} X\left(\left(t_1,\ldots,u_{n-1},u_n\right),\cdot\right) \\ +sX\left(\left(t_1,\ldots,v_{n-1},u_n\right),\cdot\right) \\ +sX\left(\left(t_1,\ldots,u_{n-1},v_n\right),\cdot\right) \\ +s^2X\left(\left(t_1,\ldots,v_{n-1},v_n\right),\cdot\right). \end{array}\right]. \qquad (16)
\end{aligned}
$$

Integrating on $[u_{n-2},v_{n-2}]$

$$
\begin{aligned}
& \frac{1}{v_{n-2}-u_{n-2}}\int_{u_{n-2}}^{v_{n-2}} X\left(\left(t_1,\ldots,\frac{u_{n-1}+v_{n-1}}{2},\frac{u_n+v_n}{2}\right),\cdot\right)dt_{n-2}. \\
\le\ & \left(\prod_{i=n-2}^{n}\frac{1}{v_i-u_i}\right)\int_{u_{n-2}}^{v_{n-2}}\int_{u_{n-1}}^{v_{n-1}}\int_{u_n}^{v_n} X_{t_n}^{n}\left(t_n,\cdot\right)dt_n dt_{n-1}dt_{n-2} \\
\le\ & \frac{1}{v_{n-2}-u_{n-2}}\int_{u_{n-2}}^{v_{n-2}}\frac{1}{(s+1)^2}\left[\begin{array}{c} X\left(\left(t_1,\ldots,u_{n-1},u_n\right),\cdot\right) \\ +sX\left(\left(t_1,\ldots,v_{n-1},u_n\right),\cdot\right) \\ +sX\left(\left(t_1,\ldots,u_{n-1},v_n\right),\cdot\right) \\ +s^2X\left(\left(t_1,\ldots,v_{n-1},v_n\right),\cdot\right) \end{array}\right]dt_{n-2}.
\end{aligned}
$$

From (10), (11) respectively

$$
\begin{aligned}
& X\left(\left(t_1,\ldots,\frac{u_{n-2}+v_{n-2}}{2},\frac{u_{n-1}+v_{n-1}}{2},\frac{u_n+v_n}{2}\right),\cdot\right) \\
\le\ & \frac{1}{v_{n-2}-u_{n-2}}\int_{u_{n-2}}^{v_{n-2}} X\left(\left(t_1,\ldots,\frac{u_{n-1}+v_{n-1}}{2},\frac{u_n+v_n}{2}\right),\cdot\right)dt_{n-2}; \qquad (17) \\
& \frac{1}{v_{n-2}-u_{n-2}}\int_{u_{n-2}}^{v_{n-2}}\frac{1}{(s+1)^2}\left[\begin{array}{c} X\left(\left(t_1,\ldots,u_{n-1},u_n\right),\cdot\right) \\ +sX\left(\left(t_1,\ldots,v_{n-1},u_n\right),\cdot\right) \\ +sX\left(\left(t_1,\ldots,u_{n-1},v_n\right),\cdot\right) \\ +s^2X\left(\left(t_1,\ldots,v_{n-1},v_n\right),\cdot\right) \end{array}\right]dt_{n-2}
\end{aligned}
$$

$$\leq \frac{1}{(s+1)^3}\begin{bmatrix} X((t_1,\ldots,u_{n-2},u_{n-1},u_n),\cdot) \\ +sX((t_1,\ldots,v_{n-2},u_{n-1},u_n),\cdot) \\ +sX((t_1,\ldots,u_{n-2},v_{n-1},u_n),\cdot) \\ +s^2X((t_1,\ldots,v_{n-2},v_{n-1},u_n),\cdot) \\ +sX((t_1,\ldots,u_{n-2},u_{n-1},v_n),\cdot) \\ +s^2X((t_1,\ldots,v_{n-2},u_{n-1},v_n),\cdot) \\ +s^2X((t_1,\ldots,u_{n-2},v_{n-1},v_n),\cdot) \\ +s^3X((t_1,\ldots,v_{n-2},v_{n-1},v_n),\cdot) \end{bmatrix}. \tag{18}$$

From (17)-(18)

$$\begin{aligned} & \frac{1}{v_{n-2}-u_{n-2}}\int_{u_{n-2}}^{v_{n-2}} X\left(\left(t_1,\ldots,\frac{u_{n-1}+v_{n-1}}{2},\frac{u_n+v_n}{2}\right),\cdot\right)dt_{n-2} \\ \leq & \left(\prod_{i=n-2}^{n}\frac{1}{v_i-u_i}\right)\int_{u_{n-2}}^{v_{n-2}}\int_{u_{n-1}}^{v_{n-1}}\int_{u_n}^{v_n} X_{t_n}^n(t_n,\cdot)\,dt_n dt_{n-1} dt_{n-2} \\ \leq & \frac{1}{(s+1)^3}\begin{bmatrix} X((t_1,\ldots,u_{n-2},u_{n-1},u_n),\cdot) \\ +sX((t_1,\ldots,v_{n-2},u_{n-1},u_n),\cdot) \\ +sX((t_1,\ldots,u_{n-2},v_{n-1},u_n),\cdot) \\ +s^2X((t_1,\ldots,v_{n-2},v_{n-1},u_n),\cdot) \\ +sX((t_1,\ldots,u_{n-2},u_{n-1},v_n),\cdot) \\ +s^2X((t_1,\ldots,v_{n-2},u_{n-1},v_n),\cdot) \\ +s^2X((t_1,\ldots,u_{n-2},v_{n-1},v_n),\cdot) \\ +s^3X((t_1,\ldots,v_{n-2},v_{n-1},v_n),\cdot) \end{bmatrix}. \end{aligned}$$

Consequently, we obtain (12) using inductive method. □

Example 8. *Let $X : \Delta^3 \times \Omega \to \mathbb{R}$ be a s-convex of first sense stochastic process and be integrated in mean-square on Δ^n. Then almost everywhere*

$$\begin{aligned} & X\left(\left(\frac{u_1+v_1}{2},\frac{u_2+v_2}{2},\frac{u_3+v_3}{2}\right),\cdot\right) \\ \leq & \frac{1}{(v_1-u_1)(v_2-u_2)(v_3-u_3)}\int_{u_1}^{v_1}\int_{u_2}^{v_2}\int_{u_3}^{v_3} X(t_1,t_2,t_3\cdot)\,dt_3dt_2dt_1 \\ \leq & \frac{1}{(s+1)^3}\begin{bmatrix} X((u_1,u_2,u_3),\cdot)+sX((v_1,u_2,u_3),\cdot) \\ +sX((u_1,v_2,u_3),\cdot)+s^2X((v_1,v_2,u_3),\cdot) \\ +sX((u_1,u_2,v_3),\cdot)+s^2X((v_1,u_2,v_3),\cdot) \\ +s^2X((u_{n-2},v_{n-1},v_n),\cdot)+s^3X((v_1,v_2,v_3),\cdot) \end{bmatrix}. \end{aligned}$$

Proof. According to Theorem 7 for n=3, we get $X_3^3(t_3,\cdot) = X((t_1,t_2,t_3),\cdot)$ and $l_i(3) := \{\kappa \in \mathbb{N}_0^3 : \kappa_i \leq 1, |\kappa| = 4-i\}, i = 1,2,3,4$. Then

$$\begin{aligned} & l_1(3) := \{(1,1,1)\};\ l_2(3) := \{(0,1,1),(1,0,1),(1,1,0)\} \\ & l_3(3) := \{(0,0,1),(0,1,0),(1,0,0)\}; l_4(3) := \{(0,0,0)\}, \end{aligned}$$

and for $u=(u_1,u_2,u_3), v=(v_1,v_2,v_3)\in\Delta^3$

$$\begin{aligned}
&\sum_{k\in l_1(3)} X(\kappa u+(1-\kappa)v,\cdot)\\
=\ & X((1,1,1)(u_1,u_2,u_3)+[(1,1,1)-(1,1,1)](v_1,v_2,v_3),\cdot)\\
=\ & X((u_1,u_2,u_3),\cdot);\\
&\sum_{k\in l_3(3)} X(\kappa u+(1-\kappa)v,\cdot)\\
=\ & X((0,1,1)(u_1,u_2,u_3)+[(1,1,1)-(0,1,1)](v_1,v_2,v_3),\cdot)\\
&+X((1,0,1)(u_1,u_2,u_3)+[(1,1,1)-(1,0,1)](v_1,v_2,v_3),\cdot)\\
&+X((1,1,0)(u_1,u_2,u_3)+[(1,1,1)-(1,1,0)](v_1,v_2,v_3),\cdot)\\
=\ & X((u_1,v_2,v_3),\cdot)+X((u_1,v_2,u_3),\cdot)+X((u_1,u_2,v_3),\cdot).
\end{aligned}$$

So

$$\begin{aligned}
&\sum_{k\in l_3(3)} X(\kappa u+(1-\kappa)v,\cdot)\\
=\ & X((v_1,v_2,u_3),\cdot)+X((v_1,u_2,v_3),\cdot)+X((u_1,v_2,v_3),\cdot);\\
&\sum_{k\in l_4(3)} X(\kappa u+(1-\kappa)v,\cdot)=X((v_1,v_2,v_3),\cdot).
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{i=1}^{4} s^{i-1}\sum_{k\in l_i(3)} X(\kappa u+(1-\kappa)v,\cdot)=X((u_1,u_2,u_3),\cdot)\\
&+s\{X((u_1,v_2,v_3),\cdot)+X((u_1,v_2,u_3),\cdot)+X((u_1,u_2,v_3),\cdot)\}\\
&+s^2\{X((v_1,v_2,u_3),\cdot)+X((v_1,u_2,v_3),\cdot)+X((u_1,v_2,v_3),\cdot)\}\\
&+s^3\{X((v_1,v_2,v_3),\cdot)\}.
\end{aligned}$$

Using all of the above equalities in (12), we obtain the desired result in this example. □

Theorem 9. *Let $X:\Delta^n\times\Omega\to\mathbb{R}$ be a s-convex of first sense stochastic process and be integrated in mean-square on Δ^n. Then almost everywhere for $u,v\in\Delta^n$*

$$\begin{aligned}
&\sum_{k=1}^{n}\frac{1}{v_k-u_k}\int_{u_k}^{v_k}\left(X_{u_n}^k(t_k,\cdot)+X_{v_n}^k(t_k,\cdot)\right)dt_k\\
&\le \frac{n}{s+1}\left[X(u,\cdot)+sX(v,\cdot)\right]+\frac{1}{s+1}\sum_{k=1}^{n}\left[X_{u_n}^k(v_k,\cdot)+sX_{v_n}^k(u_k,\cdot)\right]. \quad (19)
\end{aligned}$$

Proof. Using (7), we obtain the following inequality for $X^i_{u_n}$ and $X^i_{v_n}$ almost everywhere

$$\frac{1}{v_i-u_i}\int_{u_i}^{v_i} X^i_{u_n}(t_i,\cdot)\,dt_i \le \frac{X^i_{u_n}(u_i,\cdot)+sX^i_{u_n}(v_i,\cdot)}{s+1} \le \frac{X(u,\cdot)+sX^i_{u_n}(v_i,\cdot)}{s+1},$$
$$\frac{1}{v_i-u_i}\int_{u_i}^{v_i} X^i_{v_n}(t_i,\cdot)\,dt_i \le \frac{X^i_{v_n}(u_i,\cdot)+sX^i_{v_n}(v_i,\cdot)}{s+1} \le \frac{X^i_{v_n}(u,\cdot)+sX(v,\cdot)}{s+1}.$$

Taking summation the above inequalities

$$\begin{aligned}&\frac{1}{v_i-u_i}\int_{u_i}^{v_i}\left[X^i_{u_n}(t_i,\cdot)+X^i_{v_n}(t_i,\cdot)\right]dt_i\\ \le\ &\frac{X(u,\cdot)+sX(v,\cdot)+X^i_{v_n}(u_i,\cdot)+sX^i_{u_n}(v_i,\cdot)}{s+1},\end{aligned}$$

for $i=1,2,\ldots,n$. Adding above n inequalities we get (19). □

4. Hermite-Hadamard type integral inequalities for s-convex of second sense multidimensional stochastic process

Let $X:\Delta^n\times\Omega\to\mathbb{R}$ be a s-convex of second sense stochastic process and integrated in mean-square on Δ^n. Accordingly, $X^i_{t_n}:[u_i,v_i]\times\Omega\to\mathbb{R}$ is s-convex of second sense stochastic process and integrated in mean-square on $[u_i,v_i]$ for each $i=1,2,\ldots,n,\ n\ge 2$. Using (4)

$$\begin{aligned}&2^{s-1}X^i_{t_n}\left(\frac{u_i+v_i}{2},\cdot\right)\le\frac{1}{v_i-u_i}\int_{u_i}^{v_i}X^i_{t_n}(t_i,\cdot)\,dt_i\\ &\le\frac{X^i_{t_n}(u_i,\cdot)+X^i_{t_n}(v_i,\cdot)}{s+1},(a.e.).\end{aligned}$$

As a similar methods, using the same notations of (3) and (4), we can obtain Hermit-Hadamard inequalities for s-convex of second sense multidimensional stochastic processes as the following theorems:

Theorem 10. *Let $X:\Delta^n\times\Omega\to\mathbb{R}$ be a s-convex of second sense stochastic process and be integrated in mean-square on Δ^n. Then almost everywhere*

$$\begin{aligned}&\sum_{i=1}^{n-1}4^{s-1}X\left(\left(t_1,\ldots,t_{i-1},\frac{u_i+v_i}{2},\frac{u_{i+1}+v_{i+1}}{2},\ldots,t_n\right),\cdot\right)\\ \le\ &\sum_{i=1}^{n-1}\frac{4^{s-1}}{v_i-u_i}\int_{u_i}^{v_i}X^{i+1}_{t_n}\left(\frac{u_{i+1}+v_{i+1}}{2},\cdot\right)dt_i\\ \le\ &\sum_{i=1}^{n-1}\frac{1}{(v_i-u_i)(v_{i+1}-u_{i+1})}\int_{u_i}^{v_i}\int_{u_{i+1}}^{v_{i+1}}X^{i+1}_{t_n}(t_{i+1},\cdot)\,dt_{i+1}dt_i\end{aligned}$$

$$\leq \sum_{i=1}^{n-1} \frac{1}{v_i - u_i} \int_{u_i}^{v_i} \frac{X_{t_n}^{i+1}(u_{i+1},\cdot) + sX_{t_n}^{i+1}(v_{i+1},\cdot)}{s+1} dt_i$$

$$\leq \sum_{i=1}^{n-1} \frac{1}{(s+1)^2} \left[\begin{array}{c} X((t_1,\ldots,t_{i-1},u_i,u_{i+1},\ldots,t_n),\cdot) \\ +sX((t_1,\ldots,t_{i-1},v_i,u_{i+1},\ldots,t_n),\cdot) \\ +sX((t_1,\ldots,t_{i-1},u_i,v_{i+1},\ldots,t_n),\cdot) \\ +s^2X((t_1,\ldots,t_{i-1},v_i,v_{i+1},\ldots,t_n),\cdot) \end{array} \right]. \quad (20)$$

Remark 11. *According to Theorem 10 for $n = 2$, Hermite-Hadamard inequality for s-convex of second sense two-dimensional stochastic process can be directly derived almost everywhere:*

$$\begin{aligned} & 4^{s-1} X\left(\left(\frac{u_1+v_1}{2}, \frac{u_2+v_2}{2},\cdot\right)\right) \\ \leq\ & \frac{1}{(v_1-u_1)(v_2-u_2)} \int_{u_1}^{v_1} \int_{u_2}^{v_2} (t_1,t_2,\cdot)\, dt_2 dt_1 \\ =\ & \frac{1}{(s+1)^2} \left[X(u_1,u_2,\cdot) + X(u_1,v_2,\cdot) + X(v_1,u_2,\cdot) + X(v_1,v_2,\cdot)\right]. \end{aligned}$$

Theorem 12. *Let $X : \Delta^n \times \Omega \to \mathbb{R}$ be a s-convex of second sense stochastic process and be integrated in mean-square on Δ^n. Then almost everywhere*

$$\begin{aligned} & 2^{n(s-1)} X\left(\left(\frac{u_1+v_1}{2},\ldots,\frac{u_{n-1}+v_{n-1}}{2},\frac{u_n+v_n}{2}\right),\cdot\right) \\ \leq\ & \left(\prod_{i=1}^{n} \frac{1}{v_i-u_i}\right) \int_{u_1}^{v_1} \cdots \int_{u_n}^{v_n} X_{t_n}^{n}(t_n,\cdot)\, dt_n \ldots dt_1 \\ \leq\ & \frac{1}{(s+1)^n} \sum_{k \in l_i(n)} X(\kappa u + (1-\kappa) v,\cdot). \end{aligned}$$

Example 13. *Let $X : \Delta^3 \times \Omega \to \mathbb{R}$ be a s-convex of second sense stochastic process and be integrated in mean-square on Δ^n. Then almost everywhere*

$$\begin{aligned} & 8^{s-1} X\left(\left(\frac{u_1+v_1}{2},\frac{u_2+v_2}{2},\frac{u_3+v_3}{2}\right),\cdot\right) \\ \leq\ & \frac{1}{(v_1-u_1)(v_2-u_2)(v_3-u_3)} \int_{u_1}^{v_1} \int_{u_2}^{v_2} \int_{u_3}^{v_3} X((t_1,t_2,t_3),\cdot)\, dt_3 dt_2 dt_1 \\ \leq\ & \frac{1}{(s+1)^3} \left[\begin{array}{c} X((u_1,u_2,u_3),\cdot) + X((v_1,u_2,u_3),\cdot) \\ +X((u_1,v_2,u_3),\cdot) + X((v_1,v_2,u_3),\cdot) \\ +X((u_1,u_2,v_3),\cdot) + X((v_1,u_2,v_3),\cdot) \\ +X((v_1,u_2,v_n),\cdot) + X((v_1,v_2,v_3),\cdot) \end{array} \right]. \end{aligned}$$

Proof. Using the similar method of Example 8, this proof is completed. □

Theorem 14. *Let $X : \Delta^n \times \Omega \to \mathbb{R}$ be a s-convex of second sense stochastic process and be integrated in mean-square on Δ^n. Then almost everywhere for $u, v \in \Delta^n$*

$$\begin{aligned} & \sum_{k=1}^{n} \frac{1}{v_k - u_k} \int_{u_k}^{v_k} \left(X_{u_n}^k (t_k, \cdot) + X_{v_n}^k (t_k, \cdot) \right) dt_k \\ \leq \quad & \frac{n}{s+1} \left[X(u, \cdot) + X(v, \cdot) \right] + \frac{1}{s+1} \sum_{k=1}^{n} \left[X_{u_n}^k (t_k, \cdot) + X_{v_n}^k (t_k, \cdot) \right]. \end{aligned}$$

Interpretation

(Comparison the main results of this study with another process). The obtained results for multidimensional s-convex of first (second) sense and harmonically convex stochastic processes show that main structure belonging to each process is protected characteristically in Hermite-Hadamard's type inequality.

5. Conclusion

In this paper, we investigated s-convex of first (second) sense multidimensional stochastic processes and derived Hermite-Hadamard type inequalities for these processes. Also, we clarified these results on two and three-dimensional stochastic processes. We compare our main results with another process. We suggest that anyone defines other convex multidimensional stochastic processes and obtains some inequalities for related processes.

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