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ω-CONTINUITY ON GENERALIZED NEIGHBOURHOOD SYSTEMS

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Abstract. We introduce ω-weakly-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous functions on generalized topological spaces and study their relations with other classes of generalized continuous functions given in [1, 8]. Then, we define the notion of omega open set on generalized neighbourhood systems as ω-φ-open set. By using these sets, we generate generalized topology. Also, we introduce two kinds of continuity on generalized neighbourhood systems and investigate relationships between these two kinds, (φ, φ')-continuity and weakly-(φ, φ')-continuity.

1. Introduction

Császár introduced generalized topology and generalized neighbourhood systems, then he defined two kinds of continuity on them in [3]He gave some characterizations of (φ, φ')-continuous functions in [3, 4]. Min [8] introduced weak-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuity and weak-$(\varphi, \varphi')$-continuity, and he investigated relationships between such functions. Hdeib [6] gave the definition of ω-closed set as containing all its condensation points. Afterwards, he introduced the notion of ω-continuous functions in [7]. Besides, Al-Zoubi [2] defined ω-weakly continuous functions and showed that every ω-continuous function is ω-weakly continuous. He then studied their basic properties. Al Ghour [1] extended the concept of omega open set in ordinary topological space to generalized topological space and introduced ω-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuity as using omega open sets in generalized topology.

In this paper, we introduce ω-weakly-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous functions using ω-$\mathfrak{g}$-open sets, then obtain their relations with ω-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous functions and weakly-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous functions. Also, we define ω-φ-closed and ω-φ-open sets on generalized neighbourhood systems, and get some characterizations of these sets. Then, we give the definitions of two new operators; namely, $\imath_{\varphi_\omega}$ and γ_{φ_ω},

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and study their basic properties. Besides, we produce generalized topology via ω-φ-open sets. Afterwards, we introduce ω-(φ, φ')-continuous and ω-weakly-(φ, φ')-continuous functions on generalized neighbourhood systems and investigate relationships between these functions, (φ, φ')-continuous functions and weakly-(φ, φ')-continuous functions.

2. Preliminaries

Definition 1. [3] *Let X be a nonempty set and $\wp(X)$ be the power set of X. Then $\mathfrak{g} \subseteq \wp(X)$ is called a generalized topology (briefly GT) on X iff $\emptyset \in \mathfrak{g}$ and $H_i \in \mathfrak{g}$ for $i \in I \neq \emptyset$ implies $H = \bigcup_{i \in I} H_i \in \mathfrak{g}$. The pair $(X, \mathfrak{g})$ is called a generalized topological space (briefly GTS). The elements of $\mathfrak{g}$ are called $\mathfrak{g}$-open sets and the complements of $\mathfrak{g}$-open sets are called $\mathfrak{g}$-closed sets. If $\mathfrak{g}$ is a GT on X and $S \subseteq X$, the interior of S (denoted by $i_{\mathfrak{g}}(S)$) is the union of all $H \subseteq S, H \in \mathfrak{g}$ and the closure of S (denoted by $c_{\mathfrak{g}}(S)$) is the intersection of all $\mathfrak{g}$-closed sets containing S.*

Definition 2. [3] *Let $\varphi : X \to \wp(\wp(X))$ satisfy $a \in V$ for $V \in \varphi(a)$. Then $V \in \varphi(a)$ is called a generalized neighbourhood (briefly GN) of $a \in X$ and φ is called a generalized neighbourhood system (briefly GNS) on X. The collection of all GNSs on X is denoted by $\Phi(X)$.*
If φ is a GNS on X and $S \subseteq X$:

$$\imath_{\varphi}(S) = \{a \in S : \text{ there exists } V \in \varphi(a) \text{ such that } V \subseteq S\}$$

and

$$\gamma_{\varphi}(S) = \{a \in X : V \cap S \neq \emptyset \text{ for all } V \in \varphi(a)\}.$$

Lemma 3. [3] *Let φ be a GNS on X and $H \in \mathfrak{g}_{\varphi}$ iff $H \subseteq X$ satisfies: if $a \in H$ then there is $V \in \varphi(a)$ such that $V \subseteq H$. Then $\mathfrak{g}_{\varphi}$ is a GT.*
For $\varphi \in \Phi(X)$, $i_{\varphi} = i_{\mathfrak{g}_{\varphi}}$ and $c_{\varphi} = c_{\mathfrak{g}_{\varphi}}$.

Lemma 4. [3] *Let $\varphi \in \Phi(X)$ and $S \subseteq X$. Then,*

(1) *$\imath_{\varphi}, \gamma_{\varphi} \in \Gamma(X)$ and $\gamma_{\varphi}(S) = X - \imath_{\varphi}(X - S)$.*
(2) *$i_{\mathfrak{g}_{\varphi}}(S) \subseteq \imath_{\varphi}(S)$ and $\gamma_{\varphi}(S) \subseteq c_{\mathfrak{g}_{\varphi}}(S)$.*

Theorem 5. [3] *Let $(X, \mathfrak{g})$ be a GTS and $S \subseteq X$. Then*

(1) *$c_{\mathfrak{g}}(S) = X - i_{\mathfrak{g}}(X - S)$.*
(2) *$i_{\mathfrak{g}}(S) = X - c_{\mathfrak{g}}(X - S)$.*

Definition 6. [5] *Let (X, τ) be a topological space and $S \subseteq X$. A point $a \in X$ is called a condensation point of S if for each $H \in \tau$ with $a \in H$ the set $H \cap S$ is uncountable.*

Definition 7. [6] *Let (X, τ) be a topological space and $S \subseteq X$. S is called ω-closed if it contains all its condensation points. The complement of an ω-closed set is called ω-open.*

Definition 8. [1] *Let $(X, \mathfrak{g})$ be GTS and S be a subset of X. A point $a \in X$ is a condensation point of S if for each $H \in \mathfrak{g}$ with $a \in H$, the set $H \cap S$ is uncountable. The set of all condensation points of S is denoted by $cond(S)$. S is ω-$\mathfrak{g}$-closed if $cond(S) \subseteq S$. The complement of an ω-$\mathfrak{g}$-closed set is called ω-$\mathfrak{g}$-open. The family of all ω-$\mathfrak{g}$-open sets of $(X, \mathfrak{g})$ is denoted by $\mathfrak{g}_\omega$.*

Theorem 9. [1] *A subset S of a GTS $(X, \mathfrak{g})$ is ω-$\mathfrak{g}$-open iff for every $a \in S$, there exists a $H \in \mathfrak{g}$ such that $a \in H$ and $H - S$ is countable.*

Theorem 10. [1] *For any GTS $(X, \mathfrak{g})$, $\mathfrak{g}_\omega$ is a GT on X finer than $\mathfrak{g}$.*

Definition 11. *A function $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be*

(1) *ω-continuous* [7] *if $f^{-1}(H)$ is ω-open in (X, τ_1) for each $H \in \tau_2$.*
(2) *ω-weakly continuous* [2] *if for each $a \in X$ and for each $H \in \tau_2$ containing $f(a)$, there exists an ω-open subset G of X containing a such that $f(G) \subseteq c_{\tau_2}(H)$.*

Definition 12. [3] *A function $f : (X, \mathfrak{g}_1) \to (Y, \mathfrak{g}_2)$ is called $(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous if for every $\mathfrak{g}_2$-open set H in Y, $f^{-1}(H)$ is $\mathfrak{g}_1$-open in X.*

Theorem 13. [8] *Let $f : (X, \mathfrak{g}_1) \to (Y, \mathfrak{g}_2)$ be a function. Then the following conditions are equivalent:*

(1) *f is $(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous,*
(2) *For every $\mathfrak{g}_2$-closed set K in Y, $f^{-1}(K)$ is $\mathfrak{g}_1$-closed in X,*
(3) *For each $a \in X$ and each $\mathfrak{g}_2$-open set H containing $f(a)$, there exists a $\mathfrak{g}_1$-open set G containing a such that $f(G) \subseteq H$.*

Definition 14. *A function $f : (X, \mathfrak{g}_1) \to (Y, \mathfrak{g}_2)$ is called weakly-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous* [8] *(respectively, ω-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous* [1]*) if for each $a \in X$ and for every $\mathfrak{g}_2$-open set H containing $f(a)$, there is an $\mathfrak{g}_1$-open set (respectively, ω-$\mathfrak{g}_1$-open set) G containing a such that $f(G) \subseteq c_{\mathfrak{g}_2}(H)$ (respectively, $f(G) \subseteq H$).*

Proposition 15. [8] *If $f : (X, \mathfrak{g}_1) \to (Y, \mathfrak{g}_2)$ is $(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous at $a \in X$, then f is weakly-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous at a.*

Theorem 16. [1] *Let $f : (X, \mathfrak{g}_1) \to (Y, \mathfrak{g}_2)$ be a function. Then the following conditions are equivalent:*

(1) *f is ω-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous,*
(2) *For each $\mathfrak{g}_2$-open set $H \subseteq Y$, $f^{-1}(H)$ is ω-$\mathfrak{g}_1$-open in X,*
(3) *For each $\mathfrak{g}_2$-closed set $K \subseteq Y$, $f^{-1}(K)$ is ω-$\mathfrak{g}_1$-closed in X.*

Proposition 17. [1] *If $f : (X, \mathfrak{g}_1) \to (Y, \mathfrak{g}_2)$ is $(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous at $a \in X$, then f is ω-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous at a.*

Definition 18. *Let φ and φ' be two GNSs on X and Y, respectively. Then a function $f : (X, \varphi) \to (Y, \varphi')$ is said to be (φ, φ')-continuous* [3]*(respectively, weakly-(φ, φ')-continuous* [8]*) if for $a \in X$ and $V \in \varphi'(f(a))$, there exists $U \in \varphi(a)$ such that $f(U) \subseteq V$ (respectively, $f(U) \subseteq \gamma_{\varphi'}(V)$).*

Proposition 19. [8] *Every (φ, φ')-continuous function is weakly-(φ, φ')-continuous.*

3. ω-WEAKLY-$(\mathfrak{g}_1, \mathfrak{g}_2)$-CONTINUOUS FUNCTIONS

Definition 20. *The ω-interior (ω-closure) of a subset S of a space $(X, \mathfrak{g})$ is the interior (closure) of S in the space $(X, \mathfrak{g}_\omega)$ and is denoted by $i_{\mathfrak{g}_\omega}(S)(c_{\mathfrak{g}_\omega}(S))$. $i_{\mathfrak{g}_\omega}(S)$ is the union of all $H \subseteq S$ for $H \in \mathfrak{g}_\omega$ and $c_{\mathfrak{g}_\omega}(S)$ is the intersection of all ω-$\mathfrak{g}$-closed sets containing S.*

Remark 21. *$i_{\mathfrak{g}_\omega}(S)$ is the largest $H \in \mathfrak{g}_\omega$ such that $H \subseteq S$ and $c_{\mathfrak{g}_\omega}(S)$ is the smallest ω-$\mathfrak{g}$-closed set containing S.*

Lemma 22. *Let $(X, \mathfrak{g})$ be GTS and $S_1 \subseteq S_2 \subseteq X$.*

(1) *$c_{\mathfrak{g}_\omega}(S_1) = X - i_{\mathfrak{g}_\omega}(X - S_1)$ and $i_{\mathfrak{g}_\omega}(S_1) = X - c_{\mathfrak{g}_\omega}(X - S_1)$.*
(2) *$i_{\mathfrak{g}_\omega}(S_1) \subseteq i_{\mathfrak{g}_\omega}(S_2)$ and $c_{\mathfrak{g}_\omega}(S_1) \subseteq c_{\mathfrak{g}_\omega}(S_2)$.*
(3) *$i_{\mathfrak{g}}(S_1) \subseteq i_{\mathfrak{g}_\omega}(S_1) \subseteq S_1 \subseteq c_{\mathfrak{g}_\omega}(S_1) \subseteq c_{\mathfrak{g}}(S_1)$.*

Proof.

(1-2) It is clear from the definitions of $i_{\mathfrak{g}_\omega}$ and $c_{\mathfrak{g}_\omega}$.
(3) They are also obvious since $\mathfrak{g} \subseteq \mathfrak{g}_\omega$.

□

Proposition 23. *Let $(X, \mathfrak{g})$ be a GTS and $S \subseteq X$.*

(1) *S is ω-$\mathfrak{g}$-open in X if and only if $i_{\mathfrak{g}_\omega}(S) = S$.*
(2) *S is ω-$\mathfrak{g}$-closed in X if and only if $c_{\mathfrak{g}_\omega}(S) = S$.*

Proof. The proofs are obvious from Remark 21. □

Remark 24. *In general, $i_{\mathfrak{g}}(S) \neq i_{\mathfrak{g}_\omega}(S)$ and $c_{\mathfrak{g}}(S) \neq c_{\mathfrak{g}_\omega}(S)$ for $S \subseteq X$.*

Example 25. *Let $X = \mathbb{R}$ with GT $\mathfrak{g} = \{\emptyset, (\mathbb{R} - \mathbb{Q})^- \cup \{0\}, (\mathbb{R} - \mathbb{Q})^+ \cup \{0\}, (\mathbb{R} - \mathbb{Q}) \cup \{0\}\}$. Then $i_{\mathfrak{g}_\omega}(S_1) = \mathbb{R} - \mathbb{Q}$ and $i_{\mathfrak{g}}(S_1) = \emptyset$ for $S_1 = \mathbb{R} - \mathbb{Q}$ and $c_{\mathfrak{g}_\omega}(S_2) = \mathbb{Q}$ and $c_{\mathfrak{g}}(S_2) = \mathbb{R}$ for $S_2 = \mathbb{Q}$.*

Definition 26. *Let $(X, \mathfrak{g}_1)$ and $(Y, \mathfrak{g}_2)$ be two GTSs. Then, a function $f : (X, \mathfrak{g}_1) \to (Y, \mathfrak{g}_2)$ is called ω-weakly-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous if for each $a \in X$ and for each $\mathfrak{g}_2$-open set H containing $f(a)$, there exists an ω-$\mathfrak{g}_1$-open set G containing a such that $f(G) \subseteq c_{\mathfrak{g}_2}(H)$.*

Proposition 27. *If $f : (X, \mathfrak{g}_1) \to (Y, \mathfrak{g}_2)$ is ω-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous, then it is ω-weakly-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous.*

Proof. Let H be a $\mathfrak{g}_2$-open set containing $f(a)$ for $a \in X$. Since f is ω-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous, $f^{-1}(H)$ is ω-$\mathfrak{g}_1$-open set containing a. Therefore, there exists a ω-$\mathfrak{g}_1$-open set $f^{-1}(H)$ such that $f(f^{-1}(H)) \subseteq H \subseteq c_{\mathfrak{g}_2}(H)$. Hence, f is ω-weakly-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous. □

We can give an example to show that the converse implication of Proposition 27 may not be true.

Example 28. *Let* $X = Y = \mathbb{R}$*,* $\mathfrak{g}_1 = \{\emptyset, \mathbb{R}, \mathbb{R} - \{0\}\}$ *and* $\mathfrak{g}_2 = \{\emptyset, \mathbb{Q}, \mathbb{Q} - \{0\}\}$*. Let* $f : (\mathbb{R}, \mathfrak{g}_1) \to (\mathbb{R}, \mathfrak{g}_2)$ *be the function defined by*

$$f(a) = \begin{cases} 0 & \text{if } a \in \mathbb{R} - \mathbb{Q} \\ 1 & \text{if } a \in \mathbb{Q} \\ \end{cases}$$

Then, f *is* ω*-weakly-*$(\mathfrak{g}_1, \mathfrak{g}_2)$*-continuous but it is not* ω*-*$(\mathfrak{g}_1, \mathfrak{g}_2)$*-continuous.*

Proposition 29. *If* $f : (X, \mathfrak{g}_1) \to (Y, \mathfrak{g}_2)$ *is weakly-*$(\mathfrak{g}_1, \mathfrak{g}_2)$*-continuous, then it is* ω*-weakly-*$(\mathfrak{g}_1, \mathfrak{g}_2)$*-continuous.*

Proof. Let H be a $\mathfrak{g}_2$-open set containing $f(a)$ for $a \in X$. Since f is weakly-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous, there exists a $\mathfrak{g}_1$-open set G containing a such that $f(G) \subseteq c_{\mathfrak{g}_2}(H)$. Since $\mathfrak{g}_1 \subseteq \mathfrak{g}_{1_\omega}$, G is also ω-$\mathfrak{g}_1$-open set containing a such that $f(G) \subseteq c_{\mathfrak{g}_2}(H)$. Hence, f is ω-weakly-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous. □

We give the following example to show that the converse of Proposition 29 is not true.

Example 30. *Let* $X = \{1, 2, 3, 4\}, \mathfrak{g}_1 = \{\emptyset, \{1\}, \{1, 3\}, \{2, 4\}, \{1, 2, 4\}, X\}$ *and* $\mathfrak{g}_2 = \{\emptyset, \{1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, X\}$*. Let* $f : (X, \mathfrak{g}_1) \to (Y, \mathfrak{g}_2)$ *be the function defined by* $f(1) = f(2) = f(3) = 1$ *and* $f(4) = 2$*. Then,* f *is* ω*-weakly-*$(\mathfrak{g}_1, \mathfrak{g}_2)$*-continuous but it is not weakly-*$(\mathfrak{g}_1, \mathfrak{g}_2)$*-continuous.*

Briefly, we get the following diagram from Proposition 15 and 17 and Proposition 27 and 29.

$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous	$\Longrightarrow$	ω-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous
$\Downarrow$		$\Downarrow$
weakly-$(\mathfrak{g}_1, \mathfrak{g}_2)$-continuous	$\Longrightarrow$	ω-weakly-$(\mathfrak{g}_1, \mathfrak{g}_2)-$continuous

4. ω-(φ, φ')-continuous and ω-weakly-(φ, φ')-continuous functions

Definition 31. *Let* φ *be a GNS on* X *and* $S \subseteq X$*. A point* $a \in X$ *is called a condensation point of* S *on* φ *if for each* $V \in \varphi(a)$ *such that* $V \cap S$ *is uncountable.*

Definition 32. *Let* φ *be a GNS on* X *and* $S \subseteq X$*.* S *is called* ω*-*φ*-closed if it contains all its condensation points on* φ*. The complement of an* ω*-*φ*-closed set is called* ω*-*φ*-open.*

Theorem 33. *Let φ be a GNS on X and $S \subseteq X$. S is ω-φ-open if and only if for each $a \in S$, there exists a $V \in \varphi(a)$ such that $V - S$ countable.*

Proof.
(Necessity) Let S be ω-φ-open. Then, $X-S$ is ω-φ-closed, that is, $X-S$ contains all its condensation points on φ. Thus, for each $a \in S$, a is not a condensation point on φ of $X-S$. Therefore, there exists a $V \in \varphi(a)$ such that $V \cap (X-S)$ is countable. Hence, there exists a $V \in \varphi(a)$ such that $V - S$ is countable.
(Sufficiency) The proof can be done similarly. □

Definition 34. *Let φ be a GNS on X and $S \subseteq X$.*

$$\imath_{\varphi_\omega}(S) = \{a \in S: \text{ there exists } \omega\text{-}\varphi\text{-open set } V \text{ containing } a \text{ such that } V \subseteq S\}$$

and

$$\gamma_{\varphi_\omega}(S) = \{a \in X: \text{for all } \omega\text{-}\varphi\text{-open set } V \text{ containing } a \text{ such that } V \cap S \neq \emptyset\}.$$

Lemma 35. *Let φ be a GNS on X and $S \subseteq X$.*
(1) *If S is ω-$\mathfrak{g}_\varphi$-open, then it is ω-φ-open.*
(2) *If $a \in S \in \varphi(a)$, then it is ω-φ-open.*

Proof.
(1) Let $a \in S$ and S be ω-$\mathfrak{g}_\varphi$-open. Then, there exists a $G \in \mathfrak{g}_\varphi$ such that $a \in G$ and $G - S$ is countable. Then, there is $V \in \varphi(a)$ such that $V \subseteq G$. Since $G - S$ is countable, $V - S$ is also countable. Hence, for $a \in S$, there exists a $V \in \varphi(a)$ such that $V - S$ is countable. Thus, S is ω-φ-open.
(2) Let $a \in S \in \varphi(a)$. There exists a $V = S \in \varphi(a)$ such that $V - S = \emptyset$ is countable. Thus, S is ω-φ-open.

□

The following example is given to show that the converse implications of Lemma 35 do not hold.

Example 36. *Let $X = \mathbb{R}$ and*

$$\varphi(a) = \begin{cases} \{\mathbb{Q}\} & \text{if } a \in \mathbb{Z} \\ \{\mathbb{R}\} & \text{if } a \in \mathbb{R} - \mathbb{Z} \\ \end{cases}$$

Then, $S = \mathbb{Z}$ is ω-φ-open but it is not ω-$\mathfrak{g}_\varphi$-open and $S \notin \varphi(a)$ for $a \in S$.

Lemma 37. *Let $\varphi \in \Phi(X)$ and $S_1, S_2 \subseteq X$. Then,*
(1) *$\gamma_{\varphi_\omega}(S_1) = X - \imath_{\varphi_\omega}(X - S_1)$ and $\imath_{\varphi_\omega}(S_1) = X - \gamma_{\varphi_\omega}(X - S_1)$.*
(2) *If $S_1 \subseteq S_2$, then $\imath_{\varphi_\omega}(S_1) \subseteq \imath_{\varphi_\omega}(S_2)$ and $\gamma_{\varphi_\omega}(S_1) \subseteq \gamma_{\varphi_\omega}(S_2)$.*
(3) *$\imath_\varphi(S_1) \subseteq \imath_{\varphi_\omega}(S_1) \subseteq S_1 \subseteq \gamma_{\varphi_\omega}(S_1) \subseteq \gamma_\varphi(S_1)$.*
(4) *$i_{(\mathfrak{g}_\varphi)_\omega}(S_1) \subseteq \imath_{\varphi_\omega}(S_1)$ and $\gamma_{\varphi_\omega}(S_1) \subseteq c_{(\mathfrak{g}_\varphi)_\omega}(S_1)$.*

Proof. (1-2) The proofs are clear from the definitions of $\imath_{\varphi_\omega}$ and γ_{φ_ω}.

(3) The proofs are obvious from Lemma 35(2) and Lemma 37(1).
(4) The proofs are obvious from Lemma 35(1), Lemma 37(1) and Lemma 22(1). □

Lemma 38. *Let φ be a GNS on X and $G \in \mathfrak{g}_{(\varphi_\omega)}$ if and only if $G \subseteq X$ satisfies: if $a \in G$ then there is an ω-φ-open set V containing a such that $V \subseteq G$. Then, $\mathfrak{g}_{(\varphi_\omega)}$ is a GT.*

Proof. $\emptyset \in \mathfrak{g}_{(\varphi_\omega)}$. Let $G_i \in \mathfrak{g}_{(\varphi_\omega)}$ for each $i \in I \neq \emptyset$. Then, for $a \in \bigcup_{i\in I} G_i$, there exists $i \in I$ such that $a \in G_i$. Therefore, there is an ω-φ-open set V containing a such that $V \subseteq G_i$. Thus, we have $V \subseteq \bigcup_{i\in I} G_i$. Hence, $\bigcup_{i\in I} G_i \in \mathfrak{g}_{(\varphi_\omega)}$. □

Theorem 39. *Let φ be a GNS on X and $S \subseteq X$. $S \in \mathfrak{g}_{(\varphi_\omega)}$ if and only if $\imath_{\varphi_\omega}(S) = S$.*

Proof. Let $S \in \mathfrak{g}_{(\varphi_\omega)}$. Then, for each $a \in S$, there exists an ω-φ-open set V containing a such that $V \subseteq S$. Thus, $a \in \imath_{\varphi_\omega}(S)$ and $S \subseteq \imath_{\varphi_\omega}(S)$. Also, from Lemma 37(3), $\imath_{\varphi_\omega}(S) \subseteq S$. Hence, we have $\imath_{\varphi_\omega}(S) = S$. Conversely, let $\imath_{\varphi_\omega}(S) = S$ and $a \in S$. Then, there exists an ω-φ-open set containing a such that $V \subseteq S$. Hence, $S \in \mathfrak{g}_{(\varphi_\omega)}$. □

Definition 40. *Let φ and φ' be two GNSs on X and Y, respectively. Then a function $f : (X, \varphi) \to (Y, \varphi')$ is called ω-(φ, φ')-continuous (respectively, ω-weakly-(φ, φ')-continuous) for $a \in X$ and $V \in \varphi'(f(a))$, there exists ω-φ-open set U containing a such that $f(U) \subseteq V$ (respectively, $f(U) \subseteq \gamma_{\varphi'}(V)$).*

Proposition 41. *Every (φ, φ')−continuous function is ω-(φ, φ')–continuous.*

Proof. The proof is straightforward by Lemma 35(2). □

Proposition 42. *Every weakly-(φ, φ')-continuous function is ω-weakly-(φ, φ')–continuous.*

Proof. It is clear from Lemma 35(2). □

Proposition 43. *Every ω-(φ, φ')-continuous function is ω-weakly-(φ, φ')–continuous.*

Proof. It is obvious since $\gamma_{\varphi'}$ is enlarging. □

We can give an example to show that the converse implications of Proposition 41 and 42 do not hold.

Example 44. *Let $X = \{1, 2, 3\}$ and two GNSs φ and φ' be defined as follows: $\varphi(1) = \{X\}, \varphi(2) = \{\{2, 3\}\}, \varphi(3) = \{X\}$ $\varphi'(1) = \{\{1\}\}, \varphi'(2) = \{\{2, 3\}\}, \varphi'(3) = \{\{1, 3\}\}$.*
Let $f : (X, \varphi) \to (X, \varphi')$ be a function defined by $f(1) = f(2) = 1, f(3) = 2$. Then, f is not (φ, φ')-continuous and not weakly-(φ, φ')-continuous but it is ω-(φ, φ')-continuous and ω-weakly-(φ, φ')-continuous.

We can give an example to show that the converse of Proposition 43 does not hold.

Example 45. *Let $X = Y = \mathbb{R}$ and two GNSs φ and φ' be defined as follows:*

$$\varphi(a) = \begin{cases} \{\mathbb{R} - \mathbb{Q}\} & \text{if } a \in \mathbb{R} - \mathbb{Q} \\ \{\mathbb{R}\} & \text{if } a \in \mathbb{Q} \end{cases} \quad \text{and} \quad \varphi'(a) = \begin{cases} \{\mathbb{R}\} & \text{if } a \in \mathbb{R} - \mathbb{Q} \\ \{\mathbb{Q}\} & \text{if } a \in \mathbb{Q} \end{cases}$$

Let $f : (X, \varphi) \to (Y, \varphi')$ be a function defined by

$$f(a) = \begin{cases} \sqrt{2} & \text{if } a \in \mathbb{R} - \mathbb{Q} \\ 1 & \text{if } a \in \mathbb{Q} \end{cases}$$

Then, f is ω-weakly-(φ, φ')-continuous but it is not ω-(φ, φ')-continuous.

Therefore, we obtain the following diagram from Proposition 19 and Proposition 41, 42 and 43.

(φ, φ')-continuous	$\Longrightarrow$	ω-(φ, φ')-continuous
$\Downarrow$		$\Downarrow$
weakly-(φ, φ')-continuous	$\Longrightarrow$	ω-weakly-(φ, φ')-continuous

Theorem 46. *Let $\varphi \in \Phi(X)$, $\varphi' \in \Phi(Y)$ and $f : (X, \varphi) \to (Y, \varphi')$ be a function. If f is ω-(φ, φ')-continuous, then it is ω-$(\mathfrak{g}_{(\varphi_\omega)}, \mathfrak{g}_{\varphi'})$-continuous.*

Proof. Let $a \in X$ and $G \in \mathfrak{g}_{\varphi'}$ containing $f(a)$. Then, there exists $V \in \varphi'(f(a))$ such that $V \subseteq G$. Since f is ω-(φ, φ')-continuous, there is an ω-φ-open set U containing a such that $f(U) \subseteq V$. Since $U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(G)$ and U is ω-φ-open containing a, then $a \in f^{-1}(G) \in \mathfrak{g}_{(\varphi_\omega)}$. Thus, f is ω-$(\mathfrak{g}_{(\varphi_\omega)}, \mathfrak{g}_{\varphi'})$-continuous from $f(f^{-1}(G)) \subseteq G$. □

Theorem 47. *Let $\varphi \in \Phi(X)$, $\varphi' \in \Phi(Y)$ and $f : (X, \varphi) \to (Y, \varphi')$ be a function. If f is ω-weakly-(φ, φ')-continuous, then it is ω-weakly-$(\mathfrak{g}_{(\varphi_\omega)}, \mathfrak{g}_{\varphi'})$-continuous.*

Proof. Let $a \in X$ and $G \in \mathfrak{g}_{\varphi'}$ containing $f(a)$. Then there is $U \in \varphi'(f(a))$ such that $U \subseteq G$. Since f is ω-weakly-(φ, φ')-continuous, there exists ω-φ-open set V containing a such that $f(V) \subseteq \gamma_{\varphi'}(U)$. By Lemma 37(2), we have $f(V) \subseteq \gamma_{\varphi'}(U) \subseteq \gamma_{\varphi'}(G)$. Since $V \subseteq f^{-1}(\gamma_{\varphi'}(G))$ and V is ω-φ-open containing a, then $f^{-1}(\gamma_{\varphi'}(G))$ belongs to $\mathfrak{g}_{(\varphi_\omega)}$. Thus, we have $f(f^{-1}(\gamma_{\varphi'}(G))) \subseteq \gamma_{\varphi'}(G) \subseteq c_{\mathfrak{g}_{\varphi'}}(G)$ from Lemma 4(2). Hence, f is ω-weakly-$(\mathfrak{g}_{(\varphi_\omega)}, \mathfrak{g}_{\varphi'})$-continuous. □

We give the following example to show that the converse implications of Theorem 46 and Theorem 47 do not hold.

Example 48. *Let $X = Y = \mathbb{R}$ and two GNSs φ and φ' be defined as follows:*

$$\varphi(a) = \{\mathbb{R}\} \quad \text{and} \quad \varphi'(a) = \begin{cases} \{[a, \infty)\} & \text{if } a \in \mathbb{Q} \\ \{(-\infty, a]\} & \text{if } a \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Let $f : (X, \varphi) \to (Y, \varphi')$ be a function defined by $f(a) = a$. Then, f is ω-$(\mathfrak{g}_{(\varphi_\omega)}, \mathfrak{g}_{\varphi'})$-continuous and ω-weakly-$(\mathfrak{g}_{(\varphi_\omega)}, \mathfrak{g}_{\varphi'})$-continuous but it is not ω-(φ, φ')-continuous and not ω-weakly-(φ, φ')-continuous.

Finally, we attain the following diagram by Proposition 27 and 43 and Theorem 46 and 47.

$$\begin{array}{ccc} \omega\text{-}(\varphi, \varphi')\text{-continuous} & \Longrightarrow & \omega\text{-}(\mathfrak{g}_{(\varphi_\omega)}, \mathfrak{g}_{\varphi'})\text{-continuous.} \\ \Downarrow & & \Downarrow \\ \omega\text{-weakly-}(\varphi, \varphi')\text{-continuous} & \Longrightarrow & \omega\text{-weakly-}(\mathfrak{g}_{(\varphi_\omega)}, \mathfrak{g}_{\varphi'})\text{-continuous} \end{array}$$

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