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Research Article

4-Boyutlu 2 indeksli Yarı Öklid Uzayındaki Null Kuaterniyonik Bertrand Eğriler

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Öz

Bu çalışmada, 4-boyutlu 2 indeksli yarı Öklid uzayındaki null kuaterniyonik Bertrand eğrileri tanımlanmıştır. Ayrıca, null kuaterniyonik Bertrand eğrileri için bir karakterizasyon elde edilmiştir.

Anahtar Kelimeler: Bertrand eğri, Null kuaterniyonik eğri, Null kuaterniyonik çatı.

Null Quaternionic Bertrand Curves in Semi Euclidean 4-Space $\mathbb{R}_2^4$

Abstract

In this study, we define null quaternionic Bertrand curves in semi Euclidean spaces $\mathbb{R}_2{}^4$. We also obtain a characterization for null quaternionic Bertrand curves.

Keywords: Bertrand curves, Null quaternionic curve, Null quaternionic frame.

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1. Introduction

Bonnor introduced a Cartan frame for null curves in $\mathbb{R}_1^4$ and he proved the fundamental existence and congruence theorems. Bejancu gives a method for the general study of the geometry of null curves in lightlike manifolds and in semi-Riemannian manifolds. A. Ferrandez et al. show that geometry of null curves in semi-Riemannian manifolds of index two has been constructed. And then, Duggal and Jin studied major developments of null curves, hypersurfaces and their physical use, with voluminous bibliography in their recent book.

Bertrand curve theory is widely studied by many mathematician since it is firstly introduced by Bertrand for answering a question about the relationship between principal normals of two curves. A Bertrand curve is a curve such that its principal normal is the principal normal of the second curve. Matsuda and Yorozu proved that there is not special Bertrand curve in $E^n(n > 3)$ and they defined new type which is called $(1,3)$ −type Bertrand curve in 4-dimensional Euclidean space (Matsuza, Yorozu 2003). The study of this kind of curves has been extended to many other ambient spaces. Uçum et al. studied $(1,3)$ −type Cartan null Bertrand curves in E_2^4.

Quaternions were discovered by Hamilton as an extension to the complex number in 1843. The theory of Frenet frames for a quaternionic curve was studied and developed by several researchers in this field. In 1987, the Serret--Frenet formulas for a quaternionic curve in E^3 and E^4 were defined by Bharathi and Nagaraj and then in 2004, Serret--Frenet formulas for quaternionic curves and quaternionic inclined curves were defined in semi-Euclidean space by Çöken and Tuna. Later, Keçilioğlu and İlarslan studied quaternionic Bertrand curves in Euclidean 4-space . In 2015, Serret--Frenet formulas for null quaternionic curves were defined in semi- Euclidean spaces by Çöken and Tuna. Recently, Gök and Aksoyak defined a quaternionic Bertrand curve in E^4 and investigate its properties. Finally, A. Tuna Aksoy studied null quaternionic Bertrand curves in Minkowski space $\mathbb{R}_1^4$. Then, we established a relation of Bertrand pairs with null quaternionic Cartan helices in $\mathbb{R}_\upsilon^3$.

The main goal of this paper is to define $(1,3)$ −type of Cartan null quaternionic Bertrand curves in semi-Euclidean 4-space with index 2. In the particular case where null quaternionic curves are parametrized by the pseudo-arc length parameter. Here, by using the similar idea of Matsuda and Yorozu, we give the necessary and sufficient conditions for a Cartan null quaternionic curve to be a $(1,3)$ −Bertrand curve in $\mathbb{R}_2^4$.

2. Preliminaries

Let Q_H denotes a four dimensional vector space over the field H of characteristic grater than 2. Let e_i $(1 \le i \le 4)$ denote a basis for the vector space. Let the rule of multiplication on Q_H be defined on e_i $(1 \le i \le 4)$ and extended to the whole of the vector space by distributivity as follows:

The set of the semi real quaternions is defined by

$$Q_H = \{q \| q = ae_1 + be_2 + ce_3 + d; a,b,c,d \epsilon \mathbb{R}, e_1, e_2, e_3 \epsilon \mathbb{R}^3_{v(v=1,2)}, h_v(e_i, e_i) = \varepsilon(e_i), 1 \le i \le 3.\}$$

Where

$$e_i^2 = -\varepsilon(e_i), \qquad 1 \le i \le 3,$$

$$e_i \times e_j = \varepsilon(e_i)\varepsilon(e_j)e_k \quad in\ \mathbb{R}_1^3$$

$$e_i \times e_j = -\varepsilon(e_i)\varepsilon(e_j)e_k \quad in\ \mathbb{R}_2^4$$

and (ijk) is an even permutation of (123). The multiplication of two semi real quaternions p and q is defined by:

$$\mathrm{p} \times \mathrm{q} = S_p S_q + S_p V_q + S_q V_p + \mathrm{h}(V_p, V_q) + V_p \Lambda V_q \text{ for every } \mathrm{p}, \mathrm{q} \in Q_\upsilon$$

where we have used the inner and cross products in semi-Euclidean space $\mathbb{R}_\upsilon^3$. For a semi real quaternion

$q = ae_1 + be_2 + ce_3 + d \epsilon Q_\upsilon$, the conjugate αq of q is defined by $\alpha q = -ae_1 - be_2 - ce_3 + d$

Now, we recall the notion of null quaternionic curve in $\mathbb{R}_2^4$. Let $\beta: I \subset R \to {\mathbb{R}_2}^4$ be a null quaternionic curve in semi Euclidean spaces.
We define symmetric, non- degenerate valued bilinear form h as follows: $h_v\ :\ Q_v \times Q_v \to \mathbb{R}$ by

$$h_1(p,q) = \frac{1}{2}[\varepsilon(p)\varepsilon(\alpha q)(p \times \alpha q) + \varepsilon(q)\varepsilon(\alpha p)(q \times \alpha p)]\ for\ \mathbb{R}_1^3$$

$$h_2(p,q) = \frac{1}{2}[-\varepsilon(p)\varepsilon(\alpha q)(p \times \alpha q) - \varepsilon(q)\varepsilon(\alpha p)(q \times \alpha p)]\ for\ \mathbb{R}_2^4$$

And then, the norm of semi real quaternion q is denoted by

$$\|q\|^2 = |h_v(q,q)| = |a^2\varepsilon(e^1) + b^2\varepsilon(e^2) + c^2\varepsilon(e^3) + d^2|for\ \upsilon = \{1,2\}.$$

The concept of a spatial quaternion will be made use throughout our work. q is called a spatial quaternion whenever $q + \alpha q = 0$. It is a temporal quaternion whenever $q - \alpha q = 0$ (Bharathi & Nagaraj,1987; Çöken & Tuna, 2004; Gök & Aksoyak, 2013; Keçilioğlu & Ilarslan, 2013; Tuna Aksoy & Çöken 2018 ; Tuna Aksoy, 2016; Tuna Aksoy & Çöken, 2017).

We recall some standard facts concerning null quaternionic curves in $\mathbb{R}_2^4$. A quaternionic curve $\beta(s)$ in $\mathbb{R}_2^4$ is called a null quaternionic curve if $h(\beta'(s), \beta'(s)) = 0$ and $\beta'(s) \neq 0$ for all s. We note that a null quaternionic curve $\beta(s)$ in $\mathbb{R}_2^4$ satisfies $h(\beta''(s), \beta''(s)) = \mp 1$. We say that a null quaternionic curve $\beta(s)$ in $\mathbb{R}_2^4$ is parametrized by the pseudo-arc length s if $h(\beta''(s), \beta''(s)) = 1$, (Tuna Aksoy & Çöken, 2018).

If β is a Cartan null quaternionic curve, the Frenet formulas read (Tuna Aksoy & Çöken, 2018)

$L' = KW, N' = (\tau - p)L, U' = -(\tau - p)N + pW,\ W' = pL + KU$ and

$$L' = KW, N' = -(\tau + p)L, U' = (\tau + p)N + pW,\ W' = pL + KU \qquad (2.1)$$

where the first curve $K = 1$. Then the following conditions are satisfied:

$$h(L, L) = h(U, U) = h(L, W) = h(N, U) = h(N, W) = h(U, W) = h(L, N) = 0, h(N, N) = h(L, U) = -1 \ and \ h(W, W) = +1.$$

We give the following characterization theorems for null quaternionic Bertrand pair in $\mathbb{R}_2^4$.

3. Null Quaternionic Bertrand Curves in $\mathbb{R}_2^4$

Definition 3.1. A null quaternionic Cartan curve $\beta: I \to \mathbb{R}_2^4$ with $K(s) = 1$ is a Bertrand curve if there is a curve $\beta^*: I^* \to \mathbb{R}_2^4$ such that the principal normal vectors of $\beta(s)$ and $\beta^*(s^*)$ at $s \epsilon I$, $s^* \epsilon I^*$ are equal. In this case, $\beta^*(s^*)$ is the bertrand mate of $\beta(s)$.

Theorem 3.2. There exists no Cartan null Bertrand curve $\beta: I \to \mathbb{R}_2^4$ with curvature functions $K = 1$, $(\tau - p) \neq 0$ and $p \neq 0$.

Proof. Let $\beta(s)$ and $\beta^*(s^*)$ be null quaternionic Bertrand curves, with respect to a special paramerter s^* and suppose that $\{L, N, U, W\}$ and $\{L^*, N^*, U^*, W^*\}$ are their quaternionic Cartan frames, respectively. Then we can write

$$\beta^*(s^*) = \beta(s) + \lambda(s)N(s) \qquad (3.1)$$

Suppose s and s^* are the pseudo-arc paramerters of β and β^*, respectively, then by taking derivative of (3.1) with respect to s and using null quaternionic Cartan frame, we get

$$\frac{ds^*}{ds} L^* = (1 + \lambda(\tau - p))L + \lambda' N.$$

On the other hand, the condition

$$h(L^*, L^*) = (L^* \times \alpha L^*) = 0$$

holds for null quaternionic Bertrand curves, hence $\lambda' = 0$, we deduce that λ is a nonzero constant. This means that

$$\frac{ds^*}{ds} L^* = (1 + \lambda(\tau - p))L$$

which is a contradiction. Thus, we prove that there is no Cartan null quaternionic Bertrand curve in the semi Euclidean space $\mathbb{R}_2^4$ for $K = 1, (\tau - p) \neq 0$ and $p \neq 0$.

4. (1, 3)-Null Quaternionic Bertrand Curves in $\mathbb{R}_2^4$

Definition 4.1. Let $\beta: I \to \mathbb{R}_2^4$ be a null quaternionic Cartan Frenet curve. The plane spanned by the principal normal vector $N(s)$ and the second binormal vector $W(s)$ is called the $(1,3)$ −normal plane of β at the point $s \in I$.

Definition 4.2. Let $\beta: I \to \mathbb{R}_2^4$ and $\beta^*: I^* \to \mathbb{R}_2^4$ be a null quaternionic Cartan Frenet curves. If the Frenet $(1,3)$ −normal plane of β coincides with the $(1,3)$ −normal plane of β^* at corresponding points, then β is called a $(1,3)$ −Bertrand curve and β^* is called the $(1,3)$ −Bertrand mate curve of β.

Let $\beta: I \to \mathbb{R}_2^4$ be a $(1,3)$ −Cartan null Bertrand curve in $\mathbb{R}_2^4$ with the Frenet frame $\{L, N, U, W\}$ and the curvatures $K, (\tau - p), p$ and $\beta^*: I^* \to \mathbb{R}_2^4$ be a $(1,3)$ −Bertrand mate curve of β with the Frenet frame $\{L^*, N^*, U^*, W^*\}$ and the curvatures K^*, $(\tau^* - p^*)$, p^*.

Theorem 4.3. Let $\beta: I \to \mathbb{R}_2^4$ be a Cartan null quaternionic curve with curvature functions $K = 1$, $(\tau - p) \neq 0$ and $p \neq 0$. Then the curve β is a $(1, 3)$ −Bertrand curve with Bertrand mate β^* if and only if one of the following condition holds:

there exist constant real numbers $b \neq 0$, $a, \gamma \neq 0$ and μ satisfying

$$a(\tau - p) + bp = -1, p^2 - (\tau - p)^2 = \frac{b^2}{\gamma^4}, \mu = \frac{\lambda_1}{\lambda_2}$$

In this case, β^* is a Cartan null quaternionic curve in $\mathbb{R}_2^4$.

Proof. Assume that β is a $(1,3)$ −Bertrand Cartan null quaternionic curve parametrized by pseudo arc-length s and with Cartan first curvature always equal to 1, the second curvature and the third curvature are always a non zero, and the curve β^* is the $(1,3)$ −Bertrand mate curve with pseudo arc-length s^* of the curve β. Let β^* be a Cartan null quaternionic curve. Then, we can write the curve β^*as

$$\beta^*(s^*) = \beta(s) + a(s)N(s) + b(s)W(s), s^* = f(s) \qquad (4.1)$$

for all $s \in I$ where $a(s)$ and $b(s)$ are non zero constant. Differentiating (4.1) with respect to s and using Cartan frame (2.1), we obtain

$$L^*f' = (1 + a(\tau - p) + bp)L + a'N + bU + b'W \qquad (4.2)$$

Since L^* is null, $h(L^*, L^*) = L^* \times \alpha L^* = 0$ hold. We obtain $(1 + a(\tau - p) + bp)b = 0$. Here, $a = -(1/(\tau - p))$ if $b = 0$, this is a contradiction with our assumption, and $a(\tau - p) + bp = -1$ if $b \neq 0$. By taking the scalar product of (4.2) with N and W, respectively, we have $a' = b' = 0$. Substituting it in (4.2), we have

$$L^*f' = (1 + a(\tau - p) + bp)L + bU. \qquad (4.3)$$

If we denote

$$\delta = \frac{1+a(\tau-p)+bp}{f'} \; and \; \gamma = \frac{b}{f'} \qquad (4.4)$$

we obtain

$$L^* = \delta L + \gamma U. \qquad (4.5)$$

Since L^* is null, $h(L^*, L^*) = L^* \times \alpha L^* = 0$ hold. We obtain $\delta\gamma = 0$. Here, $L^* = \gamma U$ if $\delta = 0$ and $L^* = \delta L$ if $\gamma = 0$, which is a contraction. Thus $\gamma \neq 0$ and $\delta = 0$ which leads that $b \neq 0$ and $a(\tau - p) + bp = -1$. Then, we get $L^* = \gamma U$. Differentiating it with respect to s and using the Frenet formulas (2.1), we obtain

$$W^*f' = -\gamma(\tau - p)N + \gamma pW. \qquad (4.9)$$

On the other hand, the condition $h(W^*, W^*) = W^* \times \alpha W^* = 1$

holds for null quaternionic Bertrand curves, from (4.4) and (4.9), we obtain $p^2 - (\tau - p)^2 = \frac{b^2}{\gamma^4}$, where $p^2 \neq (\tau - p)^2$. If we denote

$$\lambda_1 = -\frac{\gamma(\tau-p)}{f'} \text{ and } \lambda^2 = \frac{\gamma p}{f'} \qquad (4.10)$$

from (4.10), we get

$$W^* = \lambda_1 N + \lambda_2 W.$$

Differentiating it with respect to s and using the Frenet formulas (2.1), we get

$$f'p^*L^* + f'U^* = (\lambda_1(\tau - p) + \lambda_2 p)L + \lambda_1' N + \lambda_2 U + \lambda_2' W \quad (4.11)$$

By taking the scalar product of (4.11) with N and W, we have $\lambda_1' = \lambda_2' = 0$. From (4.10), we find $\mu = -\frac{(\tau-p)}{p}$, where $\mu = \frac{\lambda_1}{\lambda_2} = \pm 1$.

Conversely, assume that β is a Cartan null quaternionic curve and the condition

$$a(\tau - p) + bp = -1, p^2 - (\tau - p)^2 = \frac{b^2}{\gamma^4}, \mu = \frac{\lambda_1}{\lambda_2}$$

holds for constant real numbers $b \neq 0$, a, $\gamma \neq 0$ and μ. Then, we can define a curve β^* as

$$\beta^*(s^*) = \beta(s) + a(s)N(s) + b(s)W(s), \; s^* = f(s) \quad (4.12)$$

Differentiating (4.12) with respect to s and using the Frenet formulas (2.1), we find

$$\frac{d\beta^*}{ds} = bU \qquad (4.13)$$

Differentiating (4.13) with respect to s and using the Frenet formulas (2.1), we obtain

$$\frac{d^2\beta^*}{ds^2} = -(\tau - p)bN + bpW \qquad (4.14)$$

From (4.14), we have

$$h\left(\frac{d^2\beta^*}{ds^2}, \frac{d^2\beta^*}{ds^2}\right) = \frac{b^4}{\gamma^4} \qquad (4.15)$$

Now, we find

$$f' = \left|h\left(\frac{d^2\beta^*}{ds^2}, \frac{d^2\beta^*}{ds^2}\right)\right|^{\frac{1}{4}} = m\frac{b}{\gamma} \qquad (4.16)$$

where $m = \pm 1$ such that $m\frac{b}{\gamma}\rangle 0$. Rewriting (4.13), we get

$$L^* = m\gamma U \qquad (4.17)$$

Differentiating (4.17) with respect to s and using the Frenet formulas (2.1), we obtain

$$\frac{dL^*}{ds^*} = -\frac{m\gamma(\tau-p)}{f'}N + \frac{m\gamma p}{f'}W \qquad (4.18)$$

From (4.18), we have

$$K^* = \|\frac{dL^*}{ds^*}\| = 1 \qquad (4.19)$$

Now, we can find

$$W^* = -\frac{m\gamma(\tau-p)}{f'}N + \frac{m\gamma p}{f'}W \qquad (4.20)$$

where $h(W^*, W^*) = 1$. If we denote

$$\lambda_3 = -\frac{m\gamma(\tau-p)}{f'} \text{ and } \lambda_4 = \frac{m\gamma p}{f'} \qquad (4.21)$$

we get

$$W^* = \lambda_3 N + \lambda_4 W \qquad (4.22)$$

From last two conditions of (4.1), we get (τ-p) and p are constants, which leads that $\lambda_3' = 0$ and $\lambda_4' = 0$. Differentiating (4.22) with respect to s and using the Frenet formulas (2.1), we obtain

$$(p^*L^* + U^*)f' = (\lambda_3(\tau-p) + \lambda_4 p)L + \lambda_4 U \qquad (4.23)$$

Substituting (4.21) in (4.23), we get

$$\frac{dW^*}{ds^*} = \frac{m}{\gamma}L + \frac{\mathrm{m\gamma p}}{(f')^2}U \qquad (4.24)$$

From (4.24), we can define p^* as

$$p^* = \frac{1}{2}h\left(\frac{dW^*}{ds^*}, \frac{dW^*}{ds^*}\right) = \frac{p}{(f')^2}$$

Now, we can define U^* and N^* as

$$U^* = \frac{m}{\gamma}L$$

$$N^* = \frac{(p^*m\gamma(\tau-p))}{f'(\tau-p)^*}N + \frac{m}{f'(\tau-p)^*}[p^*\gamma + \frac{1}{\gamma}]W$$

Also, $(\tau-p)^*$is defined as follows

$$(\tau-p)^* = h(\frac{dN^*}{ds^*}, U^*) = (-\frac{1}{\gamma^2 f'} + \frac{p^2}{(f')^3})^2$$

Thus β is a $(1,3)-$null quaternionic Bertrand curve and its $(1,3)-$null quaternionic Bertrand mate curve is a Cartan null quaternionic curve.

For other frame, the proofs can be done similarly.

Conclusion

Characterizations of the curves are very important in terms of differential geometry. We obtain a characterization for null quaternionic Bertrand Curves.

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