Avrupa Bilim ve Teknoloji Dergisi
Sayı 16, S. 205-210, Ağustos 2019

Araştırma Makalesi

www.ejosat.com ISSN:2148-2683

European Journal of Science and Technology
No. 16, pp. 205-210, August 2019

Research Article

Efficient Method for the Solution of Fractional-order Differential Equations with Variable Coefficients

Arzu Turan Dincel[1*]

[1] Yildiz Technical University, Faculty of Chemical and Metallurgical Engineering, Mathematical Engineering Department, Istanbul-Turkey (ORCID: 0000-0002-0416-1878)

(First received 30 March 2019 and in final form 15 May 2019)

(**DOI:** 10.31590/ejosat.547166)

ATIF/REFERENCE: Turan Dincel, A. (2019). Efficient Method for the Solution of Fractional-order Differential Equations with Variable Coefficients. *European Journal of Science and Technology*, (16), 205-210.

Abstract

In this paper, we propose the Bernoulli wavelet approximation for the solution of the fractional differential equations with variable coefficients. In the proposed method, the fractional derivatives are transformed using the operational matrix of fractional order integration and by doing that differential equation reduces to a system of algebraic equations. The operational matrix of fractional order integration is obtained via block pulse functions. Illustrative examples are presented. The examples demonstrate that the method is accurate and efficient.

Keywords: Bernoulli wavelet, Fractional-order differential equations with variable coefficients, Caputo derivative, Operational matrix.

1. Introduction

The fractional calculus is a branch of applied mathematics focused on derivatives and integrals to any arbitrary order of real or complex numbers. It is used to model several real-life phenomena in many fields of engineering and science. For example, fractional calculus or differential equations applied to model the mechanics and dynamical systems (Cajić, Karličić and Lazarević, 2015; Wang et al., 2017) and, environmental sciences (Moradi and Mehdinejadiani, 2018; Sun et al., 2014) signal and image processing (Li and Yu, 2006; Chen, Chen and Xue, 2013; Nigmatullin, Osokin and Toboev, 2011), colored noise (Mandelbrot, 1967), macroeconomics models (Tarasova and Tarasov, 2017), biology (Karaman et al., 2016) , materials (Lei, Liang and Xiao, 2018), optimal control (Karimi et al., 2005) and so on.

However, many of these fractional differential equations do not have analytical solutions, therefore various numerical algorithms are developed. Homotopy analysis method was applied to solve fractional initial value problem by Hashim, Abdulaziz and Momani, 2009. Differential transform method was presented by Arikoglu and Ozkol, 2009. Homotopy perturbation method was studied by Khader, 2017; Li and Sun, 2011. The Laplace transform method was examined by Gupta, Kumar and Singh, 2015.

In this paper, we use Bernoulli wavelet method to solve the fractional differential equations with variable coefficients in the form:

$$D^{\alpha} y(t) + \sum_{j=1}^{r-1} \gamma_j(t) D^{\beta_j} y(t) + \gamma_r(t) y(t) = g(t)\,,\ t>0\,,\ n<\alpha\le n+1 \tag{1}$$

$$y^{i}(0)=0\,,\ i=0,1,2,\cdots,n \tag{2}$$

where α is the fractional derivative-order parameter, $0<\beta_1<\beta_2<\ldots<\beta_{r-1}<\alpha$, n is an integer and D^{α} is the Caputo fractional differential operator. This method presents a procedure to reduce the fractional differential equations to a system of algebraic equations by using a family of Bernoulli wavelets. The organization the paper is as follows: Section 2 includes some necessary definitions of the fractional calculus. In Section 3, after explaining Bernoulli wavelets, the Bernoulli wavelet operational matrix of the fractional

[*] Corresponding Author: Yildiz Technical University, Faculty of Chemical and Metallurgical Engineering, Mathematical Engineering Department, Istanbul-Turkey (ORCID: 0000-0002-0416-1878),e-mail: artur@yildiz.edu.tr

integration is derived. In Section 4, we present the numerical method to solve the fractional order differential equations. Numerical example results are provided in Section 5. Last section includes concluding remarks.

2. Basic Definitions

Definition 2.1 The Riemann-Liouville fractional integral operator of order α is defined as:

$$(I^{\alpha} f)(t)=\begin{Bmatrix} \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau & \alpha>0, t>0 \\ f(t) & \alpha=0 \end{Bmatrix}. \tag{3}$$

Definition 2.2 The Caputo definition of fractional derivative operator is defined by the following expression

$$(D^{\alpha} f)(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{1-n+\alpha}}d\tau \qquad 0\le n-1<\alpha\le n. \tag{4}$$

It has the basic properties of:

$$(D^{\alpha} I^{\alpha} f)(t)=f(t) \text{ and } (I^{\alpha} D^{\alpha} f)(t)=f(t)-\sum_{k=0}^{n-1} f^{(k)}(0^{+})\frac{t^k}{k!}. \tag{5}$$

3 Bernoulli Wavelet and Operational Matrix of the Fractional Integration

3.1 Bernoulli polynomial and Bernoulli wavelets

Let $\beta_m(t)$ denote the Bernoulli polynomials of order m, which is given by Rahimkhani, Ordokhani and Babolian, 2016

$$\beta_m(t)=\sum_{i=0}^{m}\binom{m}{i}\alpha_{m-i}t^i \tag{6}$$

where α_i, $i=1,2,\cdots,m$ are Bernoulli numbers. These polynomials satisfy the following orthogonality condition (Rahimkhani, Ordokhani and Babolian, 2016).

$$\int_0^1 \beta_n(t)\beta_m(t)=(-1)^{n-1}\frac{m!n!}{(m+n)!}\alpha_{n+m}, \quad m, n\ge 1 \tag{7}$$

The Bernoulli wavelets are defined on interval [0,1) by

$$\psi_{nm}(t)=\begin{Bmatrix} 2^{\frac{k-1}{2}}\tilde{\beta}_m(2^{k-1}t-\tilde{n}) \quad , & \frac{\tilde{n}}{2^{k-1}}\le t<\frac{\tilde{n}+1}{2^{k-1}} \\ 0 \quad , & otherwise \end{Bmatrix} \tag{8}$$

with

$$\tilde{\beta}_m(t)=\begin{Bmatrix} 1 \quad , & m=0 \\ \dfrac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}\beta_m(t) \quad , & m>0 \end{Bmatrix} \tag{9}$$

where $\tilde{n}=n-1$, $n=1,2,\cdots,2^{k-1}$, $m=0,1,2,\cdots,M-1$, k is any positive integer value, t is the normalized time (Rahimkhani, Ordokhani and Babolian, 2016).

3.2 Function approximation

A function f defined over $[0,1)$ may be expressed with Bernoulli wavelets as

$$f(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\Psi_{nm}(t)=C^T\psi(t) \tag{10}$$

where C and $\psi(t)$ are $m'\times 1$ ($m'=2^{k-1}M$) vectors defined by

$$C=\left[c_{10},c_{11},\ldots c_{1(M-1)},c_{20},c_{21},\ldots c_{2(M-1)}\ldots c_{2^{k-1}0},c_{2^{k-1}1},\ldots c_{2^{k-1}(M-1)}\right]^T \tag{11}$$

$$\Psi=\left[\Psi_{10},\Psi_{11},\ldots\Psi_{1(M-1)},\Psi_{20},\Psi_{21},\ldots\Psi_{2(M-1)}\ldots\Psi_{2^{k-1}0},\Psi_{2^{k-1}1},\ldots\Psi_{2^{k-1}(M-1)}\right]^T. \tag{12}$$

The Bernoulli wavelet matrix is defined as

$$\phi_{m'xm'}=\left[\Psi(t_1)\ \Psi(t_2)\ \Psi(t_3)\ \cdots\ \Psi(t_{m'})\right] \tag{13}$$

where t_i are collocation points. We take the collocation points as $t_i=\dfrac{i-0.5}{m'}$, $i=1,2,3,\ldots,m'$.

3.3 Operational matrix of the fractional integration

We present the operational matrix of fractional order integration using the Block Pulse Functions (BPFs).The set of BPFs is defined as

$$b_i(t)=\begin{cases}1 & (i-1)/m'\le t<i/m'\\ 0 & otherwise\end{cases}\quad,\qquad i=1,2,3,\ldots,m'\ . \tag{14}$$

For $t\in[0,1)$, following properties for these functions will be used in this paper

$$b_i(t)b_j(t)=\begin{cases}0 & i\ne j\\ b_i(t) & i=j\end{cases}, \tag{15}$$

$$\int_0^1 b_i(\tau)b_j(\tau)\,d\tau=\begin{cases}0 & i\ne j\\ 1/m' & i=j\end{cases}. \tag{16}$$

The Bernoulli wavelet may be expanded to an m' terms BPFs as

$$\psi(t)=\phi_{m'xm'}\,B_{m'}(t) \tag{17}$$

where $B_{m'}(t)=\left[b_1(t),b_2(t),\ldots,b_{m'}(t)\right]^T$. The fractional integral of BPFs vector can be expressed as

$$(I^\alpha B_{m'})(t)\approx F^\alpha\,B_{m'}(t) \tag{18}$$

where F^α is given with the Bernoulli wavelet operational matrix of the fractional integration $P^\alpha_{m'xm'}$ (Kilicman, 2007):

$$P^\alpha_{m'xm'}\approx\phi_{m'xm'}F^\alpha\phi^{-1}_{m'xm'}. \tag{19}$$

The convergence analysis of the Bernoulli wavelet basis can be found in the study of Rahimkhani, Ordokhani and Babolian, 2016.

4 Numerical Method

To solve problem (1) and (2) we approximate $D^\alpha y(t)$ and $g(t)$ by the Bernoulli wavelet as

$$D^\alpha y(t)\approx C^T\psi(t)\quad,\ g(t)\approx G^T\psi(t) \tag{20}$$

where $G=\left[g_0,g_1,\cdots g_{m'-1}\right]^T$. Taking the initial conditions into account, we obtain

$$D^{\beta_j}y(t)\approx C^T P^{\alpha-\beta_j}_{m'xm'}\psi(t),\quad y(t)\approx C^T P^\alpha_{m'\times m'}\psi(t) \tag{21}$$

by substituting Eqs. (20) and (21) in Eq. (1) we get a system of algebraic equations as follows:

$$C^T\psi+\sum_{j=1}^{r-1}C^T P_{m'xm'}^{\alpha-\beta_j}\psi A_j+C^T P_{m'xm'}^{\alpha}\psi A_r=G^T\psi \tag{22}$$

where $A_i=\begin{bmatrix}\gamma_i(t_0) & 0 & \cdots & 0\\ 0 & \gamma_i(t_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \gamma_i(t_{m'})\end{bmatrix}$, $i=1,2\cdots,r$.

The Newton-Raphson iteration method is used to solve the system of algebraic equations and the unknown coefficient values of C^T , and thus the approximate solution $y(t)$ are obtained.

5 Illustirative Examples

In this section, we demonstrate some numerical examples to show the efficiency and the accuracy of the presented method. The numerical results are obtained using Matlab R2017a.

Example 5.1: We first consider the following nonlinear fractional differential equation (Atabakzadeh, Akrami and Erjaee, 2013)

$$t^{7/2}D^{3/2}y(t)+D^2y(t)+y^2(t)=\left(1+\frac{4}{\sqrt{\pi}}\right)t^4+2\,,\;\; y(0)=y'(0)=0\,,t\in[0,1]$$

whose exact solution is given by $y(t)=t^2$.

Using the method presented in Section 4, we obtain the approximate solution of the fractional differential equation given above. The comparison of the method and exact solutions for $k=2$ and $M=3$ are shown in Figure 1. It is obvious that the numerical solutions are in perfect agreement with the exact solutions.

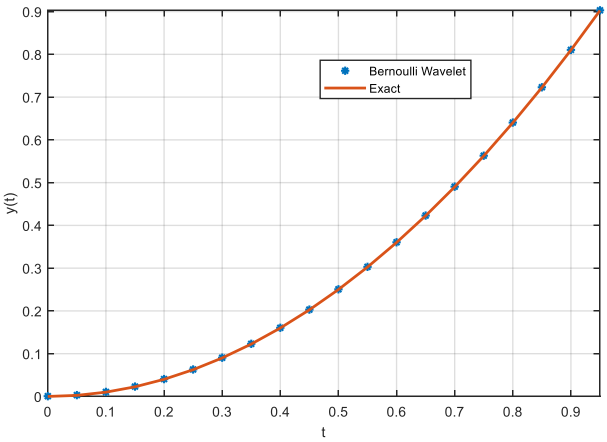

Figure 1. Bernoulli wavelet and the exact solution for Example 1.

Table 1. Comparison of the absolute errors for Example 1

t	**Present method**				**Chebyshev operational matrix method (Atabakzadeh, Akrami and Erjaee, 2013)**		
	$\boldsymbol{k=3}$	$\boldsymbol{k=5}$	$\boldsymbol{k=6}$	$\boldsymbol{k=8}$	$\boldsymbol{N=2}$	$\boldsymbol{N=3}$	$\boldsymbol{N=4}$
0.2	5.75E-04	3.62E-05	9.04E-06	5.65E-07	3.3E-3	2.2E-3	8.8E-4
0.4	5.66E-04	3.60E-05	9.01E-06	5.63E-07	1.3E-2	9.6E-3	7.1E-3
0.6	5.47E-04	3.54E-05	8.87E-06	5.55E-07	2.9E-2	1.3E-2	3.5E-3
0.8	5.13E-04	3.40E-05	8.54E-06	5.34E-07	5.3E-2	4.2E-2	2.8E-2

4 $m'=48$ $m'=96$ $m'=192$ $m'=384$

In Table 1, we present the absolute errors for several k values of Bernoulli wavelet method and Chebyshev operational matrix method (Atabakzadeh, Akrami and Erjaee, 2013). Numerical results of this initial value problem show that our method is more accurate than the Chebyshev operational matrix method.

Example 5.2 Consider the following initial value problem

$$D^{3/2}y(t)-t^{3/2}y(t)=4\sqrt{\frac{t}{\pi}}+t^{7/2} \quad y(0)=y'(0)=0\,,\, t\in(0,1]$$

with the exact solution of $y(t)=t^2$.

Again, using the method presented in Section 4, we obtain the approximate solution of the fractional differential equation given above. Table 2 shows absolute errors for $k=3,4,5,6,7$ and $M=3$. Moreover, the comparison of the numerical solutions and the exact solutions for $k=2$ and $M=3$ are plotted in Figure 2. As can be seen, numerical results demonstrate the accuracy of our method.

Table 2.The absolute errors of the Bernoulli wavelet method for $\alpha=1$ and various k values

t	*Exact*	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
0.1	0.01	7.91E-05	1.98E-05	4.94E-06	1.24E-06	3.09E-07
0.2	0.04	1.57E-04	5.01E-05	1.61E-05	5.29E-06	1.76E-06
0.3	0.09	2.01E-04	6.47E-05	2.12E-05	7.06E-06	2.38E-06
0.4	0.16	2.33E-04	7.59E-05	2.51E-05	8.43E-06	2.87E-06
0.5	0.25	2.61E-04	8.56E-05	2.85E-05	9.63E-06	3.29E-06
0.6	0.36	2.88E-04	9.48E-05	3.17E-05	1.08E-05	3.69E-06
0.7	0.49	3.14E-04	1.04E-04	3.49E-05	1.19E-05	4.08E-06
0.8	0.64	3.41E-04	1.13E-04	3.82E-05	1.30E-05	4.48E-06
0.9	0.81	3.71E-04	1.23E-04	4.17E-05	1.42E-05	4.91E-06

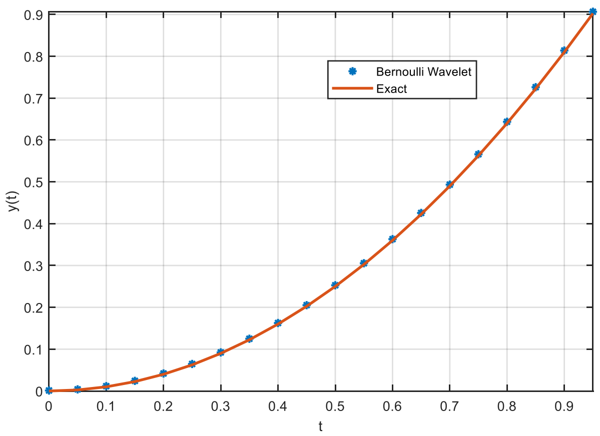

Figure 2. Bernoulli wavelet and the exact solution for Example 2.

6 Conclusion

In this paper, we intend to develop an accurate and effective method to solve the fractional differential equations with variable coefficients. To this end, the Bernoulli wavelet operational matrix of fractional order integration is used to approximate the fractional derivatives and to convert the fractional differential equations with variable coefficients into a system of algebraic equations. By solving the system of algebraic equations, we obtain the approximate solutions of the fractional differential equations. Since this transformation uses orthogonal wavelets, the corresponding operational matrix of fractional integration is a sparse matrix, which greatly contributes to have a fast and efficient solution method. The block pulse functions are employed to obtain the operational matrix of fractional order integration. Numerical results are presented to demonstrate the accuracy end the efficiency of the method.

References

Atabakzadeh M.H., Akrami M.H., & Erjaee G.H. (2013) Chebyshev operational matrix method for solving multi order fractional ordinary differential equations. Applied Mathematical Modelling, 37, 8903-8911.

Arikoglu A.,& Ozkol I. (2009), Solution of fractional integro-differential equations by using fractional differential transform method, Chaos, Solitons & Fractals, 40,521-529.

Cajić, M., Karličić D.,& Lazarević M. (2015) Nonlocal vibration of a fractional order viscoelastic nanobeam with attached nanoparticle. Theoretical and Applied Mechanics, 42(3), 167-190.

Chen D., Chen Y., & Xue D. (2013) Three fractional-order TV-models for image de-noising. Journal of Computer Information Systems, 9 (12), 4773-4780.

Gupta, S., Kumar, D.,& Singh, J.(2015) Numerical study for systems of fractional differential equations via Laplace transform. Journal of the Egyptian Mathematical Society, 23, 256–262.

Hashim I., Abdulaziz O., & Momani S. (2009) Homotopy analysis method for fractional IVPs. Communications in Nonlinear Science and Numerical Simulation, 14, 674–684.

Karaman M.M., Sui Y., Wang H., R.L. Magin, Li Y.,& Zhou X.J.(2016) Differentiating low- and high-grade pediatric brain tumors using a continuous-time random-walk diffusion model at high b-values. Magnetic Resonance in Medicine, 76, 1149-1157.

Karimi H., Moshiri B., Lohmann B., Maralani P. (2005) Haar wavelet-based approach for optimal control of second-order linear systems in time domain, Journal of Dynamical and Control Systems, 11, 237–252.

Khader M.M. (2017), Application of homotopy perturbation method for solving nonlinear fractional heat-like equations using Sumudu transform, Scientia Iranica B, 24, 648-655.

Kilicman A. (2007) Kronecker operational matrices for fractional calculus and some applications. Applied Mathematics and Computation, 187, 250-265.

Lei D., Liang Y., & Xiao R. (2018) A fractional model with parallel fractional maxwell elements for amorphous thermoplastics. Physics A, 490, 465-475.

Li Y., &Yu S.L. (2006)Fractional order difference filters and edge detection. Opto-Electronic Engineering, 33(19), 71-74.

Li Y.L.,& Sun N.(2011) Numerical solution of fractional differential equations using the generalized block pulse operational matrix, Computers & Mathematics with Applications 62 (3),1046-1054.

Mandelbrot B. (1967) Some noises with 1/f spectrum, a bridge between direct current and white noise. IEEE Transactions on Information Theory, 13(2), 289–98.

Moradi G.,& Mehdinejadiani B .(2018) Modeling solute transport in homogeneous and heterogeneous porous media using spatial fractional advection–dispersion equation. Soil and Water Research, 13, 18–28.

Nigmatullin R. R.,. Osokin S.I., & Toboev V.A. (2011) NAFASS: Discrete spectroscopy of random signals. Chaos Solitons Fract, 44, 226-240.

Rahimkhani P., Ordokhani Y. & Babolian E. (2016) An efficient approximate method for solving delay fractional optimal control problems. Nonlinear Dynamics, 86, 1649–1661.

Sun H.G., Zhang Y., Chen W. & Reeves D.M. (2014) Use of a variable-index fractional-derivative model to capture transient dispersion in heterogeneous media. Journal of Contaminant Hydrology, 157, 47–58.

Tarasova V.V.,& Tarasov V.E. (2017) Logistic map with memory from economic model. Chaos, Solitons and Fractals, 95, 84-91.

Wang X., Qi H., Yu B., Xiong Z., & Xu H. (2017) Analytical and numerical study of electroosmotic slip flows of fractional second grade fluids. Communications in Nonlinear Science and Numerical Simulation, 50, 77-87.